数理解析研究所講究録 1268 巻 2002 年 151-154

Model-robustness of equilibrium in game for modal logics *

茨城工業高等専門学校　自然科学科（数学）
松久　隆 (Takashi Matsuhisa)

Department of Liberal Arts and Sciences, Ibaraki National College of Technology
866 Nakane, Hitachinaka-shi, Ibaraki 312-8508, Japan
E-mail: `mathisa@ge.ibaraki-ct.ac.jp`

Abstract. We present the notion of robust models for a mixed strategy Nash equilibrium of a strategic form game and to introduce a group structure on the class of these models. From the algebraic point of view we give a characterization of the class of robust models by the class of epistemic models with common-knowledge of conjectures about the other players' actions.

Keywords: Robust model, Bayesian model, Nash equilibrium, Non-cooperative game, Common-knowledge, Conjectures, Rationality, Semigroup.

1. Introduction

This paper investigates the class of *robust* models for a mixed strategy Nash equilibrium of a finite strategic form game G.

The concept of Nash equilibrium has become central in game theory, economics and its related fields. R.J. Aumann and A. Brandenburger (1995) gives epistemic conditions for Nash equilibrium in the model for the modal logic **S5**. However it is still not yet clear just what classes of models leading to a Nash equilibrium in the epistemic point of view.

The purposes of this paper are two points: First to to introduce a group structure on the class of robust models for a mixed strategy Nash equilibrium of the game as models for the modal logic **S4**, and secondly to characterize the class $\mathbf{R}^{\mathbf{S4}}(G)$ of the robust models by the class $\mathbf{E}^{\mathbf{S4}}(G)$ of all the models satisfying with common-knowledge of conjectures about the other players' actions. We show:

Main Theorem. *The class* $\mathbf{E}^{\mathbf{S4}}(G)$ *is a non-empty subclass of* $\mathbf{R}^{\mathbf{S4}}(G)$. *Furthermore* $\mathbf{E}^{\mathbf{S4}}(G)$ *almost coincides with* $\mathbf{R}^{\mathbf{S4}}(G)$.

2. Knowledge structure

Let Ω be a non-empty finite set called a *state-space*, N a set of finitely many *players* $\{1, 2, \dots, i, \dots n\}$ at least two ($n \geqq 2$), and let μ be a probability measure on Ω which is common for all players. Each member of 2^{Ω} is called an *event* and each element of Ω called a *state.*

Definition 1. By a *knowledge structure* we mean a pair $\langle \Omega, (K_i)_{i\in N}\rangle$, in which Ω be a non-empty set called a *state-space*, K_i is a mapping of 2^{Ω} into itself called player i's *knowledge operator.* A *common-knowledge structure* is a quadruple $\langle \Omega, (K_i)_{i\in N}, K_E, K_C\rangle$ in which K_E is the *mutual knowledge* operator on 2^{Ω} defined

* This is an extended abstract and the final form will be published elsewhere.

by $K_E E = \cap_{i \in N} K_i E$. and K_C is a *common-knowledge* operator on 2^Ω satisfying the *fixed point property*:

FP $\quad K_C F \subseteq K_E(F \cap K_C F) \quad$ for every F of 2^Ω.

The event $K_i E$ is interpreted as the set of states of nature for which i knows E to be possible. The event $K_E E$ is interpreted as that all players know E.

The *iterated* common-knowledge operator K_C is defined in the following way: Construct the descending chain $\{K^m\}$ such that

$$K^0 F := K_E F; \qquad \bar{K}^{m-1} F := K_E(F \cap K^{m-1} F);$$
$$K^m F := \bar{K}^{m-1} F \cap K^{m-1} F.$$

The operator K_C is given by the infimum of the chain:

$$K_C E = \bigcap_{m=0,1,2,\cdots} K^m E.$$

It is plainly observed that K_C satisfies Axiom **FP**. The event $K_C E$ is interpreted as that 'all players know that all players know that $\cdots$ that all players knows E.'

