ON DEFINING EQUATIONS OF THREE VARIANTS OF THE GROTHENDIECK-TEICHMÜLLER GROUP

HIDEKAZU FURUSHO

0. Introduction

The Grothendieck-Teichmüller group $\widehat{GT}$ is a subgroup of the automorphism group $Aut\widehat{F_2}$ of the free pro-finite group of $\widehat{F_2}$ of rank 2. It is parameterized by elements of $\widehat{\mathbf{Z}}^{\times}\times[\widehat{F_2},\widehat{F_2}]$ and is defined by three relations (I), (II) and (III) (see §1). It is the pro-finite group version of the pro-algebraic group $\underline{GT}$ [Dr]. In his study of Galois representation on fundamental groups, Y. Ihara showed that the absolute Galois group $G_{\mathbf{Q}}=Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ can be embedded into $\widehat{GT}$ in [Ih94]. It is still open whether $G_{\mathbf{Q}}$ is equal to $\widehat{GT}$ or not. But recently there appeared other several new-type relations satisfied by $G_{\mathbf{Q}}$ in $\widehat{GT}$ (see [Ih00], [LNS], [NT] and also [NT*]). In this article, we show a certain relationship among them. In §1, we shall recall the definition of $\widehat{GT}$ and make a small remark (*Proposition* 1.3) which has been possibly unknown. §2 is a review of definitions of three variants $\mathbb{\Gamma}$ ([LNS]), GTK ([Ih00]) and GTA ([Ih00]) of $\widehat{GT}$. In §3, we introduce main results of the author's master's thesis [F], which is on a relationship among defining equations of $\mathbb{\Gamma}$, GTK and GTA, and give a short sketch of its proof.

1. Review of the definition of $\widehat{GT}$

Let $\widehat{F_2}$ be the pro-finite free group of rank 2 with generators x and y. We define $\underline{\widehat{GT}}$ to be the set of pairs $(\lambda,f)\in\widehat{\mathbf{Z}}^{\times}\times\widehat{F_2}'$ (where $\widehat{F_2}'$ means the topological commutator subgroup $[\widehat{F_2},\widehat{F_2}]$ of $\widehat{F_2}$) satisfying

Date: March.15, 2002.
Surikaisekikenkyusho Kokyuroku (2002): Communications in Arithmetic Fundamental Groups.

the following three relations:

$$\begin{cases} \text{(I)} \ f(x,y)f(y,x)=1 & \text{((2-cycle relation))} \\ \text{(II)} \ f(z,x)z^m f(y,z)y^m f(x,y)x^m=1 & \\ \qquad \text{for } m=\frac{1}{2}(\lambda-1),\ xyz=1 & \text{((3-cycle relation))} \\ \text{(III)} \ f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})=1 \ \text{ in } \hat{P}_5^* & \\ & \text{((5-cycle relation)).} \end{cases}$$

Here $\hat{P}_5^*$ is the pro-finite pure sphere braid group with 5 strings and $x_{i,j} := x_{i,j}^{(5)}$ $(1 \leqslant i,j \leqslant 5)$ are its standard generators [Ih94]. For $f \in \widehat{F_2}$ and elements α, β of a pro-finite group H, $f(\alpha,\beta)$ stands for the image of f by the homomorphism $\phi : \widehat{F_2} \to H$ defined by $\phi(x)=\alpha$, $\phi(y)=\beta$. An element $\sigma = (\lambda, f) \in \underline{\widehat{GT}}$ induces an endomorphism of $\widehat{F_2}$ by $\sigma(x)=x^\lambda$, $\sigma(y)=f^{-1}y^\lambda f$, from which we get an embedding $\underline{\widehat{GT}} \hookrightarrow End\widehat{F_2}$ and in fact we can regard $\underline{\widehat{GT}}$ as a sub-monoid of $End\widehat{F_2}$ ([Dr]).

