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Dynamics of Polynomial Automorphisms of $\mathbb{C}^2$: Herman ring

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Abstract

Herman ring is a periodic set which is biholomorphic to an annulus and rotates irrationally by iteration. Though the structure is known well its existence is unknown. We will show that there are no Herman rings under some conditions in the dynamics of the title.

1 Introduction

In this paper we denote $z=(x,y)\in\mathbb{C}^2$. Take an appropriate $m\in\mathbb{N}$. Let $p_j(y)$ be polynomials such that degree $d_j>1$ for $j=1,\ldots,m$. We call $f_j(x,y)=(y,p_j(y)-\delta_j x)$ *generalized Hénon mappings*, where $\delta_j\neq 0$. Moreover we define

$$f=f_m\circ\cdots\circ f_1,\quad \delta=\delta_1\cdots\delta_m,\quad d=d_1\cdots d_m.$$

Note that δ is the Jacobian determinant of f, and that f_j^{-1} are also generalized Hénon maps.

In [FM] Friedland and Milnor classified the polynomial automorphisms of $\mathbb{C}^2$ into three types:

- an affine mapping: $(x,y)\mapsto(\mu_{11}x+\mu_{12}y+\lambda_1,\mu_{21}x+\mu_{22}y+\lambda_2)$,
- an elementary mapping: $(x,y)\mapsto(\mu_1 x+\lambda,\mu_2 y+p(x))$,
- a composite of generalized Hénon mappings: $(x,y)\mapsto f(x,y)$.

Since the dynamical structures of the former two mappings are simple, they were investigated sufficiently in [FM]. So we study the last one.

We define $K^{\pm}=\{z\in\mathbb{C}^2\mid\{f^{\pm n}(z)\mid n\in\mathbb{N}\}\text{ is bounded}\}$, $J^{\pm}=\partial K^{\pm}$, $K=K^+\cap K^-$, $J=J^+\cap J^-$. They are closed invariant sets and are important objects in dynamical systems. Moreover we define $I^{\pm}=\mathbb{C}^2\setminus K^{\pm}$. For $R>0$, we define $V^+=\{z\in\mathbb{C}^2\mid |x|>\max\{|y|,R\}\}$, $V^-=\{z\in\mathbb{C}^2\mid |y|>\max\{|x|,R\}\}$ and $V=\{z\in\mathbb{C}^2\mid |x|,|y|\leq R\}$. It is known that $K^{\pm}\subset V\cup V^{\pm}$ for sufficiently large $R>0$.

We define the Green functions $G^{\pm}$ as (cf. [BS1, Section 3])

$$G^{\pm}(z)=\lim_{n\to\infty}\frac{1}{d^n}\log^+\|f^{\pm n}(z)\|.$$

$G^{\pm}$ are non-negative continuous plurisubharmonic functions such that $G^{\pm}(z)>0$ if and only if $z\in I^{\pm}$, $G^{\pm}|_{I^{\pm}}$ are pluriharmonic, and $G^{\pm}\circ f=d^{\pm 1}\cdot G^{\pm}$.

Before we define Herman ring, we state a classification of Fatou components. We proceed with the following volume property.

Proposition 1.1. ([FM, Lemma 3.7]) *Denote by* Vol() *the usual Lebesgue volume in* $\mathbb{C}^2$. *We have*

- *if* $|\delta|<1$, *then* $\mathrm{Vol}(K^+)=\infty$ *or* 0, $\mathrm{Vol}(K^-)=0$,
- *if* $|\delta|=1$, *then* $\mathrm{Vol}(K^+)=\mathrm{Vol}(K^-)<\infty$,
- *if* $|\delta|>1$, *then* $\mathrm{Vol}(K^+)=0$, $\mathrm{Vol}(K^-)=\infty$ *or* 0.

In this paper we assume $|\delta| < 1$, i.e. dissipative. Then only K^+ can have non-empty internals by the above proposition. We call each component of int K^+ *Fatou component*. Its classification theorem is as follows.

Theorem 1.2. ([BS2, Section 5]) *Each connected component of* int K^+ *is classified as follows.*

$$\begin{cases} \textit{wandering domain} \\ \textit{periodic domain} \begin{cases} \textit{non-recurrent domain} \\ \textit{recurrent domain} \begin{cases} \textit{basin of a sink} \\ \textit{Siegel cylinder} \\ \textit{Herman cylinder} \end{cases} \end{cases} \end{cases}$$

Before we define the names, we mention about existence and non-existence of the above domains.

As far as the author knows, it is unknown whether wandering domains exist or not. Non-recurrent domains exist and have been investigated only a little ([Hak, U1, U2, W]). Of course there are basins of sinks. Fornæss and Sibony investigated in [FS, Section 2] that there are Siegel cylinders. It is unknown whether Herman cylinders exist or not.

The only known fact with respect to non-existence of Herman cylinder is as follows: if f is uniformly hyperbolic on J, then Fatou components consist of basins of finite sinks (cf. [BS1, Theorem 5.6]).

Let us return to the definition of the names. Fatou component U is *wandering* if $f^n(U) \cap U = \emptyset$ for any $n \in \mathbb{N}$, *periodic* if $f^p(U) = U$ for some $p \in \mathbb{N}$. We call *p period* for the minimum p. We say U is *recurrent* if there are compact $C \subset U$ and $z \in U$ such that $f^n(z) \in C$ for infinitely many $n \in \mathbb{N}$.

For $E \subset \mathbb{C}^2$, we define

$$\begin{aligned} W^s(E) &= \{z \in \mathbb{C}^2 \mid d(f^n(z), f^n(E)) \to 0\ (n \to \infty)\}, \\ W^u(E) &= \{z \in \mathbb{C}^2 \mid d(f^n(z), f^n(E)) \to 0\ (n \to -\infty)\}, \\ W_0^s(E) &= \bigcup_{C \subset E:\text{compact}} W^s(C), \\ W_0^u(E) &= \bigcup_{C \subset E:\text{compact}} W^u(C). \end{aligned}$$

Let z_1 be a periodic point with period p. We call z_1 *sink* if both eigenvalues of $Df^p(z_1)$ are lower than 1 in modulus, *source* if greater, *saddle point* if one lower and another greater. We say U is *a basin of a sink* if $U = W^s(z_1)$ for some sink z_1.

