数理解析研究所講究録 1268 巻 2002 年 97-104

On the Unit Group of a Semigroup Ring

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A submonoid S of a torsion-free abelian (additive) group is called a grading monoid (or a g-monoid). Throughout the paper we assume that S is non-zero.

We consider the semigroup ring $R[X; S]$ of a g-monoid S over a commutative ring R.

We denote the unit group $\{s \in S \mid s+t=0 \text{ for some } t \in S\}$ of S by $H = H(S)$.

We denote the nilradical of R, that is, the set of nilpotents of R, by $N = N(R)$. If $N = 0$, then R is called reduced.

We denote the unit group of R by $U = U(R)$.

We denote the group of units $f = \sum a_s X^s$ of $R[X; S]$ with $\sum a_s = 1$ by $V(R[X; S])$. H is canonically regarded as a subgroup of $V(R[X; S])$.

Let G be an abelian group. If G has only one elements, or if G has a free basis which is not necessarily of finite number, then G is called free. Any subgroup of a free group is free.

An element x of an abelian multiplicative group G is called torsion, if $x^n = 1$ for some positive integer n. The set of torsion elements of G is a subgroup of G. If 1 is the only torsion elements of G, then G is called torsion-free.

The symbol $\otimes$ denotes direct product of groups.

Karpilovsky posed 21 research problems in [K, Chapter 7]. The 9th problem is the following:

Let G be an abelian group. Find necessary and sufficient conditions for $R[X; G]$ under which

(1) G has a torsion-free complement in $V(R[X; G])$.

(2) G has a free complement in $V(R[X; G])$.

(3) $U(R[X; G])$ is free modulo torsion.

This is an abstract and the details will appear elsewhere.

In [M1] we posed its semigroup version as follows:

Problem. Let S be a g-monoid. Find necessary and sufficient conditions for $R[X;S]$ under which

(1) H has a torsion-free complement in $V(R[X;S])$. That is, there exists a torsion-free subgroup W of $V(R[X;S])$ such that $V(R[X;S]) = H \otimes W$.

(2) H has a free complement in $V(R[X;S])$. That is, there exists a free subgroup W of $V(R[X;S])$ such that $V(R[X;S]) = H \otimes W$.

(3) $U(R[X;S])$ is free modulo torsion. That is, the residue class group $U(R[X;S])/\{f \in U(R[X;S]) \mid f \text{ is torsion}\}$ is free.

In §1, we review results in [M1, Section 1], [M2, Section 6] and [M3]. In §2, we give a preciser decomposition theorem for the unit group $U(R[X;S])$ of $R[X;S]$. And, using the decomposition theorem, we give a reduction for Problem.

§1. Review

Let $E = \{e_\lambda \mid \lambda \in \Lambda\}$ be a set of non-zero idempotents of R. If, for each λ_1 and λ_2 of Λ, there exists $\lambda_3 \in \Lambda$ such that $e_{\lambda_3} \in Re_{\lambda_1} \cap Re_{\lambda_2}$, then E is called an E-system of R. There exists a maximal (by inclusion) E-system of R by Zorn's Lemma.

Let E be a fixed maximal E-system of R. We set

$W(R[X;S])= \{\sum a_s X^s \in V(R[X;S]) \mid Ra_0 \text{ contains an element of } E\}$.

Then $W(R[X;S])$ is a subgroup of $V(R[X;S])$.

Proposition 1. (1) $V(R[X;S]) = H \otimes W(R[X;S])$.

(2) $U(R[X;S]) = U(R) \otimes H \otimes W(R[X;S])$.

For the proof of Proposition 1(2), let $f = \sum a_s X^s$ be a unit of $R[X;S]$. Then $u = \sum a_s$ is a unit of R, and $(1/u)f \in V(R[X;S])$. It follows that $U(R[X;S]) = U(R) \otimes V(R[X;S])$. Then (1) completes the proof.

Proposition 1 (1) shows that $W(R[X;S])$ does not depend on the maximal E-system of R up to isomorphisms.

We set $M = M(R) = \{x \in N \mid nx = 0 \text{ for some positive integer } n\}$.

Theorem 1 (An answer to Problem (1)). The following conditions are equivalent:

(1) H has a torsion-free complement in $V(R[X;S])$.

(2) $V(R[X;S])$ is torsion-free.

(3) $W(R[X;S])$ is torsion-free.

(4) $M = 0$.

Set $R' = R/M$. For each maximal E-system E' of R', we may define $W(R'[X;S])$.

Theorem 2 (A reduction of Problem (3) to Problem (2)). The following conditions are equivalent:

(1) $U(R[X;S])$ is free modulo torsion.

(2) $U(R)$ is free modulo torsion, and $V(R'[X;S])$ is free.

(3) $U(R)$ is free modulo torsion, and both H and $W(R'[X;S])$ are free.

(4) $U(R)$ is free modulo torsion, H is free, and H has a free complement in $V(R'[X;S])$.