Definition 2. A knowledge structure (or a common-knowledge structure) is called a ***K*** *knowledge* (or a ***K*** *common-knowledge*) structure if it satisfies the properties: For any $E, F \in 2^\Omega$,

N $\quad K_i \Omega = \Omega \quad$ and $\quad K_i \emptyset = \emptyset$;

K $\quad K_i(E \cap F) = K_i E \cap K_i F.$

It is called a ***T*** *knowledge* (or a ***T*** *common-knowledge*) structure if it satisfies in addition

T $\quad K_i(E) \subseteqq E \quad$ for every $E \in 2^\Omega$.

It is called an ***S4*** *knowledge* (or an ***S4*** *common-knowledge*) structure if it satisfies in addition

4 $\quad K_i(E) \subseteqq K_i(K_i(E)) \quad$ for every $E \in 2^\Omega$.

Finally it is called an ***S4*** *knowledge* (or an ***S4*** *common-knowledge*) structure if it satisfies in addition

5 $\quad \Omega \setminus K_i E \subseteqq K_i(\Omega \setminus K_i E) \quad$ for every $E \in 2^\Omega$.

3. Associated information structure

An *information structure* is a pair $\langle \Omega, (Q_i)_{i \in N} \rangle$ in which Ω is a non empty state-space and Q_i is a mapping on Ω into 2^Ω. It is called an *RT-information structure* if each Q_i satisfies the two conditions: For each ω of Ω,

Ref $\quad \omega \in Q_i(\omega)$; and

Trn $\quad \xi \in Q_i(\omega)$ implies $Q_i(\xi) \subseteqq Q_i(\omega)$.

Definition 3. The *associated* information structure $(P_i)_{i\in N}$ with a knowledge structure $\langle \Omega, (K_i)_{i\in N}\rangle$ is the information structure $\langle \Omega, (P_i)_{i\in N}\rangle$ defined by $P_i(\omega) = \bigcap_T \{T \in 2^\Omega \,|\, \omega \in K_i T\}$.

Remark 1. For each $L = \mathbf{K}, \mathbf{T}, \mathbf{S4} or \mathbf{S5}$, an L knowledge structure $\langle \Omega, (K_i)_{i\in N}\rangle$ is uniquely determined by the associated information structure $\langle \Omega, (P_i)_{i\in N}\rangle$ as follows: $K_i E = \{\ \omega \in \Omega \mid P_i(\omega) \subseteqq E\ \}$.

Let $K_iP_i : \Omega \to 2^\Omega$ be the mapping defined by $K_iP_i(\omega) = K_i(P_i(\omega))$. It is plainly observed that $(K_iP_i)_{i\in N}$ is an RT-information structure for any L knowledge structure $\langle \Omega, (K_i)_{i\in N}\rangle$. It is noted that the associated information structure of an **S4** knowledge structure is an RT-information structure, because $K_iP_i = P_i$. Then $P_i(\omega)$ is interpreted as the set of all the states of nature that i knows to be possible at ω.

4. Robust models

Let Ω be a non-empty *finite* state-space and μ a probability measure on Ω which is common for all players. Let $G = \langle N, (A_i)_{i\in N}, (g_i)_{i\in N}\rangle$ be a *finite* strategic form game: A_i is a finite set of i's *actions* and g_i is i's *payoff-function* of $A := \times_{i\in N} A_i$ into $\mathbb{R}$. i's *overall conjecture* ϕ_i is a probability distribution on A_{-i}. For each player j other than i, this induces the marginal on j's actions called i's *individual conjecture* about j. Let $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ be a random variable of A. If $\mathbf{x}$ is a such function and x is a value of it, we denote by $[\mathbf{x} = x]$ (or simply by $[x]$) the set $\{\omega \in \Omega | \, \mathbf{x}(\omega) = x\}$. Accordingly P_i with μ yields the i's overall conjecture defined by the marginal $\boldsymbol{\phi}_i(a_{-i}, \omega) = \mu([\mathbf{a}_{-i} = a_{-i}] | K_iP_i(\omega))$ viewed as a random variable of i's conjecture ϕ_i. Denote $[\phi_i] = \cap_{a_{-i}\in A_{-i}} [\boldsymbol{\phi}_i(a_{-i}, \cdot) = \phi(a_{-i})]$ and $[\phi] = \cap_{i\in N}[\phi_i]$. We assume here that $[a_i] \subseteqq K_i([a_i])$ for every a_i of A_i. An player i is said to be *rational* at ω if each i's actual action a_i maximizes the expectation of his actually played pay-off function g_i at ω when the other players actions are distributed according to his conjecture $\boldsymbol{\phi}_i(\omega)$. Formally, letting $a_i = \mathbf{a}_i(\omega)$, $\mathrm{Exp}(g_i(a_i, \mathbf{a}_{-i}); \omega) \geqq \mathrm{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega)$ for every b_i in A_i, where Exp is defined by $\mathrm{Exp}(g_i(b_i, \mathbf{a}_{-i}); \omega) := \sum_{a_{-i}\in A_{-i}} g_i(b_i, a_{-i})\ \boldsymbol{\phi}_i(a_{-i}, \omega)$. Let R_i be the set of all the states at which an player i is rational, and denote $R := \cap_{j\in N} R_j$.

