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Porosity of Julia sets of semi-hyperbolic fibered rational maps and rational semigroups

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Abstract

We consider fiber-preserving complex dynamics on fiber bundles whose fibers are the Riemann spheres and whose base spaces are compact metric spaces. We define the semi-hyperbolicity of dynamics on fiber bundles. We will show that if a dynamics on a fiber bundle is semi-hyperbolic, then we have that the fiberwise Julia sets are $k-$porous and that the dynamics has a kind of weak rigidity. We also show that the Julia set of a rational semigroup(semigroup generated by rational maps on $\overline{\mathbb{C}}$) which is semi-hyperbolic except at most finitely many points in the Julia set and satisfies the open set condition is porous or is equal to the closure of the open set. Note that if a set J in $\overline{\mathbb{C}}$ is k-porous then the upper Box dimension of the set J is less than $2-c(k)$ where $c(k)$ is a positive constant depending only on k. Further we get an upper estimate of the Hausdorff dimension of the Julia set.

1 Introduction

To investigate random one-dimensional complex dynamics, dynamics of semigroups generated by rational maps on the Riemann sphere $\overline{\mathbb{C}}$ and fiber-preserving holomorphic dynamics on fiber bundles which appear in complex dynamics in in several dimensions, we consider the dynamics of fibered rational maps, that is, fiber-preserving complex dynamical systems on fiber bundles whose fibers are the Riemann spheres and whose base spaces are general compact metric spaces. The notion of dynamics of fibered rational maps, which was a generalized notion of 'dynamics of fibered polynomial maps' by O.Sester([Se1], [Se2], [Se3]), was introduced by M.Jonsson in [J2]. The research on dynamics of semigroups generated by rational maps on the Riemann sphere ([HM1], [HM2], [HM3] [GR], [Bo], [St1], [St2], [St3], [S1], [S2], [S3], [S4], [S5]), the research of random iterations of rational functions([FS], [BBR]) and the research on polynomial skew products on

$\mathbb{C}^2$ ([H1], [H2], [J1]) are directly related to this subject. For the research of polynomial skew products (dynamics of fibered polynomials) whose base spaces are general compact metric spaces, see O.Sester's works [Se1], [Se2] and [Se3]. In [Se3] he investigated the quadratic case in detail. In particular, he developed a combinatorial theory for quadratic fibered polynomials and constructed an abstract space of combinatorics. Moreover he showed some realizability and rigidity for an abstract combinatorics.

1.1 Notations and definitions

Definition 1.1. ([J2]) A triplet (π, Y, X) is called a '$\overline{\mathbb{C}}$-bundle' if

1. Y and X are compact metric spaces,
2. $\pi : Y \to X$ is a continuous and surjective map,
3. There exists an open covering $\{U_i\}$ of X such that for each i there exists a homeomorphism $\Phi_i : U_i \times \overline{\mathbb{C}} \to \pi^{-1}(U_i)$ satisfying that $\Phi_i(\{x\} \times \overline{\mathbb{C}}) = \pi^{-1}(x)$ and $\Phi_j^{-1} \circ \Phi_i : \{x\} \times \overline{\mathbb{C}} \to \{x\} \times \overline{\mathbb{C}}$ is a Möbius map for each $x \in U_i \cap U_j$, under the identification $\{x\} \times \overline{\mathbb{C}} \cong \overline{\mathbb{C}}$.

Remark: By the condition 3, each fiber $Y_x := \pi^{-1}(x)$ has a complex structure. We also have that given $x_0 \in X$ we may find a continuous family $i_x : \overline{\mathbb{C}} \to Y_x$ of homeomorphisms for x close to x_0. Such a family $\{i_x\}$ will be called a 'local parameterization'. Since X is compact, we may assume that there exists a compact subset M_0 of the set of Möbius transformations of $\overline{\mathbb{C}}$ such that $i_x \circ j_x^{-1} \in M_0$ for any two local parametrizatios $\{i_x\}$ and $\{j_x\}$. In this paper we always assume that.

Moreover in this paper we always assume the following condition:

- there exists a smooth $(1,1)$-form $\omega_x > 0$ inducing a metric on Y_x and $x \mapsto \omega_x$ is continuous. That is, if $\{i_x\}$ is a local parametrization, then the pull back $i_x^* \omega_x$ is a positive smooth form on $\overline{\mathbb{C}}$ depending continuously on x.

Definition 1.2. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ and $g : X \to X$ be continuous maps. We say that f is a fibered rational map over g (or a rational map fibered over g) if

1. $\pi \circ f = g \circ \pi$
2. $f|_{Y_x} : Y_x \to Y_{g(x)}$ is a rational map for any $x \in X$. That is, $(i_{g_x})^{-1} \circ f \circ i_x$ is a rational map from $\overline{\mathbb{C}}$ to itself for any local parametrization i_x at $x \in X$ and $i_{g(x)}$ at $g(x)$.