If R has only a finite number of idempotents, then R is called almost indecomposable. If 0 and 1 are only the idempotents of R, then R is called indecomposable. If R is not indecomposable, then R is called decomposable.

Proposition 2 (A reduction of Problem for almost indecomposable rings to indecomposable rings). Let $e_1, \cdots, e_n$ be non-zero idempotents of R such that $e_1 + \cdots + e_n = 1$ and $e_ie_j = 0$ for $i \neq j$, where $n \geq 2$.

Then,

(1) H has a free complement in $V(R[X;S])$, if and only if H is free and, for each i, H has a free complement in $V(Re_i[X;S])$.

(2) $U(R[X;S])$ is free modulo torsion, if and only if, for each i, $U(Re_i[X;S])$ is free modulo torsion.

(3) $U(R[X;S])$ is a finitely generated free abelian group modulo torsion, if and only if, for each i, $U(Re_i[X;S])$ is a finitely generated free abelian group modulo torsion.

Theorem 3. (1) H has a finitely generated free complement in $V(R[X;S])$, if and only if R is reduced and either R is indecomposable or $H = 0$.

(2) $U(R[X;S])$ is a finitely generated free abelian group modulo torsion, if and only if $U(R)$ is a finitely generated free abelian group modulo torsion, H is a finitely generated free abelian group, $N = M$, and either R is indecomposable or $H = 0$.

Theorem 4 (An answer to Problem for reduced rings). Let R be reduced. Then,

(1) H has a torsion-free complement in $V(R[X;S])$.

(2) H has a free complement in $V(R[X;S])$ if and only if H is free.

(3) $U(R[X;S])$ is free modulo torsion, if and only if $U(R)$ is free modulo torsion and H is free.

§2. Results

Let $E = \{e_\lambda \mid \lambda \in \Lambda\}$ be a fixed maximal E-system of R. Then $E \ni 1$. The characteristic of R is denoted by $ch(R)$.

Proposition 3. (1) Assume that $N \neq 0$, and assume that H has a free complement in $V(R[X;S])$. Then $ch(R) = 0$, and $M = 0$.

(2) Assume that $N \neq 0$ and $ch(R) > 0$. Then $U(R[X;S])$ is free modulo torsion, if and only if $U(R)$ is free modulo torsion and H is free.

(1) follows from Theorem 1, and the necessity of (2) follows from Theorem 2.

For the sufficiency of (2), let $R' = R/M$. Then R' is reduced. By Theorem 4, H has a free complement in $V(R'[X;S])$. By Theorem 2,

$U(R[X;S])$ is free modulo torsion.

Proposition 4. (1) Assume that $N \neq 0$ and $R \supset \mathbf{Q}$, where $\mathbf{Q}$ is the field of rational numbers. Then H does not have a free complement in $V(R[X;S])$.

(2) Assume that $N \neq 0$ and $R \supset \mathbf{Q}$. Then $U(R[X;S])$ is not free modulo torsion.

For the proof of (1), take a non-zero element $x_0 \in R$ such that $x_0^2 = 0$, and take a non-zero element $s_0 \in S$. Set $W_1 = \{1 + \alpha x_0 - \alpha x_0 X^{s_0} \mid \alpha \in \mathbf{Q}\}$. Then W_1 is a subgroup of $W(R[X;S])$, and is isomorphic onto the additive group $\mathbf{Q}$. If $W(R[X;S])$ is free, and hence W_1 is free, then $\mathbf{Q}$ is free; a contradiction.

(2) Then $M = 0$, and hence $R' = R$. By (1), H does not have a free complement in $V(R[X;S])$. By Theorem 2, $U(R[X;S])$ is not free modulo torsion.

Proposition 5. $U(R[X;S])$ is free modulo torsion if and only if $U(R'[X;S])$ is free modulo torsion.

For the proof, let U be the unit group of R, and let T be the set of torsion elements of U. Let U' be the unit group of $R' = R/M$, and let T' be the set of torsion elements of U'. We can show that $U' = \{\bar{u} \mid u \in U\}$, and $T' = \{\bar{t} \mid t \in T\}$. Moreover, we can show that $U/T \cong U'/T'$. Then the proof follows from Theorem 2.

Proposition 5 reduces Problrem (3) to the case where $M = 0$.

Lemma 1. Let $E = \{e_\lambda \mid \lambda \in \Lambda\}$ be a fixed maximal E-system of R. Let $W_\Lambda = W_\Lambda(R[X;S])$ be the subgroup of $W(R[X;S])$ generated by its subset $\{e_\lambda + e'_\lambda X^\alpha \mid \lambda \in \Lambda, \alpha \in H\}$, where $e'_\lambda = 1 - e_\lambda$. Let $W_N = W_N(R[X;S]) = \{\Sigma a_s X^s \in W(R[X;S]) \mid Ra_0 \text{ contains } 1\}$. Then $W(R[X;S]) = W_\Lambda(R[X;S]) \otimes W_N(R[X;S])$.