We call $\mathcal{D} \subset \mathbb{C}^2$ *Siegel disk* if $\mathcal{D}$ satisfies the following properties: $\mathcal{D}$ is a periodic set with period p and there is a bijective holomorphic map $\varphi : \Delta \to \mathcal{D}$ such that $f^p(\varphi(\zeta)) = \varphi(b\zeta)$ for $\zeta \in \Delta$, where Δ is a unit disk on a complex plane and b is an irrational rotation, i.e. $b = e^{i\pi\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Here, we have to take the maximum $\mathcal{D}$ with respect to inclusion. We call U *Siegel cylinder* if $U = W_0^s(\mathcal{D})$ for some Siegel disk $\mathcal{D}$.

We call $\mathcal{H} \subset \mathbb{C}^2$ *Herman ring* if $\mathcal{H}$ satisfies the following properties: $\mathcal{H}$ is a periodic set with period p and there is a bijective holomorphic map $\varphi : A \to \mathcal{H}$ such that $f^p(\varphi(\zeta)) = \varphi(b\zeta)$ for $\zeta \in A$, where $A = \{\zeta \in \mathbb{C} \mid r_1 < |\zeta| < r_2\}$ is an annulus and b is an irrational rotation, i.e. $b = e^{i\pi\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Here, we have to take the maximum $\mathcal{H}$ with respect to inclusion. We call U *Herman cylinder* if $U = W_0^s(\mathcal{H})$ for some Herman ring $\mathcal{H}$.

We have arrived at a good position to describe our question.

Problem. ([BFGK, Problem 10.2.2(i)]) In case of dissipative, does a Hénon map admit a Herman ring (Herman cylinder) ?

In section 2, we will investigate several properties of a Herman cylinder. In particular, Proposition 2.9 will give a classification. In Theorem 3.1 of section 3, we will show that one of the types in the classification is impossible. Perhaps it might be a clue either to prove there are no Herman rings or to construct a Herman ring.

2 Fundamental properties of Herman cylinder

2.1 Functional properties

Note that we assume $|\delta| < 1$ in this paper. The following is a known fact.

Proposition 2.1. ([BS2, section 5]) *Define* $L(\zeta,\eta) = (b\zeta, \frac{\delta^p}{b}\eta)$. *Then there exists a biholomorphic map* $\Phi : A \times \mathbb{C} \to W_0^s(\mathcal{H})$ *such that* $\Phi(A \times \{0\}) = \mathcal{H}$ *and* $f^p \circ \Phi = \Phi \circ L$.

Proposition 2.2. *Let* U *be an arbitrary Fatou component and* M *simply connected one dimensional complex manifold in* $\mathbb{C}^2$. *Then* $M \cap U$ *is simply connected.*

Proof. Assume that $M \cap U$ is not simply connected. Then there is a point $z_1 \in M \setminus U$ which is surrounded by $M \cap U$ on M. By perturbing M to M', we can take $z_2 \in M' \setminus K^+$ which is surrounded by $M' \cap U$ on M'. We recall the Green function G^+, which vanishes on $M' \cap U$ and is positive on $M' \setminus K^+$. It contradicts the maximum principle. □

We define $u_a(\eta) = u(a,\eta) = G^- \circ \Phi(a,\eta)$ for $a \in A$, $\eta \in \mathbb{C}$. Then u_a is a subharmonic function on $\mathbb{C}$.
In general, let v be a non-negative subharmonic function. We define the *order* of v by

$$\operatorname{ord} v = \limsup_{r\to\infty} \frac{\log \max_{|\eta|=r} v(\eta)}{\log r},$$

Let ρ be the order of v, then we say v is of *mean type* of order ρ if

$$0 < \limsup_{r\to\infty} \frac{\max_{|\eta|=r} v(\eta)}{r^\rho} < \infty.$$

Proposition 2.3. *For* $a \in A$, u_a *is of mean type of order:*

$$\rho = \operatorname{ord} u_a = \frac{\log d}{\log(1/|\delta|)}.$$

Proof. It is sufficient to show that u_a is of mean type under the assumption that $\rho = \frac{\log d}{\log(1/|\delta|)}$.

$$\sup_{|\zeta|=|a|} \limsup_{r\to\infty} \frac{\max_{|\eta|=r} u_\zeta(\eta)}{r^\rho} \le \limsup_{r\to\infty} \max_{|\zeta|=|a|} \frac{\max_{|\eta|=r} u_\zeta(\eta)}{r^\rho}.$$

For $r > 1$, we take $n \in \mathbb{Z}$ such that $1/|\delta|^{pn} \le r < 1/|\delta|^{p(n+1)}$.

$$\begin{aligned}
\max_{|\zeta|=|a|} \max_{|\eta|=|a|} \frac{u_\zeta(\eta)}{r^\rho} &\le \max_{|\zeta|=|a|} \max_{|\eta|=1/|\delta|^{p(n+1)}} \frac{G^- \circ \Phi(\zeta,\eta)}{(1/|\delta|^{pn})^\rho} \\
&= \max_{|\zeta|=|a|} \max_{|\eta|=1/|\delta|^{p(n+1)}} \frac{G^- \circ f^{-p(n+1)} \circ \Phi \circ L^{n+1}(\zeta,\eta)}{d^{pn}} \\
&= \max_{|\zeta|=|a|} \max_{|\eta|=1} \frac{d^{p(n+1)} \cdot G^- \circ \Phi(b^{n+1}\zeta,\eta)}{d^{pn}} \\
&= d^p \max_{|\zeta|=|a|} \max_{|\eta|=1} u_\zeta(\eta).
\end{aligned}$$

Therefore we have

$$\limsup_{r\to\infty} \frac{\max_{|\eta|=r} u_a(\eta)}{r^\rho} < \infty.$$

Similarly we can compute as follows.

$$\inf_{|\zeta|=|a|} \limsup_{r\to\infty} \frac{\max_{|\eta|=r} u_\zeta(\eta)}{r^\rho} \ge \limsup_{r\to\infty} \min_{|\zeta|=|a|} \frac{\max_{|\eta|=r} u_\zeta(\eta)}{r^\rho}.$$

For $r > 1$, we take $n \in \mathbb{Z}$ such that $1/|\delta|^{pn} \le r < 1/|\delta|^{p(n+1)}$.

