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Fundamental groups of curves in positive characteristic

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ABSTRACT. We present some recent results (mainly of the author) concerning fundamental groups of curves over algebraically closed fields of positive characteristic.

§1. Main question (cf. [H2,4][T3]).

In this §, we introduce our main question.

Throughout this article, we let k denote an algebraically closed field of characteristic $p > 0$, X a smooth curve over k, X^* the smooth compactification of X and $\Sigma \stackrel{\text{def}}{=} X^* - X$. We define non-negative integers g and n to be the genus of the proper, smooth curve X^* and the cardinality of the point set Σ, respectively. Note that X is hyperbolic (resp. affine, resp. projective) if and only if $2 - 2g - n < 0$, i.e., $(g,n) \neq (0,0),(0,1),(0,2),(1,0)$ (resp. $n > 0$, resp. $n = 0$).

We denote by $k(X)^\sim$ the maximal separable algebraic extension of $k(X)$ in which the discrete valuation ring $\mathcal{O}_{X,x}$ is unramified for all $x \in X$, and by $k(X)^{\sim\mathrm{t}}$ the maximal separable algebraic extension of $k(X)$ in which $\mathcal{O}_{X,x}$ is unramified for all $x \in X$ and at most tamely ramified for all $x \in \Sigma$. Then, the fundamental group $\pi_1(X)$ (resp. the tame fundamental group $\pi_1^{\mathrm{t}}(X)$) of X is nothing but the Galois group $\mathrm{Gal}(k(X)^\sim/k(X))$ (resp. $\mathrm{Gal}(k(X)^{\sim\mathrm{t}}/k(X))$).

Definition.
(i) For a (discrete) group Γ, we denote by $\Gamma^\wedge$ its profinite completion

$$\varprojlim_{\mathrm{N}\triangleleft\Gamma,\ (\Gamma:\mathrm{N})<\infty} \Gamma/\mathrm{N}.$$

(ii) For a profinite group G, we denote by $G^{p'}$ its maximal pro-prime-to-p quotient

$$\varprojlim_{N\triangleleft G \text{ closed},\ p\nmid(G:N)<\infty} G/N.$$

(iii) For non-negative integers g and n, we denote by $\Pi_{g,n}$ the topological fundamental group of a compact orientable surface of genus g with n points deleted. More concretely,

$$\Pi_{g,n} = \langle \alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g,\gamma_1,\dots,\gamma_n \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\dots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\gamma_1\dots\gamma_n = 1\rangle$$

In particular, if $n > 0$, $\Pi_{g,n}$ is a free group of rank $2g + n - 1$.

The following fact concerning $\pi_1(X)$ and $\pi_1^{\mathrm{t}}(X)$ is more or less well-known:

Lemma.
We have the following three surjections (i)–(iii)*:*

$$\begin{array}{ccccc} & & \pi_1(X) & & \\ & & \downarrow \text{(i)} & & \\ (\Pi_{g,n})^{\wedge} & \overset{\text{(ii)}}{\to} & \pi_1^{\mathrm{t}}(X) & \overset{\text{(iii)}}{\to} & (\Pi_{g,n})^{\wedge p'}. \end{array}$$

Moreover, we have:

$$\text{(i) } \textit{is an isomorphism} \iff n = 0,$$

$$\text{(ii) } \textit{is an isomorphism} \iff (g,n) = (0,0),(0,1),$$

and

$$\text{(iii) } \textit{is an isomorphism} \iff \begin{cases} \textit{either } (g,n) = (0,0),(0,1),(0,2), \\ \textit{or } (g,n) = (1,0) \textit{ and } X \textit{ is supersingular.} \end{cases}$$

Proof. Surjection (i) comes from the definitions of $\pi_1(X)$ and $\pi_1^{\mathrm{t}}(X)$ (or the above descriptions of $\pi_1(X)$ and $\pi_1^{\mathrm{t}}(X)$ in terms of Galois groups). For surjections (ii) and (iii), see [SGA1].

If $n = 0$, i.e., X is projective, then the natural surjection $\pi_1(X) \to \pi_1^{\mathrm{t}}(X)$ is an isomorphism by definition. On the other hand, if $n > 0$, i.e., X is affine, $\pi_1(X)$ is not topologically finitely generated (as its maximal pro-p quotient $\pi_1(X)^p$ is a free pro-p group of rank $|k|$), while $\pi_1^{\mathrm{t}}(X)$ is always topologically finitely generated as a quotient of $(\Pi_{g,n})^{\wedge}$. Thus, in this case, (i) cannot be an isomorphism.