Notation: If $f : Y \to Y$ is a fibered rational map over $g : X \to X$, then we put $f_x^n = f^n|_{Y_x}$ for any $x \in X$ and $n \in \mathbb{N}$. Furthermore we put $d_n(x) = \deg(f_x^n)$ and $d(x) = d_1(x)$ for any $x \in X$ and $n \in \mathbb{N}$.

Definition 1.3. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ is a fibered rational map over $g : X \to X$. Then for any $x \in X$ we denote by $F_x(f)$(simply F_x) the set of points $y \in Y_x$ which has a neighborhood U of y in Y_x satisfying that $\{f_x^n\}_{n\in\mathbb{N}}$ is a normal family in U, that is, $y \in F_x$ if and only if the family $Q_x^n = i_{x_n}^{-1} \circ f_x^n \circ i_x$ of rational maps on $\overline{\mathbb{C}}$ (x_n denotes $g^n(x)$) is normal near $i_x^{-1}(y)$: note that by remark in the definition of $\overline{\mathbb{C}}$-bundle, this does not depend on the choices of local parametrizations at x and x_n. Still equivalently, F_x is the open subset of Y_x where the family $\{f_x^n\}$ of mappings from Y_x into Y is local equicontinuous. We put $J_x(f)$(simply J_x)$= Y_x \setminus F_x$. Furthermore, we put

$$\tilde{J}(f) = \overline{\bigcup_{x\in X} J_x}, \ \tilde{F}(f) = Y \setminus \tilde{J}(f),$$

and $\hat{J}_x(f)$(simply $\hat{J}_x$)$= \tilde{J}(f) \cap Y_x$ for each $x \in X$.

Remark 1. There exists a fibered rational map $f : Y \to Y$ satysfying that $\bigcup_{x\in X} J_x$ is NOT compact.

We give some notations and definitions on dynamics of rational semigroups.

For a Riemann surface S, let End(S) denote the set of all holomorphic endomorphisms of S. It is a semigroup with the semigroup operation being composition of maps. A *rational semigroup* is a subsemigroup of End($\overline{\mathbb{C}}$) without any constant elements. We say that a rational semigroup G is a *polynomial semigroup* if each element of G is a polynomial. The researches on dynamics of rational semigroups were started by A.Hinkkanen and G.J.Martin ([HM1]), who were interested in the role of dynamics of polynomial semigroups in the research of various one-complex-dimensional moduli spaces for discrete groups, and F.Ren's group([GR]).

Definition 1.4. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}, \ J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the *Fatou set* for G and $J(G)$ is called the *Julia set* for G. The backward orbit $G^{-1}(z)$ of z and the set of exceptional points $E(G)$ are defined by: $G^{-1}(z) = \cup_{g\in G} g^{-1}(z)$ and $E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp G^{-1}(z) \le 2\}$. For any subset A of $\overline{\mathbb{C}}$, we set $G^{-1}(A) = \cup_{g\in G} g^{-1}(A)$. We denote by $\langle h_1, h_2, \dots \rangle$ the rational semigroup generated by the family $\{h_i\}$.

Lemma 1.5 ([S4]). *Let G be a rational semigroup and assume G is generated by a precompact subset Λ of End($\overline{\mathbb{C}}$). Then*

$$J(G) = \overline{\bigcup_{f\in\Lambda} f^{-1}(J(G))} = \bigcup_{h\in\overline{\Lambda}} h^{-1}(J(G)).$$

In particular if Λ is compact then we have $J(G) = \bigcup_{f\in\Lambda} f^{-1}(J(G))$.

We call this property the backward self-similarity of the Julia set.

Remark 2. By the backward self-similarity, the research on the Julia sets of rational semigroups may be considered as a kind of generalization of the research on self-similar sets constructed by some similitudes from $\mathbb{C}$ to itself, which can be regarded as the Julia sets of some rational semigroups. It is easily seen that the Sierpiński gasket is the Julia set of a rational semigroup $G = \langle h_1, h_2, h_3 \rangle$ where $h_i(z) = 2(z - p_i) + p_i, i = 1, 2, 3$ with $p_1p_2p_3$ being a regular triangle.

Example 1.6. 1. ([S4].) Let $h_1, \ldots, h_m$ be non-constant rational maps. Let $\Sigma_m = \{1, \ldots, m\}^{\mathbf{N}}$ be the space of one-sided infinite sequences of m symbols and $g : \Sigma_m \to \Sigma_m$ be the shift map: that is, g is defined by $g((w_1, w_2, \ldots)) = (w_2, w_3, \ldots)$. Let X be a compact subset of Σ_m such that $g(X) \subset X$. Let $Y = X \times \overline{\mathbb{C}}$ and $\pi : Y \to X$ be the natural projection. Then (π, Y, X) is a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a map defined by: $f((w, y)) = (g(w),\ h_{w_1}(y))$. Then $f : Y \to Y$ is a fibered rational map over $g : X \to X$.