$$\begin{aligned}
\min_{|\zeta|=|a|} \max_{|\eta|=|a|} \frac{u_\zeta(\eta)}{r^\rho} &\ge \min_{|\zeta|=|a|} \max_{|\eta|=1/|\delta|^{pn}} \frac{G^- \circ \Phi(\zeta,\eta)}{(1/|\delta|^{p(n+1)})^\rho} \\
&= \min_{|\zeta|=|a|} \max_{|\eta|=1/|\delta|^{pn}} \frac{G^- \circ f^{-pn} \circ \Phi \circ L^n(\zeta,\eta)}{d^{p(n+1)}} \\
&= \min_{|\zeta|=|a|} \max_{|\eta|=1} \frac{d^{pn} \cdot G^- \circ \Phi(b^n\zeta,\eta)}{d^{p(n+1)}} \\
&= d^{-p} \min_{|\zeta|=|a|} \max_{|\eta|=1} u_\zeta(\eta).
\end{aligned}$$

Let us show that the last side is positive. Assume for some a' with $|a'| = |a|$, $u_{a'}|_{|\eta|\le 1} \equiv 0$. Then

$$\begin{aligned} &u(\{b^{-n}a'\} \times \{\eta \in \mathbb{C} \mid |\eta| \le 1/|\delta|^{pn}\}) \\ &= u(L^{-n}(\{a'\} \times \{\eta \in \mathbb{C} \mid |\eta| \le 1\})) \\ &= d^n u(\{a'\} \times \{\eta \in \mathbb{C} \mid |\eta| \le 1\}) = 0 \end{aligned}$$

Since $\bigcup_{n=0}^{\infty}\{b^{-n}a'\} \times \{\eta \in \mathbb{C} \mid |\eta| \le 1/|\delta|^{pn}\}$ is dense in $\{\zeta \in \mathbb{C} \mid |\zeta| = |a|\} \times \mathbb{C}$, $u_a \equiv 0$.

On the other hand, $G^-|_{V^+} > 0$ and the range of the non-constant holomorphic map Φ_a is contained in $V \cup V^+$. So $u_a \not\equiv 0$. It is a contradiction.

Therefore we have

$$\limsup_{r\to\infty} \frac{\max_{|\eta|=r} u_a(\eta)}{r^{\rho}} > 0.$$

□

2.2 Formal classification

Let $C = \{c_1, \dots, c_n\}$ be a finite ordered subset of a metric space with a metric d. We call C ε-chain if $d(c_j, c_{j+1}) < \varepsilon$ for any $1 \le j < n$.

Lemma 2.4. *Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a positive decreasing sequence converging to 0. Take ε_j-chain $C_j = \{c_{j1}, \dots, c_{jn_j}\}$. Assume $\{c_{j1}\}_{j\in\mathbb{N}}$ converges and $\overline{\bigcup_{j=1}^{\infty} C_j}$ is compact. Then the ω-limit set:*

$$\bigcap_{k=1}^{\infty}\overline{\bigcup_{j=k}^{\infty} C_j}$$

is a connected compact set.

Proof. Assume $E = \bigcap_{k=1}^{\infty}\overline{\bigcup_{j=k}^{\infty} C_j}$ is not connected. Then there exist compact sets E_1 and E_2 such that $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. Observe that $d(E_1, E_2) > 0$. We may assume $\{c_{j1}\}_{j\in\mathbb{N}}$ converges in E_1. Then there is a sequence $\{c_{jk_j}\}_{j\in\mathbb{N}}$ which accumulates on E_2.

On the other hand, because $\{\varepsilon_j\}$ decreases to 0, there is a sequence $\{c_{jl_j}\}_{j\in\mathbb{N}}$ which accumulates on $\{w \in \overline{\bigcup C_j} \mid \min\{d(w, E_1), d(w, E_2)\} \ge d(E_1, E_2)/3\}$. It is a contradiction. □

Lemma 2.5. *Let $X \subset \mathbb{R}^2$ be a closed subset and Y a compact component of X. Then there is a simple closed curve $\Gamma \subset \mathbb{R}^2 \setminus X$ which winds Y once.*

Proof. At first we show that there is $\varepsilon > 0$ such that the subset of X which can be joined to Y by ε-chain on X is compact.

Assume the contrary. Take a positive decreasing sequence $\{\varepsilon_j\}$ converging to 0 and $w_1 \in Y$ and $r > 0$ with $Y \subset B(w_1, r)$. By the assumption, for any $j \in \mathbb{N}$ we can take ε_j-chain $C_j \subset X$ such that the start point of C_j is w_1 and $C_j \setminus B(w_1, r) \ne \emptyset$ and $C_j \subset B(w_1, 2r)$. By the previous lemma, we can conclude that the connected component of X containing w_1 exceeds $B(w_1, r)$. It is a contradiction.

Let $Y' \subset X$ be the compact set which can be joined to Y by ε-chain. Each point on $X \setminus Y'$ is at least ε far from Y'. It is not difficult to find a simple closed curve $\Gamma \subset \mathbb{C} \setminus X$ which winds Y' once. □

We proceed with investigating the structure of a Herman cylinder.

We define $\widetilde{K} = \Phi^{-1}(K^-)$, $\widetilde{K}_a = \{\eta \in \mathbb{C} \mid \Phi(a, \eta) \in K^-\}$ for $a \in A$. Note that $\widetilde{K} = \{(\zeta, \eta) \in A \times \mathbb{C} \mid u(\zeta, \eta) = 0\}$, $\widetilde{K}_a = \{\eta \in \mathbb{C} \mid u_a(\eta) = 0\}$.

Definition 2.6. We say $\widetilde{K}_a$ is *bridged* if the component of $\widetilde{K}_a$ containing 0 is unbounded.

Lemma 2.7. *The following are equivalent.*

(1) $\widetilde{K}_a$ is bridged.

(2) The component of $\widetilde{K}_a$ containing 0 is not a point.

(3) $\widetilde{K}_a$ has an unbounded component.

Proof. (2)⇒(1). Assume the component of $\widetilde{K}_a$ containing 0 is bounded, i.e. the component is contained in $B(0,r) = \{\eta \in \mathbb{C} \mid |\eta| < r\}$ for some $r > 0$. Take an increasing sequence $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $b^{-n_j}a$ converges to a. Then

$$\{b^{-n_j}a\} \times \widetilde{K}_{b^{-n_j}a} = L^{-n_j}(\{a\} \times \widetilde{K}_a) = \{b^{-n_j}a\} \times (b/\delta^p)^{n_j}\widetilde{K}_a \subset \widetilde{K}.$$

Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a positive sequence decreasing to 0. We take ε_j-chain C_j in $\{b^{-n_j}a\} \times (b/\delta^p)^{n_j}\widetilde{K}_a$ so that the starting point of C_j is $(b^{-n_j}a, 0)$ and $C_j \subset \{b^{-n_j}a\} \times B(0,2r)$. Moreover we can assume $C_j \not\subset \{b^{-n_j}a\} \times B(0,r)$ for any sufficiently large j because of the hypothesis (2). By Lemma 2.4 we can conclude that the component of $\widetilde{K}_a$ containing 0 exceeds $B(0,r)$. It is a contradiction.

