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Numerical semigroups of toric type of higher dimension

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Let k be an algebraically closed field of characteristic 0 and n an integer at least 2. We set $T=\mathbf{G}_m^n$ where $\mathbf{G}_m=\mathrm{Spec}\ k[X,X^{-1}]$ is the multiplicative group. Moreover, we denote by M (resp. N) the group $\mathrm{Hom}_{Alg.Groups}(T,\mathbf{G}_m)$ of characters of T (resp. the group $\mathrm{Hom}_{Alg.Groups}(\mathbf{G}_m,T)$ of 1-parameter subgroups of T). Then we have a non-singular canonical pairing $<\ ,\ >: M\times N\longrightarrow\mathbf{Z}$ where $\mathbf{Z}$ is the ring of intergers. We set $N_{\mathbf{R}}=N\otimes_{\mathbf{Z}}\mathbf{R}$ and $M_{\mathbf{R}}=M\otimes_{\mathbf{Z}}\mathbf{R}$ where $\mathbf{R}$ is the set of real numbers. Let σ be a strongly convex rational polyhedral cone in $N_{\mathbf{R}}$, i.e., there exist a finite number of vectors $x_i\in N_{\mathbf{R}}$ defined over the ring $\mathbf{Q}$ of rational numbers such that

$$\sigma=\{\sum_{i=1}^{N'}\lambda_ix_i|\lambda_i\geq 0,\text{ all } i\}=\sum_{i=1}^{N'}\mathbf{R}_+x_i$$

and it contains no line through the origin where $\mathbf{R}_+$ is the set of non-negative real numbers. We set

$$\check{\sigma}=\{r\in M_{\mathbf{R}}|<r,a>\geq 0,\ \text{all } a\in\sigma\}.$$

Then $\check{\sigma}\cap M$ becomes a subsemigroup of M. An *n-dimensional affine toric variety* is expressed as Spec $k[\check{\sigma}\cap M]$. Let $\mathbf{M}(\check{\sigma}\cap M)$ be the minimal set of generators for the semigroup $\check{\sigma}\cap M$. Then we can embed the affine toric variety $X_\sigma=$ Spec $k[\check{\sigma}\cap M]$ into the affine m-space $\mathbf{A}^m=\mathrm{Spec}\ k[Y_1,\ldots,Y_m]$ using the k-algebra homomorphism $k[Y_1.\ldots,Y_m]\longrightarrow k[\check{\sigma}\cap M]$ which sends Y_i to $\mathcal{T}^{b_i}$ where we set $\mathbf{M}(\check{\sigma}\cap M)=\{b_1,\ldots,b_m\}$.

Let H be a *numerical semigroup*, i.e., a subsemigroup of the additive semigroup $\mathbf{N}$ of non-negative integers such that its complement in $\mathbf{N}$ is finite. We denote by $g(H)$ the cardinality of $\mathbf{N}\backslash H$, which is called the *genus* of H. We set

$$c(H)=\mathrm{Min}\{c\in\mathbf{N}\mid c+\mathbf{N}\subseteq H\},$$

which is called the *conductor* of H. Then we get $c(H)\leq 2g(H)$. Let $\mathbf{M}(H)$ be the minimal set of generators for H. If $\mathbf{M}(H)=\{a_1,a_2,\ldots,a_l\}$, then we set

$$\alpha_i=\mathrm{Min}\{\alpha\mid\alpha a_i\in<a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_l>\}$$

for $i=1,\ldots,l$, where for any positive integers $b_1,\ldots,b_l$ we denote by $<b_1,\ldots,b_l>$ the subsemigroup of $\mathbf{N}$ generated by $b_1,\ldots,b_l$. Moreover, H is called a *Weierstrass semigroup* if there exist a complete non-singular irreducible algebraic curve C over k and its point P such that

$$H=\{\nu\in\mathbf{N}|\text{there is a rational function } f \text{ on } C \text{ such that } (f)_\infty=\nu P\}.$$