Next, if $(g,n) = (0,0),(0,1)$, then we have $\Pi_{g,n} = \{1\}$. Thus, in this case, surjections (ii) and (iii) must be isomorphisms. If $(g,n) = (0,2)$ (resp. $(g,n) = (1,0)$ and X is supersingular), then $\pi_1^{\mathrm{t}}(X)$ coincides with $(\Pi_{g,n})^{\wedge p'} = \widehat{\mathbb{Z}}^{p'}$ (resp. $\widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'}$). Thus, surjection (iii) is then an isomorphism.

On the other hand, assume that (ii) is an isomorphism. Then, the abelianizations $(\Pi_{g,n})^{\wedge \mathrm{ab}}$ and $\pi_1^{\mathrm{t}}(X)^{\mathrm{ab}}$ are also isomorphic to each other. Observing the pro-p parts, we obtain $2g + n - 1 + b^{(2)} = \gamma$, where $b^{(2)}$ denotes the second Betti number of X (i.e., $b^{(2)} = 1$ for $n = 0$ and $b^{(2)} = 0$ for $n > 0$), and γ denotes the p-rank (or Hasse-Witt invariant) of X. Since $\gamma \leq g$, this equality implies $(g,n) = (0,0),(0,1)$.

Finally, assume that (iii) is an isomorphism, which implies that $\pi_1^{\mathrm{t}}(X)$ has a trivial pro-p-Sylow subgroup. If either $(g,n) = (0,0),(0,1),(0,2)$, or $(g,n) = (1,0)$ and X is supersingular, nothing remains to be proved. If $(g,n) = (1,0)$ and X is ordinary, then $\pi_1^{\mathrm{t}}(X) = \pi_1(X)$ is isomorphic to $\widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'} \times \mathbb{Z}_p$, hence its pro-$p$-Sylow subgroup is non-trivial. Finally, if $(g,n) \neq (0,0),(0,1),(0,2),(1,0)$, i.e., X is hyperbolic, then there exists a tame covering $Y \to X$ such that the genus of Y^* is not less than 2. By [R1], Corollaire 4.3.2, $\pi_1(Y^*)$ has a non-trivial pro-p-Sylow subgroup. Since $\pi_1(Y^*)$ is a subquotient of $\pi_1^{\mathrm{t}}(X)$, this implies that $\pi_1^{\mathrm{t}}(X)$ admits a non-trivial pro-p-Sylow subgroup. This completes the proof. □

Example.
The following are the only cases (for the present) in which we can describe $\pi_1(X)$ or $\pi_1^{\mathrm{t}}(X)$ explicitly.
(i) If $(g, n) = (0, 0)$, then we have

$$\pi_1(X) = \pi_1^{\mathrm{t}}(X) = \{1\}.$$

(ii) If $(g, n) = (0, 1)$, then we have $\pi_1^{\mathrm{t}}(X) = \{1\}$.
(iii) If $(g, n) = (0, 2)$, then we have

$$\pi_1^{\mathrm{t}}(X) \simeq \widehat{\mathbb{Z}}^{p'}.$$

(iv) If $(g, n) = (1, 0)$, then we have

$$\pi_1(X) = \pi_1^{\mathrm{t}}(X) \simeq \begin{cases} \widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'} \times \mathbb{Z}_p, & X\text{: ordinary,} \\ \widehat{\mathbb{Z}}^{p'} \times \widehat{\mathbb{Z}}^{p'}, & X\text{: supersingular.} \end{cases}$$

Now, our main question is as follows.

Main question.
Exactly what information on the geometry of X does $\pi_1(X)$ (or $\pi_1^{\mathrm{t}}(X)$) carry?