(3)⇒(2). Take an increasing sequence $\{n_j\}_{j\in\mathbb{N}}$ such that $b^{n_j}a$ converges to a. Then

$$\{b^{n_j}a\} \times \widetilde{K}_{b^{n_j}a} = \{b^{n_j}a\} \times (\delta^p/b)^{n_j}\widetilde{K}_a.$$

Let E be an unbounded component of $\widetilde{K}_a$. We can take $r > 0$ such that $B(0,r) \cap E \neq \emptyset$. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a positive sequence decreasing to 0. We take ε_j-chain C_j in $\{b^{n_j}a\} \times (\delta^p/b)^{n_j}E$ so that the starting point of C_j converges $(a, 0)$ and $C_j \subset \{b^{n_j}a\} \times B(0,2r)$ and $C_j \not\subset \{b^{n_j}a\} \times B(0,r)$. By Lemma 2.4, we can conclude that the component of $\widetilde{K}_a$ containing 0 is not a point. □

Lemma 2.8. *For $a \in A$ the following hold.*

(1) If $\widetilde{K}_a$ has no compact components, then so is $\widetilde{K}_\zeta$ for any $|\zeta| = |a|$.

(2) If $\widetilde{K}_a$ is bridged, then so is $\widetilde{K}_\zeta$ for any $|\zeta| = |a|$.

Proof. The proof of (2) is similar to the previous lemma. So we give only the proof of (1).

To prove (1), we show that if $\widetilde{K}_a$ has a compact component then so is $\widetilde{K}_{a'}$, for $|a'| = |a|$. By Lemma 2.5, there is a curve Γ which surrounds the component and never intersects $\widetilde{K}_a$. Take $\eta_1 \in \widetilde{K}_a$ surrounded by Γ. Then there exists $\varepsilon > 0$ such that $\Gamma_\varepsilon = \{\zeta \in A \mid |\zeta - a| \le \varepsilon\} \times \Gamma$ never intersects $\widetilde{K}$.

Consider u_ζ. Note that $u_\zeta(\eta) = 0$ if and only if $\eta \in \widetilde{K}_\zeta$. We define $c = \min_{(\zeta,\eta)\in\Gamma_\varepsilon} u_\zeta(\eta) > 0$. When $\varepsilon > 0$ is sufficiently small, $u_\zeta(\eta_1) < c$ for any ζ with $|\zeta - a| \le \varepsilon$. Recall that u_ζ is harmonic in $\mathbb{C} \setminus \widetilde{K}_\zeta$ and continuous on $\mathbb{C}$. So u_ζ has zero points inside of Γ, i.e. for ζ with $|\zeta - a| \le \varepsilon$, $\widetilde{K}_\zeta$ has at least one compact component inside of Γ.

Take $n \in \mathbb{N}$ such that $|b^n a' - a| \le \varepsilon$. Then

$$\widetilde{K}_{a'} = (b/\delta^p)^n \widetilde{K}_{b^n a'}.$$

Because $\widetilde{K}_{b^n a'}$ has a compact component, so is $\widetilde{K}_{a'}$. □

We obtain the following classification.

Proposition 2.9. *For $a \in A$, $\widetilde{K}_a$ is classified into the following three types:*

(1) $\widetilde{K}_a$ has no compact components,

(2) $\widetilde{K}_a$ is bridged and has compact components,

(3) each component of $\widetilde{K}_a$ is compact.

Moreover, for any ζ with $|\zeta| = |a|$, $\widetilde{K}_a$ and $\widetilde{K}_\zeta$ are classified into the same category.

2.3 Continuity about irrational rotation

In the following, we show several kinds of continuities of sets along irrational rotation.

Lemma 2.10. *Take $a, a' \in A$ with $|a| = |a'|$. Let $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ be a sequence such that $ab^{n_j} \to a'$ as $j \to \infty$. Then*

$$\bigcap_{k=1}^{\infty} \overline{\bigcup_{j=k}^{\infty} \left(\frac{\delta^p}{b}\right)^{n_j} \widetilde{K}_a} \subset \widetilde{K}_{a'}.$$

Proof. Since

$$\{ab^{n_j}\} \times \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{K}_a = L^{n_j}(\{a\} \times \widetilde{K}_a) \subset \widetilde{K},$$

we have

$$\overline{\bigcup_{j=k}^{\infty} \{ab^{n_j}\} \times \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{K}_a} \subset \widetilde{K},$$

$$\bigcap_{k=1}^{\infty} \overline{\bigcup_{j=k}^{\infty} \{ab^{n_j}\} \times \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{K}_a} \subset \{a'\} \times \widetilde{K}_{a'}.$$

On the other hand, we take

$$\eta \in \bigcap_{k=1}^{\infty} \overline{\bigcup_{j=k}^{\infty} \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{K}_a}.$$

This means that for any $\varepsilon > 0$ and $k \in \mathbb{N}$, there is $l \geq k$ such that

$$d\left(\eta, \left(\frac{\delta p}{b}\right)^{n_l} \widetilde{K}_a\right) < \frac{\varepsilon}{2}.$$

Then, there exists $j \geq k$ such that

$$d\left((a', \eta), \{ab^{n_j}\} \times \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{K}_a\right) < \varepsilon.$$

This implies

$$(a', \eta) \in \bigcap_{k=1}^{\infty} \overline{\bigcup_{j=k}^{\infty} \{ab^{n_j}\} \times \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{K}_a}.$$

Therefore

$$\{a'\} \times \bigcap_{k=1}^{\infty} \overline{\bigcup_{j=k}^{\infty} \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{K}_a} \subset \bigcap_{k=1}^{\infty} \overline{\bigcup_{j=k}^{\infty} \{ab^{n_j}\} \times \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{K}_a}.$$

We obtain the assertion. □

We define $\widetilde{I} = A \times \mathbb{C} \setminus \widetilde{K}$ and $\widetilde{I}_a = \mathbb{C} \setminus \widetilde{K}_a$ for $a \in A$. Under the hypothesis of the above lemma, we have

$$\bigcup_{k=1}^{\infty} \operatorname{int} \bigcap_{j=k}^{\infty} \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{I}_a \supset \widetilde{I}_{a'}.$$

More precisely we obtain the following.