Definition 1.1 ([Dr]). The *Grothendieck-Teichmüller group* $\widehat{GT}$ is the group of invertible elements of $\underline{\widehat{GT}}$:

$$\widehat{GT} := \left\{ \sigma = (\lambda, f) \in Aut\,\widehat{F_2} \mid (\lambda, f) \text{ satisfies (I)} \sim \text{(III).} \right\}.$$

Remark 1.2. The above 5-cycle relation (III) is different from the original relation of $\widehat{GT}$ ([Dr])

$$\text{(III)}_{\text{DR}} \quad f(x_{12}^{(4)}, x_{23}^{(4)}x_{24}^{(4)})f(x_{13}^{(4)}x_{23}^{(4)}, x_{34}^{(4)}) = f(x_{23}^{(4)}, x_{34}^{(4)})f(x_{12}^{(4)}x_{13}^{(4)}, x_{24}^{(4)}x_{34}^{(4)})f(x_{12}^{(4)}, x_{23}^{(4)}) \quad \text{in } \hat{B}_4$$

which appeared in [NT] ([NT*]), where $\hat{B}_n$ $(n \in \mathbf{N})$ stands for the pro-finite braid group with n-strings. But we can show that (I)+(III) is equivalent to (I)+(III)$_{\text{DR}}$.

Proposition 1.3. *The relation* (III) *implies* (I) .

Proof . Recall that there is a basic projection $p : \hat{P}_5^* \to \hat{P}_4^*$ by sending $x_{i,j} = x_{i,j}^{(5)} \in \hat{P}_5^*$ to $x_{i,j}^{(4)} \in \hat{P}_4^*$ $(1 \leqslant i,j \leqslant 4)$ and $x_{i,5} \in \hat{P}_5^*$ to $1 \in \hat{P}_4^*$ $(1 \leqslant i \leqslant 5)$. It is immediate to see that (III) implies (I) because

$$p\big(f(x_{12},x_{23})f(x_{34},x_{45})f(x_{51},x_{12})f(x_{23},x_{34})f(x_{45},x_{51})\big) = f(x_{12}^{(4)},x_{23}^{(4)})f(x_{23}^{(4)},x_{34}^{(4)}) = f(x_{12}^{(4)},x_{23}^{(4)})f(x_{23}^{(4)},x_{12}^{(4)})$$

and $\hat{P}_4^*$ is a free group of rank 2 with generators $x_{12}^{(4)}$ and $x_{23}^{(4)}$. □

With the $G_{\mathbf{Q}}$-action on the geometric fundamental group $\pi_1(\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0,1,\infty\})$ of the projective line minus 3 points, we associate a pro-finite group homomorphism $\varphi : G_{\mathbf{Q}} \to Aut\widehat{F_2}$ (recall that $\widehat{F_2} \simeq \pi_1(\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0,1,\infty\})$). It follows from Belyĭ's theorem [Be] that φ is injective. In [Ih90] and [Ih94], it was shown that $\varphi(G_{\mathbf{Q}})$ lies in $\widehat{GT}$. Now to determine whether $G_{\mathbf{Q}}$ is equal to $\widehat{GT}$ or not is a basic open problem, which is also related to a project posed by A. Grothendieck in [Gr]. Recently there appeared other new-type relations which $G_{\mathbf{Q}}$ satisfies in $\widehat{GT}$, for example (I′), (II′), (III′), (IV), (IV′), (V), (A_n), (K_n) for $n = 1, 2, 3, \ldots$ (see [Ih00], [LNS], [NS], [NT] and also [NT*]). Although they do not seem to be deduced from defining relations (I), (II) and (III) of $\widehat{GT}$, it has not been shown yet whether they are really new ones and whether they are enough to characterize $G_{\mathbf{Q}}$ as a subgroup of $\widehat{GT}$.