As for this question, the best situation we can expect is:

Hope.
Assume $k = \overline{\mathbb{F}}_p$. (For general k, see [T3]*.)*
(i) *$\pi_1(X)$ determines the isomorphism class of X (as a scheme), unless $(g, n) = (1, 0)$.*
(ii) *$\pi_1^{\mathrm{t}}(X)$ determines the isomorphism class of X (as a scheme), unless $(g, n) = (0, 0), (0, 1), (1, 0)$.*

Remark (which shows that our problem is rather subtle).
(i) In characteristic 0, we have $\pi_1(X) = \pi_1^{\mathrm{t}}(X) \simeq (\Pi_{g,n})^{\wedge}$, which carries very little information about X.
(ii) Let $\pi_A(X)$ denote the set of isomorphism classes of finite quotient groups of $\pi_1(X)$. Then, the Abhyankar conjecture (proved by Raynaud [R2] and Harbater [H1]) asserts that, if $n > 0$, we have

$$\pi_A(X) = \{G \mid G^{p'} \text{ is generated by (at most) } 2g + n - 1 \text{ elements.}\}/\simeq .$$

Thus, in this case, $\pi_A(X)$ carries very little information about X.
(iii) The geometric Shafarevich conjecture (proved by Harbater [H3] and Pop [P]) asserts that the absolute Galois group $G_{k(X)}$ is a free profinite group of rank $|k|$. Thus, $G_k(X)$, which is an extension group of $\pi_1(X)$, carries very little information

§2. Some results (which support our hope).

In this §, we present some results which support our hope in the last §.

Theorem 0.
(i) ([T1]) $\pi_1(X)$ *determines* (g,n).
(ii) ([T2]) $\pi_1^{\mathrm{t}}(X)$ *determines* (g,n), *unless* $(g,n)=(0,0),(0,1)$.

Theorem 1.
Assume $k=\overline{\mathbb{F}}_p$ *and* $g=0$.
(i) ([T1]) $\pi_1(X)$ *determines the isomorphism class of* X *(as a scheme).*
(ii) ([T2]) $\pi_1^{\mathrm{t}}(X)$ *determines the isomorphism class of* X *(as a scheme), unless* $n=0,1$.

Theorem 2 ([PS], [R4], [T4]).
Assume $k=\overline{\mathbb{F}}_p$.
(i) $\pi_1(X)$ *determines the isomorphism class of* X *up to finite possibilities, unless* $(g,n)=(1,0)$.
(ii) $\pi_1^{\mathrm{t}}(X)$ *determines the isomorphism class of* X *up to finite possibilities, unless* $(g,n)=(1,0)$.

One of the main ingredients of the proofs of the above theorems is Raynaud's theory of theta divisors ([R1], cf. [R3], [M]).

§3. Applications.

In this §, we present two applications of the results in the last §.

Corollary to Theorem 1(ii) (Tamagawa, unwritten yet).
Let L *be a subfield of* $\overline{\mathbb{Q}}$, *and assume that there exist infinitely many rational primes* p, *such that* $I_{\overline{\mathfrak{p}}}\cap G_L\not\subset I_{\overline{\mathfrak{p}}}^{\mathrm{w}}$ *holds for some prime* $\overline{\mathfrak{p}}$ *of* $\overline{\mathbb{Q}}$ *above* p. *(Here,* $I_{\overline{\mathfrak{p}}}$ *(resp.* $I_{\overline{\mathfrak{p}}}^{\mathrm{w}}$*) denotes the inertia subgroup (resp. the pro-p-Sylow subgroup of the inertia subgroup) at* $\overline{\mathfrak{p}}$.*) Let* X *be a smooth, hyperbolic curve over* L, *and assume that the smooth compactification of* X *is of genus* 0. *Then, the outer Galois representation* $\rho: G_L\to \mathrm{Out}(\pi_1(X_{\overline{\mathbb{Q}}}))$ *determines the isomorphism class of* X *as an* L*-scheme.*

Corollary to Theorem 2(ii) ([T4]).
Let F *be a function field of one variable over a finite field of characteristic* p. *Let* X *be a smooth, hyperbolic curve over* F, *and assume that* X *is non-isotrivial. Let* ρ^{t} *denote the outer Galois representation* $G_F\to\mathrm{Out}(\pi_1^{\mathrm{t}}(X_{\overline{F}}))$. *Then, for all primes* P *of* F, *we have*

$$\mathrm{Ker}(\rho^{\mathrm{t}})\cap D_P\subset I_P^{\mathrm{w}}.$$

(Here, D_P *(resp.* I_P^{w}*) denotes the decomposition subgroup (resp. the pro-p-Sylow subgroup of the inertia subgroup) at* P, *defined up to conjugacy.)*

Remark.
Let F be an arbitrary field of characteristic >0, and X a smooth, affine curve over F. Then, the outer Galois representation $\rho: G_F\to\mathrm{Out}(\pi_1(X_{\overline{F}}))$ is injective (see

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