Proposition 2.11. *Take $a, a' \in A$ with $|a| = |a'|$. Let $\{n_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ be a sequence such that $ab^{n_j} \to a'$ as $j \to \infty$. Then each component of*

$$\bigcup_{k=1}^{\infty} \operatorname{int} \bigcap_{j=k}^{\infty} \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{I}_a$$

is either in agreement with a component of $\widetilde{I}_{a'}$ or contained in $\widetilde{K}_{a'}$.

Proof. Take an arbitrary component I_1 from

$$\bigcup_{k=1}^{\infty} \operatorname{int} \bigcap_{j=k}^{\infty} \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{I}_a.$$

We may assume I_1 contains some component of $\widetilde{I}_{a'}$. Take a compactly contained open set $V \subset \widetilde{I}_{a'}$. Then there is $k \in \mathbb{N}$ such that for any $j \geq k$,

$$\left(\frac{\delta p}{b}\right)^{n_j} \widetilde{I}_a \supset V, \text{ i.e. } \widetilde{I}_{ab^{n_j}} \supset V.$$

Since u is continuous on $A \times \mathbb{C}$,

$$u_{ab^{n_j}}|_V \longrightarrow u_{a'}|_V$$

uniformly as $j \to \infty$. On the other hand, since $u_{ab^{n_j}}$ is harmonic on $\widetilde{I}_{ab^{n_j}}$,

$$u_{ab^{n_j}} \in \mathcal{H}(V)$$

for any $j \geq k$. Because $V \Subset I_1$ is arbitrary,

$$u_{a'} \in \mathcal{H}(I_1).$$

Recall that I_1 contains some component of $\widetilde{I}_{a'}$, on which $u_{a'}$ is a positive harmonic function, and $u_{a'}$ vanishes on $\mathbb{C} \setminus \widetilde{I}_{a'}$. Because I_1 is connected, I_1 coincides with some component of $\widetilde{I}_{a'}$. □

For $c > 0$, we define

$$\widetilde{I}'_a = \{\eta \in \mathbb{C} \mid u_a(\eta) > c\},$$

which is a subset of $\widetilde{I}_a$. The next lemma tells that $\widetilde{I}'_a$ plays a role similar to $\widetilde{I}_a$.

Lemma 2.12. *Take $a, a' \in A$ with $|a| = |a'|$. Let $\{n_j\}_{n \in \mathbb{N}} \subset \mathbb{N}$ be an increasing sequence such that $ab^{n_j} \to a'$ as $j \to \infty$. Then*

$$\widetilde{I}_{a'} \subset \bigcup_{k=1}^{\infty} \operatorname{int} \bigcap_{j=k}^{\infty} \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{I}'_a \subset \bigcup_{k=1}^{\infty} \operatorname{int} \bigcap_{j=k}^{\infty} \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{I}_a.$$

Moreover, each component of the middle side is either in agreement with some component of $\widetilde{I}_{a'}$ or contained in $\widetilde{K}_{a'}$.

Proof. It is sufficient to show the left inclusion.

$\eta \in \widetilde{I}'_a$ if and only if $u_a(\eta) > c$. Therefore

$$\begin{aligned} \eta \in \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{I}'_a &\iff (ab^{n_j}, \eta) \in \{ab^{n_j}\} \times \left(\frac{\delta p}{b}\right)^{n_j} \widetilde{I}'_a \\ &\iff (ab^{n_j}, \eta) \in L^{n_j}(\{a\} \times \widetilde{I}'_a) \\ &\iff L^{-n_j}(ab^{n_j}, \eta) \in \{a\} \times \widetilde{I}'_a \\ &\iff u(L^{-n_j}(ab^{n_j}, \eta)) > c \\ &\iff d^{n_j} u(ab^{n_j}, \eta) > c. \end{aligned}$$

Take $\eta_1 \in \widetilde{I}_{a'}$. Then there is $\varepsilon_1 > 0$ with $B(\eta_1, 3\varepsilon_1) \subset \widetilde{I}_{a'}$. Note that

$$u_{a'}|_{B(\eta_1, 3\varepsilon_1)} > 0.$$

Since u is continuous, there is $k_1 \in \mathbb{N}$ such that for any $j \geq k_1$,

$$u_{ab^{n_j}}|_{B(\eta_1, 2\varepsilon_1)} > 0.$$

Moreover there exists $k_2 \in \mathbb{N}$ for arbitrary $j \geq k_2$,

$$u_{ab^{n_j}}|_{B(\eta_1,\varepsilon_1)} > \frac{c}{d^{n_j}}, \text{ i.e. } B(\eta_1, \varepsilon_1) \subset \left(\frac{\delta^p}{b}\right)^{n_j} \widetilde{I}'_a.$$

Therefore we have

$$\eta_1 \in \operatorname{int} \bigcap_{j=k_2}^{\infty} \left(\frac{\delta^p}{b}\right)^{n_j} \widetilde{I}'_a.$$

This implies the left inclusion in the assertion. □

3 In case that $\widetilde{K}_a$ has no compact components

We have the following non-existence of Herman rings.

Theorem 3.1. *The case (1) in Proposition 2.9 is impossible.*

Corollary 3.2. *The case that $\widetilde{K}_a$ is connected is impossible.*

To prove the theorem we use a Böttcher function φ^- (cf. [MNTU, Section 7.3]). φ^- is holomorphic on V^+ for sufficiently large $R > 0$, and satisfies $\varphi^- \circ f^{-1}(z) = (\varphi^-(z))^d$ and $\log|\varphi^-(z)| = G^-(z)$ for $z \in V^+$. There is $M > 1$ such that $1/M \leq |\varphi^-(x,y)|/|x| \leq M$ for $(x,y) \in V^+$. When $|w|$ ($w \in \mathbb{C}$) is sufficiently large, $\{z \in V^+ \mid \varphi^-(z) = w\}$ is a simply connected one dimensional complex manifold in V^+.