2. Review of definitions of $\mathbb{I}\Gamma$, GTK and GTA

To characterize $G_{\mathbf{Q}}$ in $\widehat{GT}$, three subgroups $\mathbb{I}\Gamma$, GTK and GTA are considered in [Ih00] and [LNS]. The subgroup $\mathbb{I}\Gamma$ (resp. GTK) was geometrically constructed by P. Lochak, H. Nakamura, and L. Schneps in [LNS] (resp. by Y. Ihara in [Ih00]). On the other hand, GTA was arithmetically constructed by Y. Ihara in [Ih00]. They all contain $G_{\mathbf{Q}}$, but it has not been known whether they are really proper subgroups of $\widehat{GT}$ and whether they are equal to $G_{\mathbf{Q}}$.

2.1. $\mathbb{I}\Gamma$.

Definition 2.1 ([LNS]). The *new Grothendieck-Teichmüller group* $\mathbb{I}\Gamma$ is the subset defined as follows:

$$\mathbb{I}\Gamma := \left\{\sigma = (\lambda, f) \in \widehat{GT} \mid (\lambda, f) \text{ satisfies (III}') \text{ and (IV) below.}\right\}.$$

$$\begin{cases} (\mathrm{III}') \quad g(x_{45}, x_{51}) f(x_{12}, x_{23}) f(x_{34}, x_{45}) = f(\sigma_1\sigma_3, \sigma_2^2) & \text{in } \hat{B}_5 \\ (\mathrm{IV}) \quad f(\sigma_1, \sigma_2^4) = \sigma_2^{-8\Psi_2^{(0)}(\sigma)} f(\sigma_1^2, \sigma_2^2) \sigma_1^{-4\Psi_2^{(0)}(\sigma)} (\sigma_2\sigma_1)^{6\Psi_2^{(0)}(\sigma)} & \text{in } \hat{B}_3. \end{cases}$$

Here $g(x, y) \in \widehat{F_2}$ is the auxiliary parameter (depending on $\sigma \in \widehat{GT}$) satisfying $f(x, y) = g(y, x)^{-1} g(x, y)$ which was introduced in [LS]. For the definition of $\Psi_2^{(0)}(\sigma)^\dagger$, see §§2.2. In [LNS] and [NS], it was shown that actually $\mathbb{I}\Gamma$ forms a subgroup of $\widehat{GT}$. Note that (III′) implies (III). These two relations, (III′) and (IV), describe the condition for elements

†The 1-cocycle $\rho_2(\sigma)$ and $\rho_3(\sigma)$ (see [LNS], [LS], [NS], [NT] and also [NT*]) are equal to $-\Psi_2^{(0)}(\sigma)$ and $-\Psi_3^{(0)}(\sigma)$ respectively.

of $\widehat{GT}$ to act (as $G_{\mathbf{Q}}$ does) on all types of pro-finite Teichmüller modular groups in a certain consistent way (for more details, see [LNS]).

2.2. *GTK*. For a natural number n, let H_n be the index n normal subgroup of $\widehat{F_2}$ which is freely generated by $n+1$ elements x^n, y, $x^{-1}yx, \cdots, x^{-(n-1)}yx^{n-1}$. For $\sigma = (\lambda, f) \in \widehat{GT}$, f belongs to H_n, since $H_n \supset [\widehat{F_2}, \widehat{F_2}]$. In [Ih99], Ihara constructed the extended Kummer 1-cocycle $\Psi_n^{(0)}(\sigma)$ for $\sigma \in \widehat{GT}$, which is the image of f by the continuous group homomorphism $H_n \to \widehat{\mathbf{Z}}$ defined by $x^n \mapsto 0$, $y \mapsto 1$, $x^{-j}yx^j \mapsto 0$ $(1 \leqslant j < n)$. We remark that especially for $\sigma \in G_{\mathbf{Q}}$, $\Psi_n^{(0)}(\sigma)$ is the Kummer 1-cocycle which is characterized by $\sigma(\sqrt[k]{n}) = \sqrt[k]{n}\zeta_k^{-\Psi_n^{(0)}(\sigma)}$ for $k \in \mathrm{N}$, where $\zeta_k = exp(\frac{2\pi i}{k})$. Suppose that $\varphi_n : H_n \to \widehat{F_2}$ is the continuous group homomorphism defined by $x^n \mapsto x$, $y \mapsto y$, $x^{-j}yx^j \mapsto 1$ $(1 \leqslant j < n)$.

