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Dynamics of modular groups acting on infinite dimensional Teichmüller spaces

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1 Introduction

This note is an abstract of my recent papers [1], [2] and [3], which will be published elsewhere.

For a hyperbolic Riemann surface R, the reduced Teichmüller modular group $\mathrm{Mod}^{\#}(R)$ is a group of automorphisms on the reduced Teichmüller space $T^{\#}(R)$. The action of $\mathrm{Mod}^{\#}(R)$ is isometric with respect to the Teichmüller distance d_T. We focus our attention on the proper discontinuity of $\mathrm{Mod}^{\#}(R)$, which is defined as follows.

Definition 1 We say that a subgroup $G \subset \mathrm{Mod}^{\#}(R)$ is *properly discontinuous* at a point $p \in T^{\#}(R)$ if there exists a neighborhood U of p such that the set $\{\chi \in G \mid \chi(U) \cap U \neq \emptyset\}$ consists of only finitely many elements.

If R is of analytically finite type, $\mathrm{Mod}^{\#}(R)$ and $T^{\#}(R)$ are nothing but the ordinary Teichmüller modular group $\mathrm{Mod}(R)$ and the ordinary Teichmüller space $T(R)$ respectively, and $T^{\#}(R)$ is finite dimensional. In this case, the definition of proper discontinuity is well known and $\mathrm{Mod}^{\#}(R)$ is properly discontinuous at any point in $T^{\#}(R)$.

On the other hand, if R is of topologically infinite type, $T^{\#}(R)$ is infinite dimensional and is not locally compact. However the above definition is suitable also in infinite dimensional cases. It is different from the case of finite type that $\mathrm{Mod}^{\#}(R)$ is not necessarily properly discontinuous. On the basis of this fact, in [3], we have given a sufficient condition for $\mathrm{Mod}^{\#}(R)$ to be properly discontinuous at any point in $T^{\#}(R)$. Further, in [1], we divide the Teichmüller space into the limit set and the region of discontinuity for the Teichmüller modular group as an analogy to the theory of Kleinian groups.

2 Limit sets and regions of discontinuity

Definition 2 For a subgroup $G \subset \mathrm{Mod}^{\#}(R)$, we define $\Omega(G)$ as the set of points $p \in T^{\#}(R)$ such that G is properly discontinuous at p, and $\Lambda(G)$ as the set of points $p \in T^{\#}(R)$ such that there exists a sequence $\{\chi_n\}$ of distinct elements of G such that $\lim_{n\to\infty} d_T(\chi_n(p), p) = 0$. We call $\Omega(G)$ the *region of discontinuity* of G, and $\Lambda(G)$ the *limit set* of G.

Remark 1 For a Riemann surface R of analytically finite type, $\Lambda(\mathrm{Mod}(R)) = \Lambda(\mathrm{Mod}^{\#}(R)) = \emptyset$. On the other hand, for a Riemann surface R whose Fuchsian model is of the second kind, we always have $\Omega(\mathrm{Mod}(R)) = \emptyset$. This is the reason why we consider the reduced modular group $\mathrm{Mod}^{\#}(R)$, not the ordinary modular group $\mathrm{Mod}(R)$, for Riemann surfaces R of infinite type.

Lemma 1 *$\Lambda(G)$ is G-invariant and closed.*

We classify the points in $\Lambda(G)$ into three types $\Lambda_0(G)$, $\Lambda^1_\infty(G)$ and $\Lambda^2_\infty(G)$ according to their stabilizer.

Definition 3 In a subgroup G of $\mathrm{Mod}^{\#}(R)$, the *stabilizer* of a point $p \in T^{\#}(R)$ is defined by $\mathrm{Stab}_G(p) = \{\chi \in G \mid \chi(p) = p\}$.

We define $\Lambda_0(G)$ as the set of points $p \in \Lambda(G)$ such that there exists a sequence $\{\chi_n\}$ of distinct elements of G that satisfies $\lim_{n\to\infty} d_T(\chi_n(p), p) = 0$ and that $\chi_n(p) \neq p$ for all n, and $\Lambda_\infty(G)$ as the set of points $p \in \Lambda(G)$ such that $\mathrm{Stab}_G(p)$ consists of infinitely many elements. Furthermore we divide $\Lambda_\infty(G)$ into two disjoint subsets $\Lambda^1_\infty(G)$ and $\Lambda^2_\infty(G)$. The $\Lambda^1_\infty(G)$ is the set of points $p \in \Lambda_\infty(G)$ such that there exists an element in $\mathrm{Stab}_G(p)$ that is of infinite order, and the $\Lambda^2_\infty(G)$ is the set of points $p \in \Lambda_\infty(G)$ such that all elements in $\mathrm{Stab}_G(p)$ are of finite order.