For a profile of conjectures $\phi = (\phi_i)_{i\in N}$, we denote by $\mathbf{B}^{\mathbf{S4}}_{\phi}(G)$ the class of all the epistemic models $\mathcal{B} = \langle \Omega, \mu, (K_i)_{i\in N}, (\mathbf{a}_i)_{i\in N}\rangle$ on the game G for the logic **S4**, and denote $\mathbf{B}^{\mathbf{S4}}(G) = \cup_{\phi \in \times_{i\in N}\Delta(A_{-i})} \mathbf{B}^{\mathbf{S4}}_{\phi}(G)$. We call $\mathbf{B}^{\mathbf{S4}}(G)$ the class of all *Bayesian* models on G. Let $\mathbf{E}^{\mathbf{S4}}_{\phi}(G)$ be the subclass consisting of all the models $\mathcal{B} \in \mathbf{B}^{\mathbf{S4}}_{\phi}(G)$ with $K_C([\phi]) \cap K_E R \neq \emptyset$, and denote $\mathbf{E}^{\mathbf{S4}}(G) = \cup_{\phi \in \times_{i\in N}\Delta(A_{-i})} \mathbf{E}^{\mathbf{S4}}_{\phi}(G)$, called the class of models with *epistemic conditions* for G. Let $\mathrm{NE}(G)$ be the set of all mixed strategy Nash equilibria and let $\sigma = (\sigma_i)_{i\in N} \in \mathrm{NE}(G)$. We set by $\mathbf{R}^{\mathbf{S4}}_{\sigma}(G)$ the class of all the models $\mathcal{B}$ such that there exists a profile of conjectures $\phi \in \times_{i\in N}\Delta(A_{-i})$ with the property that for every $i, j \in N, i \neq j$ and for all $a_i \in A_i$, $\phi_j(a_i) = \sigma_i(a_i)$. We will call $\mathbf{R}^{\mathbf{S4}}_{\sigma}(G)$ the class of *robust* models for a Nash equilibrium σ of G. Denote $\mathbf{R}^{\mathbf{S4}}(G) = \cup_{\sigma\in \mathrm{NE}(G)} \mathbf{R}^{\mathbf{S4}}_{\sigma}(G)$.

5. Proof of Main theorem

For the first part that $\mathbf{E}^{\mathbf{S4}}(G) \subset \mathbf{R}^{\mathbf{S4}}(G)$: It can be plainly observed that for $\sigma \in \mathrm{NE}(G)$, $\mathbf{E}^{\mathbf{S4}}_{\sigma}(G) \neq \emptyset$ and so $\mathbf{E}^{\mathbf{S4}}(G) \neq \emptyset$. Let $\mathcal{B} = \langle \Omega, \mu, (K_i)_{i\in N}, (\mathbf{a}_i)_{i\in N} \rangle \in \mathbf{E}^{\mathbf{S4}}(G)$ and $\mathcal{B} \in \mathbf{E}^{\mathbf{S4}}_{\phi}(G)$. Take $\omega \in K_C([\phi]) \cap K_E R$. Set $\sigma_i(a_i) := \phi_j(a_i)$ with $j \neq i$ and $\sigma = (\sigma_i)_{i\in N}$. We can observe that for every $a \in \prod_{i\in N} \mathrm{Supp}(\sigma_i)$, $\phi_i(a_{-i}) = \sigma_1(a_1)\cdots\sigma_{i-1}(a_{i-1})\sigma_{i+1}(a_{i+1})\cdots\sigma_n(a_n)$, and thus we obtain that each action a_i appearing with positive probability in σ_i at ω maximizes g_i against $\phi_i(a_{-i})$. This implies that $\sigma = (\sigma_i)_{i\in N} \in \mathrm{NE}(G)$, and so $\mathcal{B} \in \mathbf{R}^{\mathbf{S4}}_{\sigma}(G)$.