Definition 2.2 ([Ih00]). The *Grothendieck-Teichmüller-Kummer group* *GTK* is the subset defined as follows:

$$GTK_n := \left\{ \sigma = (\lambda, f) \in \widehat{GT} \mid (\lambda, f) \text{ satisfies } (\mathrm{K}_n) \text{ below.} \right\}$$

$$GTK := \bigcap_{n \in \mathbf{N}} GTK_n$$

$$(\mathrm{K}_n) \qquad \varphi_n(f) = y^{\Psi_n^{(0)}(\sigma)} f \quad .$$

It follows immediately from [Ih00] *Proposition 1* that GTK_n and GTK actually form subgroups of $\widehat{GT}$. Relation (K_n) describes the condition for elements of $\widehat{GT}$ to act (as $G_{\mathbf{Q}}$ does) on H_n and $\widehat{F_2}$ consistently with two algebraic morphisms, the Kummer covering $\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0, \mu_n, \infty\} \twoheadrightarrow \mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0, 1, \infty\}$ defined by $t \mapsto t^n$ and the natural inclusion $\mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0, \mu_n, \infty\} \hookrightarrow \mathbf{P}^1_{\overline{\mathbf{Q}}} - \{0, 1, \infty\}$ defined by $t \mapsto t$ (for more details, see [Ih00]).

2.3. *GTA*. By the $\widehat{F_2}^{ab} (:= \widehat{F_2}/\widehat{F_2}')$-action on $(\widehat{F_2}')^{ab} := \widehat{F_2}'/[\widehat{F_2}', \widehat{F_2}']$ induced from the conjugation $n \mapsto fnf^{-1}$ for $n \in \widehat{F_2}'$ and $f \in \widehat{F_2}$, we can regard $(\widehat{F_2}')^{ab}$ as a free A_2 $(:= \widehat{\mathbf{Z}}[[\widehat{F_2}^{ab}]])$-module of rank 1, generated by the class $\overline{[x, y]} \in (\widehat{F_2}')^{ab}$ of $[x, y] := xyx^{-1}y^{-1} \in \widehat{F_2}'$ (for more details, see [Ih99]). Thus the action of $\sigma = (\lambda, f) \in \widehat{GT}$ on $(\widehat{F_2}')^{ab}$ induced from that on $\widehat{F_2}$ is expressed $\sigma(\overline{[x, y]}) = B'_\sigma \cdot \overline{[x, y]}$, where $B'_\sigma \in A_2^\times$. The *adelic beta function* $B_\sigma \in A_2^\times$ was defined by $B'_\sigma = \frac{x^\lambda - 1}{x-1}\frac{y^\lambda - 1}{y-1} B_\sigma$ in [A] (for $\sigma \in G_{\mathbf{Q}}$) and [Ih99] (for $\sigma \in \widehat{GT}$). By the embedding constructed by G. W. Anderson in [A], B_σ can be regarded as a function on $(\mathbf{Q}/\mathbf{Z})^{\oplus 2}$ valued in $\widehat{\mathbf{Z}} \otimes \mathbf{Q}^{ab}$, where $\mathbf{Q}^{ab}$ stands for the maximal abelian extension field over $\mathbf{Q}$. In [Ih99] *Proposition 1.6.1*, it was shown that the adelic beta function has much analogy with the classical beta function. Especially it is remarkable that the adelic beta function $B_\sigma(s_1, s_2)$ ($\sigma \in \widehat{GT}$, $(s_1, s_2) \in (\mathbf{Q}/\mathbf{Z})^{\oplus 2}$) can be split into the product of the *adelic gamma function* Γ_σ, $B_\sigma(s_1, s_2) = \frac{\Gamma_\sigma(s_1)\Gamma_\sigma(s_2)}{\Gamma_\sigma(s_1+s_2)}$, where Γ_σ is a function on $\mathbf{Q}/\mathbf{Z}$ valued in the product $\prod_{p:\text{prime}} \mathbb{W}(\overline{\mathbb{F}_p})$ of the Witt vector ring $\mathbb{W}(\overline{\mathbb{F}_p})$ of $\overline{\mathbb{F}_p}$. In [A](i) *Corollary 8.6.3*, Anderson showed the n-th multiplication formula of the adelic gamma function Γ_σ *for elements* σ *of* $G_{\mathbf{Q}}$ as an analogy of Gauß' n-th multiplication formula of the classical gamma function:

