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Remarks on locally inverse $*$-semigroups

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A semigroup S with a unary operation $*: S \to S$ is called a *regular $*$-semigroup* if it satisfies

$$\text{(i) } (x^*)^* = x; \qquad \text{(ii) } (xy)^* = y^*x^*; \qquad \text{(iii) } xx^*x = x.$$

Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if $e^* = e$. For a subset A of S, denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively.

Let S be a regular $*$-semigroup. Define a relation $\leq$ on S as follows:

$$a \leq b \iff a = eb = bf \quad \text{for some } \ e, f \in P(S).$$

A regular $*$-semigroup S is called a *locally inverse $*$-semigroup* if eSe is an inverse semigroup for any $e \in E(S)$.

Let G be a non-empty set with a partial product $\cdot$, a unary operation $*$ and a partial order $\leq$. We simply write ab instead of $a \cdot b$. If ab is defined for $a, b \in G$, we sometimes write $\exists ab$. An element $e \in G$ is called an *idempotent* if $\exists ee$ and $ee = e$. If an idempotent e satisfies $e^* = e$, it is called a *projection*. Denote the sets of idempotents and projections of G by $E(G)$ and $P(G)$, respectively.

If G satisfies the following axioms, it is called an *ordered $*$-groupoid.*

(A1) $a(bc)$ exists if and only if $(ab)c$ exists, in which case they are equal.

(A2) $a(bc)$ exists if and only if ab and bc exist.

(A3) $(a^*)^* = a$.

(A4) If ab exists, then b^*a^* exists and $(ab)^* = b^*a^*$.

(A5) For any $a \in G$, a^*a exists and a^*a is the unique projection of G such that $\exists a(a^*a)$ and $a(a^*a) = a$. We write $a^*a = d(a)$ and call it the *domain identity.*

(A6) $a \leq b$ implies $a^* \leq b^*$.

(A7) For $a, b, c, d \in G$, if $a \leq b$, $c \leq d$, $\exists ac$ and $\exists bd$, then $ac \leq bd$.

(A8) Let $a \in G$ and $e \in P(G)$ such that $e \leq d(a)$. Then there exists a unique element $(a|e)$, called the *restriction* of a to e, such that $(a|e) \leq a$ and $d(a|e) = e$.

(A9) $E(G)$ is an order ideal.

Lemma 1. [3] *Let G be an ordered $*$-groupoid.*

(1) *For any $a \in G$, aa^* exists and aa^* is the unique element of $P(G)$ such that $\exists(aa^*)a$ and $(aa^*)a = a$. We write $aa^* = r(a)$ and call it the* range identity.

(2) *Let $a \in G$ and $e \in P(G)$ such that $e \leq r(a)$. Then there exists a unique element $(e|a)$, called the* corestriction *of a to e, such that $(e|a) \leq a$ and $r(e|a) = e$.*

An ordered $*$-groupoid G is called a *locally inductive $*$-groupoid* if it satisfies

(LG) For any $e, f \in P(G)$, there exists the maximum element in $< e, f >= \{(g,h) \in P(G) \times P(G) : g \leq e,\ h \leq f \text{ and } \exists gh\}$.

Let S be a locally inverse $*$-semigroup. The representation in [4] raise us a new partial product $\cdot$ on S, which is called a *restricted product*, as follows:

$$a \cdot b = \begin{cases} ab & ab \in R_a \cap L_b \\ \text{undefined} & \text{otherwise} \end{cases}$$

where R_a and L_a denote the $\mathcal{R}$-class and the $\mathcal{L}$-class containing a, respectively.

Lemma 2. [3] *$S(\cdot, *, \leq)$ is a locally inductive $*$-groupoid, which is denoted by $\mathbf{G}(S)$.*

Conversely, let $G(\cdot, *, \leq)$ be a locally inductive $*$-groupoid. For any $a, b \in G$, there exists the maximun element (e, f) in $< d(a), r(b) > = \{(g,h) \in P(S) \times P(S) : g \leq d(a),\ h \leq r(b),\ \exists gh\}$. We define a new product $\otimes$ on G as follows:

$$a \otimes b = (a|e)(f|b),$$

and we call it a *pseudoproduct* of a and b.

Lemma 3. [3] *$G(\otimes, *)$ is a locally inverse $*$-semigroup, which is denoted by $\mathbf{S}(G)$.*

Lemma 4. [3] (1) *For a locally inverse $*$-semigroup S, we have $\mathbf{S}(\mathbf{G}(S)) = S$.*

(2) *For a locally inductive $*$-groupoid $G(\cdot, *, \leq)$, we have $\mathbf{G}(\mathbf{S}(G(\cdot, *, \leq))) = G(\cdot, *, \leq)$.*

Let S and T be regular $*$-semigroups. A mapping $\phi : S \to T$ is called a *prehomomorphism* if it satisfies

(i) $(ab)\phi \leq (a\phi)(b\phi)$,

(ii) $(a\phi)^* = a^*\phi$,

for all $a, b \in S$.

Lemma 5. [2] *Let S and T be locally inverse $*$-semigroups and $\phi : S \to T$ a mapping.*

(1) *ϕ is a prehomomorphism if and only if it preserves the restricred product and the natural order.*

(2) *ϕ is a homomorphism if and only if it is a prehomomorphism which satisfies $(ef)\phi = (e\phi)(f\phi)$ for all $e, f \in E(S)$.*

(3) *The product of prehomomorphisms between locally inverse $*$-semigroups is also a prehomomorphism.*

A functor between two ordered $*$-groupoids is said to be *ordered* if it is order-preserving. An ordered functor between two locally inductive $*$-groupoids is said to be *locally inductive* if it preserves the pseudoproduct.

Now, we have the main result.

Theorem 6. *The category of locally inverse $*$-semigroups and prehomomorphisms is isomorphic to the category of locally inductive $*$-groupoids and ordered functors. Moreover, the category of locally inverse $*$-semigroups and homomorphisms is isomorphic to the category of locally inductive $*$-groupoids and locally inductive functors.*

References

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