We use a notation $\psi_\zeta(\eta) = \psi(\zeta, \eta) = \varphi^- \circ \Phi(\zeta, \eta)$. Note that $\log|\psi_a| = u_a$.

Lemma 3.3. *If $\widetilde{K}_a$ has no compact components, then*

$$\nabla u_a(\eta) \neq (0,0)$$

for any $\eta \in \widetilde{I}_a$.

Proof. Assume the contrary i.e. there is $\eta_0 \in \widetilde{I}_a$ such that $\frac{\partial u_a}{\partial \eta}(\eta_0) = 0$.

By Proposition 2.3, for any $c > 0$ the number of components of $\{\eta \in \mathbb{C} \mid u_a(\eta) > c\}$ is at most $\max\{1, 2p\}$ (cf. [Hay, Theorem 8.9]). Since the number is monotone increasing along $c > 0$, we can take $c > 0$ so that the number attains its maximum.

Then $\frac{\partial u_a}{\partial \eta}$ has no zero points in $\widetilde{I}'_a = \{\eta \in \mathbb{C} \mid u_a(\eta) > c\}$. In fact, let us assume the contrary, i.e. there is $\eta_1 \in \widetilde{I}'_a$ with $\nabla u_a(\eta_1) = (0,0)$. Define $c' = u_a(\eta_0)$. There are $n \geq 2$, $0 \leq \theta < 2\pi$ and $\varepsilon_1 > 0$ such that

$$u_a(\eta_0 + t\exp(i(\theta + 2\pi j/n))) > c',$$
$$u_a(\eta_0 + t\exp(i(\theta + 2\pi(j+1/2)/n))) < c',$$

for any $0 \leq j < n$ and $0 < t < 2\varepsilon_1$. Define

$$I_1 = \{\eta \in \mathbb{C} \mid u_a(\eta) > c', \eta \text{ is in the component of } \widetilde{I}'_a \text{ containing } \eta_0\}.$$

Because of the definition of c, I_1 is a connected open set. Moreover, I_1 is simply connected because so is $\widetilde{I}_a$. Therefore there is an arc $\Gamma \subset I_1$ which joins

$$\eta_0 + \varepsilon_1 \exp(i\theta) \text{ and } \eta_0 + \varepsilon_1 \exp(i(\theta + 2\pi/n)).$$

We can extend Γ in a neighborhood of η_0 and obtain a closed curve Γ' so that $u_a \geq c'$ on Γ'. But there is a point inside of Γ' on which $u_a < c'$. It is a contradiction.

Let $\{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ be a sequence such that $ab^{n_j} \to a$ as $j \to \infty$. Then for $\eta \in (\delta^p/b)^{n_j}\widetilde{I}'_a$,

$$\begin{aligned} u_{ab^{n_j}}(\eta) &= u(ab^{n_j}, \eta) \\ &= u(L^{n_j}(a, (b/\delta^p)^{n_j}\eta)) \\ &= d^{-n_j}u(a, (b/\delta^p)^{n_j}\eta) \\ &= d^{-n_j}u_a((b/\delta^p)^{n_j}\eta). \end{aligned}$$

$$\frac{\partial u_{ab^{n_j}}}{\partial \eta}(\eta) = d^{-n_j}\left(\frac{b}{\delta^p}\right)^{n_j}\frac{\partial u_a}{\partial \eta}((b/\delta^p)^{n_j}\eta) \neq 0.$$

On the other hand, by Lemma 2.12, there are $\varepsilon_0 > 0$ and $k \in \mathbb{N}$ such that for any $j \geq k$,

$$B(\eta_0, \varepsilon_0) \subset \left(\frac{\delta^p}{b}\right)^{n_j} \widetilde{I}'_a.$$

Because u is continuous, $u_{ab^{n_j}}$ converge to u_a uniformly in $B(\eta_0, \varepsilon_0)$ as $j \to \infty$.

When harmonic functions $u_{ab^{n_j}}$ converge to a non-constant harmonic function u_a uniformly on $B(\eta_0, \varepsilon_0)$, then antiholomorphic functions $\frac{\partial u_{ab^{n_j}}}{\partial \eta}$ converge to $\frac{\partial u_a}{\partial \eta}$ uniformly. By Hurwitz theorem, each zero point of $\frac{\partial u_a}{\partial \eta}$ is an accumulation point of zero points of $\frac{\partial u_{ab^{n_j}}}{\partial \eta}$. But $\frac{\partial u_{ab^{n_j}}}{\partial \eta}$ has no zero points in $B(\eta_0, \varepsilon_0)$ for any $j \geq k$. It is a contradiction. □

We use a notation $\mathbb{H} = \{\xi \in \mathbb{C} \mid \operatorname{Re}\xi > 0\}$.

Lemma 3.4. *Assume $\widetilde{K}_a$ has no compact components. Then for any component I_0 of $\widetilde{I}_a$ and for any $c > 0$ the number of components of*

$$\{\eta \in I_0 \mid u_a(\eta) > c\}$$

is exactly one.

Proof. Since I_0 is simply connected, there is $g \in \mathcal{O}(I_0)$ such that $\operatorname{Re} g = u_a$. Then

$$g : I_0 \to \mathbb{H}.$$

By the previous lemma, for each $c > 0$ its level set

$$\{\eta \in I_0 \mid u_a(\eta) = c\}$$

is a set of smooth simple arcs, whose all ends go to infinity. Therefore $g : I_0 \to \mathbb{H}$ is locally biholomorphic and proper. This implies g is bijective. Hence the above each level set consists of single arc. We obtain the required result. □

Lemma 3.5. *Assume $\widetilde{K}_a$ has no compact components. Let I_0 be an arbitrary component of $\widetilde{I}_a$. Then it is possible to define*

$$\log \psi_a : I_0 \to \mathbb{H}$$

by analytic continuation. Moreover it is biholomorphic.

Proof. For sufficiently large $c > 0$, ψ_a is defined on

$$I'_0 = \{\eta \in I_0 \mid u_a(\eta) > c\},$$

because the set is contained in $\Phi^{-1}(V^+)$. Since I'_0 is simply connected, $\log \psi_a$ is well-defined on I'_0.

We take g used in the previous proof. Because $\operatorname{Re} \log \psi_a = u_a$, $\log \psi_a - g$ is a purely imaginary constant. Then $\log \psi_a$ can be analytic continued to I_0 because I'_0 is connected. Since $g : I_0 \to \mathbb{H}$ is biholomorphic, so is $\log \psi_a$. □

Let us investigate the structure of $\widetilde{I}$.