For the second part: We can introduce the operation $\bigoplus : \mathbf{B}^{\mathbf{S4}}(G) \times \mathbf{B}^{\mathbf{S4}}(G) \to \mathbf{B}^{\mathbf{S4}}(G)$ such that the structure $\langle \mathbf{B}^{\mathbf{S4}}(G), \bigoplus \rangle$ is an abelian semigroup. Now we can explicitly state the second part of the main theorem as follows:

Theorem 1. *Let the notation be the same as above. Then the following statements are true:*

(i) *Both structures $\langle \mathbf{E}^{\mathbf{S4}}(G), \bigoplus \rangle$ and $\langle \mathbf{R}^{\mathbf{S4}}(G), \bigoplus \rangle$ are sub-semigroups of $\langle \mathbf{B}^{\mathbf{S4}}(G), \bigoplus \rangle$. Furthermore, $\mathbf{E}^{\mathbf{S4}}(G)$ is a sub-semigroup of $\mathbf{R}^{\mathbf{S4}}(G)$.*

(ii) *The quotient group $\mathcal{G}(\mathbf{E}^{\mathbf{S4}}(G))$ of $\mathbf{E}^{\mathbf{S4}}(G)$ is isomorphic to the quotient group $\mathcal{G}(\mathbf{R}^{\mathbf{S4}}(G))$ of $\mathbf{R}^{\mathbf{S4}}(G)$.*

Proof. For (i): It is easy to verify (i). For (ii): First we note that the *cancellation law* holds in the abelian semigroups $\langle \mathbf{E}^{\mathbf{S4}}(G), \bigoplus \rangle$ and $\langle \mathbf{R}^{\mathbf{S4}}(G), \bigoplus \rangle$ respectively. This implies that the quotient groups $\mathcal{G}(\mathbf{E}^{\mathbf{S4}}(G))$ and $\mathcal{G}(\mathbf{R}^{\mathbf{S4}}(G))$ of $\langle \mathbf{E}^{\mathbf{S4}}(G), \bigoplus \rangle$ and $\langle \mathbf{R}^{\mathbf{S4}}(G), \bigoplus \rangle$ respectively are constructed, and it can be verified that $\mathcal{G}(\mathbf{E}^{\mathbf{S4}}(G)) \cong \mathcal{G}(\mathbf{R}^{\mathbf{S4}}(G))$.

6. Concluding remarks

1. A model $\mathcal{B}$ for the **S4** logic is called a model for the logic **S5** provided that for any $E \in 2^{\Omega}$,

   5 $\quad \Omega \setminus K_i E \subseteqq K_i(\Omega \setminus K_i E)$.

   Or equivalently, the associated information structure $(P_i)_{i\in N}$ makes an *information partition*; that is, for each $\omega \in \Omega$,

   Sym $\quad \xi \in P_i(\omega)$ implies $P_i(\xi) = P_i(\omega)$.

   In view of R.J. Aumann and A. Brandenburger (1995) it can be shown that $\mathbf{E}^{\mathbf{S5}}(G) \subseteqq \mathbf{R}^{\mathbf{S4}}(G)$ in the models for the logic **S5**.
2. Furthermore, the main theorem is still true in the class of models for each modal logics **K**, **KT**, **S4** = **KT4** and **S5** = **KT45**.

References

R. J. Aumann and A. Brandenburger, *Epistemic conditions for Nash equilibrium*, Econometrica **63** (1995), 1161–1180.

T. Matsuhisa, *Robust models for Nash equilibrium*, Ibaraki National College of Technology Working Paper (In preparation.)