To show that $W_\Lambda \cap W_N = 1$, assume that $W_\Lambda \cap W_N \ni f = \prod_1^n (e_{i1} + e_{i2}X^{\alpha_i}) = 1 + x_0 + x_1X^{s_1} + \cdots + x_mX^{s_m}$, where $e_{i1} \in E, e_{i2} = 1 - e_{i1}, \alpha_i \in H$, and $0 \neq x_i \in N$ for $i \geq 1$. If $m = 0$, then $f = 1$. If $m \geq 1$, and if $(i_1, \cdots, i_n) \neq (j_1, \cdots, j_n)$, then $e_{1i_1}e_{2i_2}\cdots e_{ni_n}$ and $e_{1j_1}e_{2j_2}\cdots e_{nj_n}$ are orthogonal. It follows that x_1 is an idempotent; a contradiction.

Let $f = \epsilon_0 + \epsilon_1 X^{\alpha_1} + \cdots + \epsilon_n X^{\alpha_n} + x_1X^{s_1} + \cdots + x_mX^{s_m}$ be an element of $W(R[X;S])$, where ϵ_i is non-zero idempotent, $\alpha_i \in H, x_i \in N$. We can show that $f \in W_\Lambda \otimes W_N$ by induction on n.

Lemma 2. Let $\mathrm{W}_H = W_H(R[X;S]) = \{\Sigma a_\alpha X^\alpha \in W_N(R[X;S]) \mid \alpha \in H\}$, and $\mathrm{W}_m = W_m(R[X;S]) = \{\Sigma a_s X^s \in W_N \mid s$ is 0 or a non-unit of $S\}$. Then $\mathrm{W}_N = W_H \otimes W_m$.

For the proof, let $f \in W_N$. f may be written as $f = \Sigma a_i X^{\alpha_i} + \Sigma b_j X^{s_j}$, where $\alpha_i \in H$ and s_j is a non-unit of S. Set $f_1 = \Sigma a_i X^{\alpha_i}$ and $f_2 = \Sigma b_j X^{s_j}$. Then f_1 is a unit of $R[X;S]$. Set $g_1 = \sum c_k X^{t_k} = 1 + f_1^{-1} f_2, u_1 = \sum a_i, v_1 = \sum c_k$. Then $u_1v_1 = 1, f = (v_1f_1)(u_1g_1), v_1f_1 \in W_H$ and $u_1g_1 \in W_m$.

Theorem 5. (1) $W(R[X;S]) = W_\Lambda(R[X;S]) \otimes W_H(R[X;S]) \otimes W_m(R[X;S])$.

(2) $\mathrm{W}_\Lambda(R[X;S]) = 1$, if and only if either R is indecomposable or $H = 0$.

(3) $\mathrm{W}_H = W_m = 1$, if and only if $\mathrm{W}_N = 1$, if and only if R is reduced.

(4) Both $\mathrm{W}_H = 1$ and $\mathrm{W}_m \neq 1$, if and only if both $N \neq 0$ and $H = 0$.

(5) Both $\mathrm{W}_H \neq 1$ and $\mathrm{W}_m = 1$, if and only if both $N \neq 0$ and $H = S$.

Lemma 3. Assume that R is decomposablble. Then $W_\Lambda(R[X;S])$ is free if and only if H is free.

For the proof, let $E = \{e_\lambda \mid \lambda \in \Lambda\}$ be a maximal E-system of R with $e_0 = 1$. Let $e = e_\lambda$ with $\lambda \neq 0$. Then $W_1 = \{e + e'X^\alpha \mid \alpha \in H\}$ is a subgroup of W_Λ, and $W_1 \cong H$. If W_Λ is free, then W_1 is free, and hence H is free.

The converse follows from [M3, §2, Proposition 1].

Proposition 6. Let $S_1 = (S - H) \cup \{0\}$. Then $W(R[X;S])$ is free, if and only if both $W(R[X;H])$ and $W(R[X;S_1])$ are free.

For the proof, denote $V(S) = V(R[X;S])$, $W_\Lambda(S) = W_\Lambda(R[X;S])$, and etc. We have;

$V(S) = H \otimes W_\Lambda(S) \otimes W_H(S) \otimes W_m(S)$.

$V(H) = H \otimes W_\Lambda(H) \otimes W_H(H)$.

$V(S_1) = W_\Lambda(S_1) \otimes W_m(S_1)$.

$W_H(S) \cong W_H(H)$.

$W_m(S) \cong W_m(S_1)$.

The proof follows from the above formulas.

Proposition 6 shows that Problem (2) reduces to the case where either every elements of S is unit, or every element of S is non-unit except zero.

Proposition 6 implies the following,

Proposition 7. Let $S_1 = (S - H) \cup \{0\}$. Then $U(R[X;S])$ is free modulo torsion, if and only if both $U(R[X;H])$ and $U(R[X;S_1])$ are free modulo torsion.

Proposition 7 shows that Problem (3) reduces to the case where either every elements of S is unit, or every element of S is non-unit except zero.

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