Proposition 1 *Let G be a subgroup of $\mathrm{Mod}^{\#}(R)$. For any point p in $T^{\#}(R) - \Lambda_0(G)$, there exists a constant $r > 0$ such that $\chi(B(p,r)) \cap B(p,r) = \emptyset$ for any $\chi \in G - \mathrm{Stab}_G(p)$.*

Corollary 1 *$T^{\#}(R) - \Lambda(G) = \Omega(G)$ for any subgroup $G \subset \mathrm{Mod}^{\#}(R)$.*

Hence $T^{\#}(R)$ is divided into two disjoint subset, the limit set and the region of discontinuity, as an analogy to the theory of Kleinian groups acting on the Riemann sphere. It seems that the essential natures of limit sets and regions of discontinuity for Teichmüller modular groups are different from the case of Kleinian groups. However we expect that they satisfy similar properties to that of limit sets and regions of discontinuity for Kleinian groups. We see that if the limit set has an isolated point, the isolated point belongs to $\Lambda^2_\infty(G)$. However, we do not know whether the limit set has an isolated point or not.

Theorem 1 ([1]) *For a subgroup $G \subset \mathrm{Mod}^{\#}(R)$, $\Lambda(G) - \Lambda^2_{\infty}(G)$ does not have an isolated point.*

Corollary 2 *For a subgroup $G \subset \mathrm{Mod}^{\#}(R)$ such that $\Lambda(G) - \Lambda^2_{\infty}(G)$ is not empty, the limit set $\Lambda(G)$ is an uncountable set.*

3 Teichmüller modular group of the second kind

We consider sufficient conditions for $\mathrm{Mod}^{\#}(R)$ to have a non-empty region of discontinuity. The conditions are given in terms of hyperbolic geometry on R.

Definition 4 For a subgroup G of $\mathrm{Mod}^{\#}(R)$, we say that G is of the *first kind* if $\Omega(G) = \emptyset$, and otherwise of the *second kind.*

Definition 5 For a given $M > 0$, we say that a point p of R belongs to a subset R_M of R if there exists a non-trivial simple closed curve c_p containing p such that the hyperbolic length of c_p is less than M.

The condition mentioned above are given as follows.

Definition 6 We say that R satisfies the *lower bound condition* if there exists an $\epsilon > 0$ such that R_ϵ consists only of cusp neighborhoods. Further we say that R satisfies the *upper bound condition* if there exist a constant $M > 0$ and a connected component R^*_M of R_M such that a homeomorphism of $\pi_1(R^*_M)$ to $\pi_1(R)$ that is induced by the inclusion map of R^*_M into R is surjective.

Remark 2 The lower and upper bound conditions are invariant under quasiconformal deformations.

Theorem 2 ([1]) *Let R be a Riemann surface which does not satisfy the lower bound condition (That is, R has a sequence of disjoint simple closed geodesics that are not peripheral (i.e. that are not freely homotopic to a boundary component) and that these hyperbolic lengths tend to 0). Then $\mathrm{Mod}^{\#}(R)$ is of the first kind.*

Theorem 3 ([1]) *If R satisfies the lower and upper bound conditions, then $\mathrm{Mod}^{\#}(R)$ is of the second kind.*

The following proposition gives examples of Riemann surfaces that satisfy the lower and upper bound conditions.

Proposition 2 *Let $\hat{R}$ be an analytically finite Riemann surface, and R a normal covering surface of $\hat{R}$ which is not a universal cover. Then R satisfies the lower and upper bound conditions.*

By Theorem 3 and Proposition 2, the following corollary is obtained.

Corollary 3 *Let $\hat{R}$ be an analytically finite Riemann surface, and R a normal covering surface of $\hat{R}$ which is not a universal cover. Then* $\mathrm{Mod}^{\#}(R)$ *is of the second kind.*

Example 1 Let $\hat{R}$ be a compact Riemann surface of genus $g \geq 2$, and R a normal covering surface of $\hat{R}$ whose covering transformation group is a cyclic group $\langle\phi\rangle$ generated by a conformal automorphism ϕ of R. Then $p_0 = [R, id] \in T^{\#}(R)$ and $[\phi] \in \mathrm{Mod}^{\#}(R)$ satisfy $[\phi](p_0) = p_0$. Hence p_0 belongs to $\Lambda^1_\infty(\mathrm{Mod}^{\#}(R))$. On the other hand, $\mathrm{Mod}^{\#}(R)$ is of the second kind by Corollary 3. Thus both $\Omega(\mathrm{Mod}^{\#}(R)) \neq \emptyset$ and $\Lambda(\mathrm{Mod}^{\#}(R)) \neq \emptyset$ are satisfied. A simple example of such Riemann surface is $R = \mathbf{C} - \{n \mid n \in \mathbf{Z}\}$.