Proposition 3.6. *For any $a \in A$ there are $N \in \mathbb{N}$ and a closed curve*

$$\gamma : [0, N] \to \widetilde{I}$$

such that

$$\pi_A \circ \gamma(t) = a \exp(2\pi i t),$$

where $\pi_A : A \times \mathbb{C} \to A$ is a natural projection.

Note that we take the minimum $N \geq 1$ when we use the proposition.

Proof. Assume $\widetilde{I}_a$ has q'. We know $q' \leq \max\{1, 2\rho\} < \infty$. Take $\eta_1, \ldots, \eta_{q'} \in \widetilde{I}_a$ so that any two of them belong to distinct components of $\widetilde{I}_a$. There is small $\varepsilon_0 > 0$ such that

$$T = \bigcup_{j=1}^{q'} \{(ae^{it}, \eta_j) \mid 0 \leq t \leq 2\varepsilon_0\} \subset \widetilde{I}.$$

We define a sequence $\{n_j\}_j \subset \mathbb{Z}$ as follows. We take $n_1 \in \mathbb{Z}$ such that

$$\varepsilon_0 \leq \arg ab^{n_1} - \arg a \leq 2\varepsilon_0 \pmod{2\pi}.$$

Then we take $n_2 \in \mathbb{Z}$ such that

$$\varepsilon_0 \leq \arg ab^{n_2} - \arg ab^{n_1} \leq 2\varepsilon_0 \pmod{2\pi}.$$

By repeating the procedure, we return to the starting point, i.e. there is $k \in \mathbb{N}$ such that

$$0 < \arg a - \arg ab^{n_k} \leq 2\varepsilon_0 \pmod{2\pi}.$$

Then we can draw arcs in $\widetilde{I}$ as follows. For η_{j_0}, draw an arc from (a, η_{j_0}) to (ab^{n_1}, η_{j_0}) along T. Choose $(\delta^p/b)^{n_1}\eta_{j_1}$ so that η_{j_0} and $(\delta^p/b)^{n_1}\eta_{j_1}$ are in the same component of $\widetilde{I}_{ab^{n_1}}$, then draw arcs joining the two points in the component. In the sequel draw an arc from (ab^{n_1}, η_{j_1}) to (ab^{n_2}, η_{j_1}) along $L^{n_1}(T)$. By repeating the procedure, we can draw an arc from each $\eta_1, \ldots, \eta_{q'}$ to $\widetilde{I}_a$.

If for some η_j the arc returns to the component of $\widetilde{I}_a$ containing the same η_j, we can draw an arc in the component joining the start point and the end point, and obtain a closed curve. Otherwise repeat N times the above procedure and at last some end point arrives at the same component of its start point. So we can draw a closed curve.

Finally by perturbing the closed curve we obtain γ as required. □

Lemma 3.7. *If $\widetilde{K}_a$ has no compact components, γ in Proposition 3.6 is unique in the following sense. Let $\gamma' : [0, N'] \to \widetilde{I}$ be another closed curve satisfying the same condition, and $\gamma(0)$ and $\gamma'(0)$ are in the same component of $\widetilde{I}_a$. Then $N = N'$ and for any $t \in [0, N]$, $\gamma(t)$ and $\gamma'(t)$ are in the same component of $\widetilde{I}_{a\exp(2\pi it)}$.*

Proof. We may suppose $N \leq N'$. Let us assume for some $t \in [0, N]$, $\gamma(t)$ and $\gamma'(t)$ are in distinct components of $\widetilde{I}_{a\exp(2\pi it)}$, and derive a contradiction.

By iteration of L^{-1}, we may assume $\gamma, \gamma' \subset \Phi^{-1}(V^+)$. Then ψ is defined in a neighborhood of γ and γ'.

The set

$$\{t \in [0, N] \mid \gamma(t) \text{ and } \gamma'(t) \text{ are in the same component of } \widetilde{I}_{a\exp(2\pi it)}\}$$

is open. In fact, take t_1 from the set. Then there is an arc $C \subset \widetilde{I}_{a\exp(2\pi it_1)}$ joining $\gamma(t_1)$ and $\gamma'(t_1)$. Because

$$\bigcup_{|\zeta|=|a|} \widetilde{I}_\zeta$$

is open in $\{\zeta \in A \mid |\zeta| = |a|\} \times \mathbb{C}$, a neighborhood of C is also contained in the above open set. So in a neighborhood of t_1, $\gamma(t)$ and $\gamma'(t)$ are joined by an arc in $\widetilde{I}_{a\exp(2\pi it)}$.

Define

$$t_2 = \min\{t \in [0, N] \mid \gamma(t) \text{ and } \gamma'(t) \text{ are in distinct components of } \widetilde{I}_{a\exp(2\pi it)}\},$$

$(a_2, \eta_2) = \gamma(t_2)$ and $(a_2, \eta_2') = \gamma'(t_2)$. Take $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\{a_2\exp(2\pi it) \mid -\varepsilon_1 < t \leq 0\} \times B(\eta_2, \varepsilon_2) \subset \Phi^{-1}(V^+) \subset \widetilde{I},$$
$$\{a_2\exp(2\pi it) \mid -\varepsilon_1 < t \leq 0\} \times B(\eta_2', \varepsilon_2) \subset \Phi^{-1}(V^+) \subset \widetilde{I},$$
$$(\{a_2\exp(2\pi it) \mid -\varepsilon_1 < t \leq 0\} \times \partial B(\eta_2, \varepsilon_2)) \cap \gamma = \emptyset,$$
$$(\{a_2\exp(2\pi it) \mid -\varepsilon_1 < t \leq 0\} \times \partial B(\eta_2', \varepsilon_2)) \cap \gamma' = \emptyset.$$

By Lemma 3.5, for each $\zeta \in \{a_2\exp(2\pi it) \mid -\varepsilon_1 < t < 0\}$, $\log\psi_\zeta$ is well-defined in the component of $\widetilde{I}_\zeta$ containing η_2 and η_2'. We choose the branches of the logarithms so that $\log\psi_\zeta(\eta_2)$ varies continuously with respect to ζ. Then

$$\log\psi_\zeta(\eta_2')$$

varies continuously on $\zeta \in \{a_2\exp(2\pi it) \mid -\varepsilon_1 < t < 0\}$. Moreover since ψ_ζ converges to ψ_{a_2} as $\zeta \to a_2$ uniformly in a neighborhood of η_2', there is ξ_2 such that

$$\log\psi_\zeta(\eta_2') \to \xi_2$$

as $\zeta \to a_2$.