Let $\lambda \in \sigma \cap N$ such that $< r, \lambda > > 0$ for any non-zero $r \in \check{\sigma} \cap M$. Take a numerical semigroup H containing the semigroup $< \check{\sigma} \cap M, \lambda >$. We can define the morpshim $\mathbf{A}^l \longrightarrow \mathbf{A}^m$ by the k-algebera homomorphism

$$k[Y_1, \ldots, Y_m] \longrightarrow k[X_1, \ldots, X_l]$$

which sends Y_i to $\mathcal{X}^{<b_i, \lambda>} = X_1^{\nu_1} \cdots X_l^{\nu_l}$ where $\mathbf{M}(H) = \{a_1 \ldots, a_l\}$ and $< b_i, \lambda > = \nu_1 a_1 + \cdots + \nu_l a_l$ for some non-negative integers ν_i's. The above morphism $\mathbf{A}^l \longrightarrow \mathbf{A}^m$ is said to be *induced by* λ. A numerical semigroup H is *constructed from* X_σ and λ if $\sharp \mathbf{M}(H) = \sharp \mathbf{M}(\check{\sigma} \cap M) - n + 1$ and Spec $k[H]$ is isomorphic to the fiber product

$$\mathbf{A}^l \times_{\mathbf{A}^{l+n-1}} \text{Spec } k[\check{\sigma} \cap M]$$

where $l = \sharp \mathbf{M}(H)$, Spec $k[\check{\sigma}_{a,b} \cap M] \longrightarrow \mathbf{A}^{l+n-1}$ is the embedding using $\mathbf{M}(\check{\sigma} \cap M)$, and $\mathbf{A}^l \longrightarrow \mathbf{A}^{l+n-1}$ is the morphism induced by λ. In this case we also call H a *numerical semigroup of* (n*-dimensional*) *toric type*. Then we can show that H is Weierstrass (see Komeda [2]). Here we pose the following problem :

Problem 1. Let X_σ be an affine toric variety. Give a numerical semigroup H which is constructed from X_σ and some $\lambda \in \sigma \cap N$.

In the case where X_σ is 2-dimensional we get the following :

Fact 2. Let X_σ be a 2-dimensional affine toric variety. Then σ is expreesed as $\sigma = \mathbf{R}_+(1,0) + \mathbf{R}_+(a,b)$ where a and b are integers with $b > 0$ and $(a,b) = 1$. If $b = 1$, then we may assume that $a = 0$. If $b > 1$, then we may assume that $0 < a < b$. The above cone σ is denoted by $\sigma_{a,b}$. If $a \leq 9$, we can give a numerical semigroup $H_{a,b}$ which is constructed from $X_{\sigma_{a,b}}$ and $\lambda = (a^2, (a-1)b)$ (see Komeda [3]).

We would like to consider Problem 1 in a higher dimensional case. This paper is aimed at the following :

Aim 3. For any $n \geq 3$ we give a numerical semigroup H of n-dimensional toric type. Namely, we find an n-dimensional affine toric variety X_σ such that there exists a numerical semigroup H which is constructed from X_σ and some $\lambda \in \sigma \cap N$.

Example 4. Consider the 4-dimensional cone

$$\sigma = \mathbf{R}_+(1,0,0,0) + \mathbf{R}_+(0,0,0,1) + \mathbf{R}_+(1,0,1,0) + \mathbf{R}_+(0,1,1,0) + \mathbf{R}_+(0,1,0,1).$$

Let $X_\sigma = $ Spec $k[\check{\sigma} \cap M]$ be the 4-dimensional affine toric variety associated to σ. We note that

$$\check{\sigma} \cap M = < (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,-1,0), (0,-1,1,1) > .$$

Let H be a numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$. Assume that $c(H) < 2g(H)$, i.e., H is *non-symmetric.* Then we have

$$\alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3, \alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3, \alpha_3 a_3 = \alpha_{31} a_1 + \alpha_{32} a_2,$$

where $0 < \alpha_{ij} < \alpha_j$, $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{12} + \alpha_{32}$ and $\alpha_3 = \alpha_{13} + \alpha_{23}$ (see Herzog [1]). Take $\lambda = (\alpha_{31}a_1, \alpha_{21}a_1, \alpha_{12}a_2, \alpha_{32}a_2) \in \sigma \cap N$. Then we can show that the numerical semigroup H is of 4-dimensional toric type which is constructed from X_σ and λ.