$$(\mathrm{A}_n) \quad \prod_{nc=0} \frac{\Gamma_\sigma(s+c)}{\Gamma_\sigma(c)} \cdot \frac{1}{\Gamma_\sigma(ns)} = 1 \otimes exp\left[2\pi i \cdot n\Psi_n^{(0)}(\sigma)s\right]$$

by using Deligne's theory of absolute Hodge cycles. Ihara guessed that this arithmetic relation (A_n) could be a key condition to distinguish $G_{\mathbf{Q}}$ from $\widehat{GT}$ and considered the following new subgroup of $\widehat{GT}$ containing $G_{\mathbf{Q}}$.

Definition 2.3 ([Ih00]). The *Grothendieck-Teichmüller-Anderson group* GTA is the subset defined as follows:

$$GTA_n := \left\{\sigma = (\lambda, f) \in \widehat{GT} \mid \sigma \text{ satisfies } (\mathrm{A}_n). \right\}$$
$$GTA := \bigcap_{n \in \mathbf{N}} GTA_n .$$

It can be checked directly from definitions of $\Psi_n^{(0)}(\sigma)$ and $\Gamma_\sigma(s)$ that GTA and GTA_n actually form subgroups of $\widehat{GT}$.

The relationship among the above three subgroups $\mathbb{\Gamma}$, GTK and GTA has not been fully understood yet. But we remark that it was shown in [Ih99] that the relation (K_n) implies $(Dlog\mathrm{A}_n)$ which stands for the logarithmic derivative of the equation (A_n).

3. Main results

Theorem 3.1. *Relations* (I) , (II), (IV) *and* (K_2) *imply* (A_{2^n}) *for* $n = 1, 2, 3, \cdots$.

Sketch of the proof. At first, we introduce a new parameter f_+ on the rank 3 free group $\widehat{F_3}$ with generators W, X and Y. Recall that the index 2 subgroup H_2 (§§2.2) of $\widehat{F_2}$ generated by xyx^{-1}, x^2 and y can be identified with $\widehat{F_3}$ by sending xyx^{-1}, x^2, y into W, X, Y respectively. Since $y^{-2\Psi_2^{(0)}(\sigma)}f$ lies on H_2, we can associate an element $f_+(W,X,Y)$ of $\widehat{F_3}$ such that $y^{-2\Psi_2^{(0)}(\sigma)}f = f_+(xyx^{-1},x^2,y)$. It is immediate that the equation (K_2) is equivalent to

$$(1)\qquad y^{\Psi_2^{(0)}(\sigma)}f_+(xyx^{-1},x^2,y) = f_+(1,x,y)\ .$$

It can be checked by calculation that (IV) is also re-expressed in terms of $f_+ \in \widehat{F_3}$ as follows:

$$(2)\qquad f_+(W,X,Y)f_+(X^{-1}W^{-1}Y^{-1},Y,X) = 1\ .$$

From equations (I), (II) and (IV), we can also deduce the following equation

$$(3)\quad Y^m f_+(W,X,Y)X^m f_+(Z,W,X)W^m f_+(Y,Z,W) \\ Z^m f_+(X,Y,Z) = 1 \qquad \text{for} \quad WXYZ=1$$