We have a sufficient condition for $\Omega(\mathrm{Mod}^{\#}(R))$ to coincide with $T^{\#}(R)$, as a special case of Theorem 3.

Theorem 4 ([3]) *Let R be a Riemann surface satisfying the lower and upper bound conditions. Further suppose that either the genus of R, the number of cusps or the number of holes of R is positive finite. Then $\Lambda(\mathrm{Mod}^{\#}(R)) = \emptyset$.*

Weakening the assumption and the conclusion in Theorem 4, we have the following.

Theorem 5 *Suppose that R satisfies the lower and upper bound conditions. Let G be a subgroup of* $\mathrm{Mod}^{\#}(R)$ *satisfying the following: there exist two compact subsets C_1 and C_2 on R such that, for every $g \in G$, there is a conformal self-map f of R satisfying $f \circ g(C_1) \cap C_2 \neq \emptyset$. Then $p_0 = [R, id] \notin \Lambda_0(G)$.*

Let $\hat{R}$ be an analytically finite Riemann surface. For a normal covering surface R of $\hat{R}$ which is not a universal covering and for $G = \mathrm{Mod}^{\#}(R)$, the assumptions of Theorem 5 are satisfied.

Indeed, by Proposition 2, R satisfies the lower and upper bound conditions. Let g be an arbitrary element in $\mathrm{Mod}^{\#}(R)$. Since $\hat{R}$ is an analytically finite Riemann surface, $\hat{R}_{\geq\epsilon} = \hat{R} - \hat{R}_\epsilon$ is compact. Here $\hat{R}_\epsilon$ is the ϵ-thin part of $\hat{R}$. Let Γ be the covering transformation group for a normal covering surface R of $\hat{R}$, and C a compact subset of R that satisfies $C/\Gamma \supset \hat{R}_{\geq\epsilon}$. Then there exists an element $\gamma \in \Gamma$ such that $g(C) \cap \gamma(C) \neq \emptyset$. Thus $\gamma^{-1} \circ g(C) \cap C \neq \emptyset$, and the assumptions of Theorem 5 are satisfied.

Hence, by Proposition 1 and Theorem 5, there exists a neighborhood U of $p_0 = [R, id]$ which is precisely invariant under $\mathrm{Stab}_G(p_0)$. Therefore, in U, we have only to consider the action of $\mathrm{Stab}_G(p_0)$ as the action of $\mathrm{Mod}^{\#}(R)$.

4 A conjecture and a partial solution

In connection with Theorems 2 and 3, we have the following conjecture.

Conjecture If R satisfies the lower bound condition, then $\mathrm{Mod}^{\#}(R)$ is of the second kind. That is, considering Theorem 2, we conjecture that $\mathrm{Mod}^{\#}(R)$ is of the second kind if and only if R satisfies the lower bound condition.

We show a partial solution of this conjecture, giving a weaker condition than the upper bound condition.

Theorem 6 ([2]) *Let R be a Riemann surface with the non-abelian fundamental group. Suppose that R satisfies the following two conditions:*

1. *R satisfies the lower bound condition.*
2. *There exists a constant $M > 0$ such that, for any connected component V of the complement of R_M, V is either simply or doubly connected and $R - \overline{V}$ consists of finitely many connected components.*

Then $\mathrm{Mod}^{\#}(R)$ *is of the second kind.*

Example 2 Set

$$R = \mathbf{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbf{Z}} \left\{ \frac{m}{n} + (2n+1)\sqrt{-1} \right\}.$$

This Riemann surface R satisfies the assumptions of Theorem 6 (but does not satisfy the upper bound condition). Then $\mathrm{Mod}^{\#}(R)$ is of the second kind. On the other hand, $p_0 \doteq [R, id]$ belongs to $\Lambda(\mathrm{Mod}^{\#}(R))$. Indeed, set

$$f_n(z) = \begin{cases} x - (y - 2n - 2)/n + y\sqrt{-1} & (2n+1 \leq y < 2n+2) \\ x + (y - 2n)/n + y\sqrt{-1} & (2n \leq y < 2n+1) \\ x + y\sqrt{-1} & \text{elsewhere.} \end{cases}$$

Then f_n are quasiconformal self-maps of R and the maximal dilatations of $\{f_n\}$ tend to 1. Then we conclude that R satisfies both $\Omega(\mathrm{Mod}^{\#}(R)) \neq \emptyset$ and $\Lambda(\mathrm{Mod}^{\#}(R)) \neq \emptyset$.

References

[1] E. Fujikawa, Limit sets and regions of discontinuity of Teichmüller modular groups, preprint.

[2] E. Fujikawa, Teichmüller modular groups with non-empty regions of discontinuity, preprint.

[3] E. Fujikawa, H. Shiga and M. Taniguchi, On the action of the mapping class group for Riemann surfaces of infinite type, preprint.