We can generalize the above cone to an n-dimensional cone σ such that there exists a numerical semigroup which is constructed from X_σ and some $\lambda \in \sigma \cap N$.

Proposition 5. *Let $n \geq 4$. For any i with $1 \leq i \leq n$ let e_i be the vector in $\mathbf{R}^n$ whose j-th component is δ_{ij} where δ_{ij} is Kronecker symbol. We set*

$$\sigma = \mathbf{R}_+ e_1 + \sum_{i=4}^{n} \mathbf{R}_+ e_i + \mathbf{R}_+(1,0,1,0,\ldots,0) + \mathbf{R}_+(0,1,1,0,\ldots,0) + \mathbf{R}_+(0,1,0,1,\ldots,1).$$

Consider the n-dimensional affine toric variety $X_\sigma = \mathrm{Spec}\ k[\check{\sigma} \cap M]$. We note that

$$\check{\sigma} \cap M = < e_i\ (1 \leq i \leq n),\ (1,1,-1,0,\ldots,0),\ e_{-2,3,j}\ (4 \leq j \leq n) >$$

where $e_{-2,3,j}$ is the vector in $\mathbf{R}^n$ whose second component is -1, third and j-th components are 1, and the other components are 0. Let H_n be a numerical semigroup with

$$M(H_n) = \{a_1 = n, a_2 = n+1, a_3 = 2n+3, a_4 = 2n+4, \ldots, a_{n-1} = 2n+n-1\}.$$

Then we have relations

$$\alpha_1 a_1 = 4a_1 = a_2 + a_{n-1},\ \alpha_2 a_2 = 3a_2 = a_1 + a_3,\ \alpha_3 a_3 = 2a_3 = 2a_2 + a_4,$$

$$\alpha_i a_i = 2a_i = a_{i-1} + a_{i+1}\ (4 \leq i \leq n-2),\ \alpha_{n-1}a_{n-1} = 2a_{n-1} = 3a_1 + a_{n-2}.$$

Take $\lambda = (3a_1, a_1, a_2, 2a_2, a_3, a_4, \ldots, a_{n-3}, a_{n-2})$. Then $\lambda \in \sigma \cap N$. We can show that the numerical semigroup H_n is of n-dimensional toric type which is constructed from X_σ and λ.

A desired 3-dimensional affine toric variety is given by the following :

Example 6. Let $\sigma_{1,1,2} = \mathbf{R}_+(1,0,0) + \mathbf{R}_+(0,1,0) + \mathbf{R}_+(1,1,2)$. Consider the 3-dimensional affine toric variety $X_\sigma = \mathrm{Spec}\ k[\check{\sigma} \cap M]$. We note that

$$\check{\sigma} \cap M = < (1,0,0), (0,1,0), (0,0,1), (2,0,-1), (1,1,-1), (0,2,-1) > .$$

For any $m \in \mathbf{N}$ with $m \geq 1$, let H be a numerical semigroup with

$$M(H) = \{a_1 = 4, a_2 = 4m+1, a_3 = 4m+3, a_4 = 4m+2\}.$$

Then we have relations

$$\alpha_1 a_1 = (2m+1)a_1 = a_2 + a_3,\ \alpha_2 a_2 = 2a_2 = ma_1 + a_4$$

$$\alpha_3 a_3 = 2a_3 = (m+1)a_1 + a_4,\ \alpha_4 a_4 = 2a_4 = a_2 + a_3.$$

Take $\lambda = (4m+1, 4m+3, 4m+2)$. Then $\lambda \in \sigma \cap N$. We can show that the numerical semigroup H is of 3-dimensional toric type which is constructed from $X_{\sigma_{1,1,2}}$ and λ.

References

[1] J. Herzog, *Generators and relations of abelian semigroups and semigroup rings.* Manuscripta Math. **3** (1970), 175-193.

[2] J. Komeda, *On Weierstrass points whose first non-gaps are four.* J. Reine Angew. Math. **341** (1983), 68-86.

[3] J. Komeda, *Numerical semigroups of 2-dimensional toric type.* In preparation.