On Anderson's duplication formula (A_2), we can show the following equivalence by using pro-finite free differential calculus developed in the appendix of [Ih99].

$$(A_2)\quad \frac{\Gamma_\sigma(s)\Gamma_\sigma(s+\frac{1}{2})}{\Gamma_\sigma(\frac{1}{2})\Gamma_\sigma(2s)} = 1\otimes exp\big[2\pi i\cdot\Psi_2^{(0)}(\sigma)s\big]$$

$$\Longleftrightarrow B_\sigma(x,x) = x^{-2\Psi_2^{(0)}(\sigma)}B_\sigma(x,-1)$$

$$\underset{*}{\Longleftrightarrow}\frac{x^\lambda+1}{x^m(x+1)}B_\sigma(x,\frac{1}{x^2}) = x^{-2\Psi_2^{(0)}(\sigma)}B_\sigma(-1,x)$$

$$\Longleftrightarrow\frac{x^\lambda+1}{x^m(x+1)}\Big\{1-(x-1)\frac{df}{dx}(x,\frac{1}{x^2})\Big\} = x^{-2\Psi_2^{(0)}(\sigma)}\Big\{1+2\frac{df}{dx}(-1,x)\Big\}$$

$$\Longleftrightarrow\frac{x^\lambda+1}{x^m(x+1)}\Big[1-(x-1)\Big\{x^{-2\Psi_2^{(0)}(\sigma)}\frac{df_+}{dW}(\frac{1}{x^2},x^2,\frac{1}{x^2})\cdot(1-\frac{1}{x^2}) \\ +\frac{df_+}{dX}(\frac{1}{x^2},x^2,\frac{1}{x^2})\cdot 2x\Big\}\Big] \\ -x^{-2\Psi_2^{(0)}(\sigma)}\Big[1+2x^{2\Psi_2^{(0)}(\sigma)}\Big\{(1-x)\frac{df_+}{dW}(x,1,x)-2\frac{df_+}{dX}(x,1,x)\Big\}\Big]=0$$

By (1) and Anderson's identity ([Ih99] *Theorem A.1*),

$$\Longleftrightarrow\frac{x^\lambda+1}{x^m(x+1)}\Big\{1-x^{-2\Psi_2^{(0)}(\sigma)}\cdot(x-1)\frac{df_+}{dX}(1,x,\frac{1}{x^2})\Big\} \\ -x^{-2\Psi_2^{(0)}(\sigma)}\Big[1+2x^{2\Psi_2^{(0)}(\sigma)}\Big\{(1-x)\frac{df_+}{dW}(x,1,x)-2\frac{df_+}{dX}(x,1,x)\Big\}\Big]=0$$

By (2) and Anderson's identity,

$$\Longleftrightarrow \frac{x^{\lambda}+1}{x^m(x+1)} x^{-2\Psi_2^{(0)}(\sigma)} \left\{1-(x^2-1)\frac{df_+}{dW}(1,x,\frac{1}{x^2})\right\}$$
$$- x^{-2\Psi_2^{(0)}(\sigma)} \left\{1-2(\frac{x-1}{x})\frac{df_+}{dW}(\frac{1}{x^2},x,1)\right\} = 0 .$$

By direct calculation, it is possible to deduce the validity of the above last equation from (2), (3) and Anderson's identity.

Since we can prove that conditions (A_n) and (A_m) implies (A_{n+m}) for $n, m \in \mathbb{N}$, it is easy to see that one relation (A_2) implies infinite ones (A_{2^n}) for $n = 1, 2, 3, \cdots$. Thus we get the claim. □

Therefore, we see that arithmetic relations (A_{2^n}) for $n = 1, 2, 3 \cdots$ follows from geometric relations, (I), (II), (IV) and (K_2).