On the other hand, there is η_3 in the component of $\widetilde{I}_{a_2}$ containing η_2 such that

$$\log\psi_{a_2}(\eta_3) = \xi_2,$$

where $\log\psi_{a_2}$ is defined so that $\log\psi_\zeta(\eta_2) \to \log\psi_{a_2}(\eta_2)$ as $\zeta \to a_2$. Observe that

$$\log\psi_\zeta(\eta_3) \to \xi_2$$

as $\zeta \to a_2$, because ψ is continuous in a neighborhood of (a_2, η_3).

Therefore, both $\log\psi_\zeta(\eta_2')$ and $\log\psi_\zeta(\eta_3)$ converge to ξ_2. It contradicts with the injectivity of $\log\psi_\zeta$.

Hence $\gamma(N)$ and $\gamma'(N)$ are in the same component of $\widetilde{I}_a$. We can draw a curve in the component from $\gamma'(N)$ to $\gamma'(0)$, i.e. $N = N'$. □

Lemma 3.8. *Assume $\widetilde{K}_a$ has no compact components. Take an arbitrary closed curve γ as in Proposition 3.6. Then $\Phi(\gamma)$ is trivial in $\pi_1(I^-)$.*

Proof. Since $\widetilde{I}_a$ has finite components, there is $q \in \mathbf{N}$ such that bothe γ and $L^q(\gamma)$ intersect a common component of $\widetilde{I}_a$. We know by Lemma 3.7 that for each $t \in [0, N]$

$$\gamma(t) \text{ and } L^q(\gamma(t + t_0)) \pmod N$$

are in the same component of $\widetilde{I}_{\exp(2\pi it)}$ for some $t_0 \in \mathbf{R}$.

We can draw a curve in $\widetilde{I}_{a\exp(2\pi it)}$ between $\gamma(t)$ and $L^q(\gamma'(t + t_0))$. We can extend the curve along t to a strip, i.e. $\gamma(t)$ and $L^q(\gamma(t + t_0))$ are locally homotopic. since each component of $\widetilde{I}_\zeta$ is simply connected, we can join the homotopies and have that γ and $L^q(\gamma)$ are homotopic in $\widetilde{I}$.

On the other hand, there is an isomorphism $\alpha : \pi_1(I^-) \to \mathbf{Z}[\frac{1}{d}]$ such that

$$\alpha(f(C)) = \frac{1}{d}\alpha(C)$$

for any $C \in \pi_1(I^-)$ (cf. [MNTU, Section 7.3]).

Since γ and $L^q(\gamma)$ are homotopic in $\widetilde{I} = \Phi^{-1}(I^-)$, $\Phi(\gamma)$ and $\Phi(L^q(\gamma))$ are homotopic in I^-. We obtain

$$\frac{1}{d^{pq}}\alpha(\Phi(\gamma)) = \alpha(f^{pq}(\Phi(\gamma))) = \alpha(\Phi(L^q(\gamma))) = \alpha(\Phi(\gamma)).$$

Therefore $\alpha(\Phi(\gamma)) = 0$. □

Proof of Theorem 3.1. Take γ as in Proposition 3.6. By iteration of L^{-1}, we may assume ψ is defined on the curve. Let $\pi_{\mathbf{C}} : A \times \mathbf{C} \to \mathbf{C}$ be a natural projection. For each $t \in [0, N]$, let I_t be the component of $\widetilde{I}_{a\exp(2\pi it)}$ containing $\pi_{\mathbf{C}} \circ \gamma(t)$. By Lemma 3.5, we have

$$\log\psi_{a\exp(2\pi it)} : I_t \to \mathbf{H}.$$

We choose the branch of the logarithm so that $\log\psi_{a\exp(2\pi it)}(\pi_{\mathbf{C}} \circ \gamma(t))$ varies continuously. Here, we regard $\psi_{a\exp(2\pi it)}$ and $\psi_{a\exp(2\pi i(t+1))}$ as different functions.

Then in general, $\log \psi_{a\exp(2\pi i\cdot 0)}$ and $\log \psi_{a\exp(2\pi i\cdot N)}$ do not have to be coincide. But since $\Phi(\gamma)$ is trivial in $\pi_1(I^-)$, they coincide. In fact, there is a 1-form ω in I^- such that

$$\pi_1(I^-) \ni C \mapsto \int_C \omega = \alpha(C) \in \mathbb{Z}[\frac{1}{d}]$$

and $\int \omega = \log \varphi^-$ (indefinite integral) (cf. [MNTU, Section 7.3]). Therefore

$$\begin{aligned} &\log \psi_{a\exp(2\pi i\cdot N)}(\pi_{\mathbb{C}} \circ \gamma(0)) \\ &= \log \psi_{a\exp(2\pi i\cdot N)}(\pi_{\mathbb{C}} \circ \gamma(N)) \\ &= \log \psi_{a\exp(2\pi i\cdot 0)}(\pi_{\mathbb{C}} \circ \gamma(0)) + \int_\gamma \Phi^*\omega \\ &= \log \psi_{a\exp(2\pi i\cdot 0)}(\pi_{\mathbb{C}} \circ \gamma(0)). \end{aligned}$$

Take an appropriate $\xi \in \mathbb{H}$ so that $\operatorname{Re}\xi$ is sufficiently large. Then for each $t \in [0, N]$, there is a unique point $\eta_t \in I_t$ such that

$$\log \psi_{a\exp(2\pi i t)}(\eta_t) = \xi.$$

Then

$$[0, N] \ni t \mapsto (a\exp(2\pi i t), \eta_t) \in \widetilde{I}$$

is a closed curve and satisfies

$$\psi(a\exp(2\pi i t), \eta_t) = e^\xi.$$

for any $t \in [0, N]$.

Therefore

$$[0, N] \ni t \mapsto \Phi(a\exp(2\pi i t), \eta_t) \in W_0^s(\mathcal{H})$$

is a non-contractible closed curve, and satisfies

$$\{\Phi(a\exp(2\pi i t), \eta_t) \mid t \in [0, N]\} \subset (\varphi^-)^{-1}(e^\xi).$$

This contradicts with Proposition 2.2. □

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