Corollary 3.2. *$GTK \cap \mathbb{I}\Gamma \subseteq GTA_{2^\infty}$, where $GTA_{2^\infty} := \bigcap_{n\in\mathbb{N}} GTA_{2^n}$.*

Proof . It follows immediately from *Theorem 3.1.* □

Thus we get a relationship among the arithmetic subgroup GTA_{2^∞} and the geometric subgroups GTK and $\mathbb{I}\Gamma$.

References

[A] Anderson, G. W.; (i) The hyperadelic gamma function: a précis. Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986), 1–19, Adv. Stud. Pure Math., 12, North-Holland, Amsterdam, 1987.
(ii) The hyperadelic gamma function. Invent. Math. 95 (1989), no. 1, 63–131.

[Be] Belyĭ, G. V.; Galois extensions of a maximal cyclotomic field. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 2, 267–276, 479.

[Dr] Drinfel'd, V. G.; On quasitriangular quasi-Hopf algebras and a group closely connected with Gal($\overline{Q}/Q$). ; Leningrad Math. J. 2 (1991), no. 4, 829–860

[F] Furusho, H.; Geometric and arithmetic subgroups of the Grothendieck-Teichmüller group. (in Japanese), Master's thesis, RIMS, Kyoto University, March 2000, in preparation to translate into English.

[Gr] Grothendieck, A.; Esquisse d'un programme. Geometric Galois actions, 1, 5–48, Edited by Leila Schneps and Pierre Lochak. London Mathematical Society Lecture Note Series, 242. Cambridge University Press, Cambridge, 1997

[Ih90] Ihara, Y.; Braids, Galois groups, and some arithmetic functions. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 99–120, Math. Soc. Japan, Tokyo, 1991.

[Ih91] ______ Automorphisms of pure sphere braid groups and Galois representations. The Grothendieck Festschrift, Vol. II, 353–373, Progr. Math., 87, Birkhäuser Boston, Boston, MA, 1990.

---

*Here, we use (I) and (II).

[Ih94] ——— On the embedding of Gal($\overline{Q}/Q$) into $\widehat{GT}$. London Math. Soc. Lecture Note Ser., 200, The Grothendieck theory of dessins d'enfants (Luminy, 1993), 289–321, Cambridge Univ. Press, Cambridge, 1994.

[Ih99] ——— On beta and gamma functions associated with the Grothendieck-Teichmüller groups. Aspects of Galois theory, 144–179, London Math. Soc. Lecture Note Ser., 256, Cambridge Univ. Press, Cambridge, 1999.

[Ih00] ——— On beta and gamma functions associated with the Grothendieck-Teichmüller group II. J. Reine Angew. Math.527 (2000), 1–11.

[LNS] Lochak, P., Nakamura, H., Schneps, L. ; On a new version of the Grothendieck-Teichmüller group. C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 1, 11–16.

[LS] Lochak, P., Schneps, L. ; A cohomological interpretation of the Grothendieck-Teichmüller group. With an appendix by C. Scheiderer. Invent. Math. 127 (1997), no. 3, 571–600.

[NS] Nakamura, H., Schneps, L. ; On a subgroup of the Grothendieck-Teichmüller group acting on the tower of profinite Teichmüller modular groups. Invent. Math. 141 (2000), no. 3, 503–560.

[NT] Nakamura, H., Tsunogai, H, ; Harmonic and equianharmonic equations in the Grothendieck-Teichmüller group. Forum Math. (to appear).

[NT*] Nakamura, H., Tsunogai, H, ; Harmonic equations in the Grothendieck-Teichmüller group. (in this volume).

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 Japan

E-mail address: furusho@kurims.kyoto-u.ac.jp