In the above if $X = \Sigma_m$ then we say that $f : Y \to Y$ is the **fibered rational map associated with the generator system** $\{h_1, \ldots h_m\}$ of the rational semigroup $G = \langle h_1, \ldots, h_m \rangle$. Then by Proposition 3.2 in [S5](See also §8:Note in [S7]) we have

$$\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G),$$

where $\pi_{\overline{\mathbb{C}}} : Y \to \overline{\mathbb{C}}$ is the projection. See [S4] for more details.

2. Let Y be a ruled surface over a Riemann surface X: that is, Y is a smooth projective variety of complex dimension 2 which is also a holomorphic $P^1(\mathbb{C})$-bundle over X. Every Y_x has a unique conformal structure and a positive form $\omega_x = \omega|_{Y_x}$, where ω is the Kähler form on Y. Let $\pi : Y \to X$ be the projection. Then (π, Y, X) is a $\overline{\mathbb{C}}$-bundle.

   Dabija [D] showed that (almost) every holomorphic selfmap f of Y is a fibered rational map over a holomorphic map $g : X \to X$.

3. Let $p(x) \in \mathbb{C}[x]$ be a polynomial with degree at least two and $q(x, y) \in \mathbb{C}[x, y]$ a polynomial of the form: $q(x, y) = y^n + a_1(x)y^{n-1} + \cdots$. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a map defined by

$$f((x, y)) = (p(x), q(x, y)).$$

   This is called a polynomial skew product in $\mathbb{C}^2$. Dynamics of maps of this form were investigated by S.-M.Heinemann in [H1] and [H2] and by M.Jonsson in [J1].

   Let X be a compact subset of $\overline{\mathbb{C}}$ such that $p(X) \subset X$. (e.g. the Julia set of p.) Let $(\pi, Y = X \times \overline{\mathbb{C}}, X)$ be a trivial $\overline{\mathbb{C}}$-bundle. Then the map $\tilde{f} : Y \to Y$ defined by $\tilde{f}((x, y)) = (p(x), q(x, y))$ is a fibered rational map over $p : X \to X$.

Notation :

- Let Z_1 and Z_2 be two topological spaces and $g : Z_1 \to Z_2$ be a map. For any subset A of Z_2, we denote by $c(g, A)$ the set of all connected components of $g^{-1}(A)$.
- for any $y \in \overline{\mathbb{C}}$ and $\delta > 0$, we put $B(y, \delta) = \{y' \in \overline{\mathbb{C}} \mid d(y, y') < \delta\}$, where d is the spherical metric. Similarly, for any $y \in \mathbb{C}$ and $\delta > 0$ we put $D(y, \delta) = \{y' \in \mathbb{C} \mid |y - y'| < \delta\}$.
- Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. For any $y \in Y$ and $r > 0$ we set

$$\tilde{B}(y, r) = \{y' \in Y_{\pi(y)} \mid d_{\pi(y)}(y', y) < r\},$$

where for each $x \in X$ we denote by d_x the metric on Y_x induced by the form ω_x.

Now we define the semi-hyperbolicity of fibered rational maps.

Definition 1.7. (semi-hyperbolicity on fibered rational maps) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Let $N \in \mathbb{N}$. We denote by $SH_N(f)$ the set of points $z \in Y$ satisfying that there exists a positive number δ, a neighborhood U of $\pi(z)$ and a local parametrization $\{i_x\}$ in U such that for any $x \in U$, any $n \in \mathbb{N}$, any $x_n \in g^{-1}(x)$ and any $V \in c(i_x(B(i_{\pi(z)}^{-1}(z),\ \delta)),\ f_x^n)$, we have

$$\deg(f_x^n : V \to i_x(B(i_{\pi(z)}^{-1}(z),\ \delta))) \leq N.$$

We set

$$UH(f) = Y \setminus \bigcup_{N \in \mathbb{N}} SH_N(f).$$

A point $z \in SH_N(f)$ is called a *semi-hyperbolic point of degree N*. We say that f is *semi-hyperbolic (along fibers)* if $\tilde{J}(f) \subset \bigcup_{N \in \mathbb{N}} SH_N(f)$. This is equivalent to $\tilde{J}(f) \subset SH_N(f)$ for some $N \in \mathbb{N}$.

Similarly we define the semi-hyperbolicity on rational semigroups.

Definition 1.8. (semi-hyperbolicity on rational semigroups) Let G be a rational semigroup and N a positive integer. We denote by $SH_N(G)$ the set of points $z \in \overline{\mathbb{C}}$ satisfying that there exists a positive number δ such that for any $g \in G$ and any $V \in c(B(z, \delta),\ g)$, we have

$$\deg(g : V \to B(z, \delta)) \leq N.$$

Further we set $UH(G) = \overline{\mathbb{C}} \setminus (\cup_{N \in \mathbb{N}} SH_N(G))$. A point $z \in SH_N(G)$ is called a *semi-hyperbolic point of degree N*. We say that G is *semi-hyperbolic* if $J(G) \subset \bigcup_{N \in \mathbb{N}} SH_N(G)$. This is equivalent to $J(G) \subset SH_N(G)$ for some xample 1.9. 1. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. We set

$$P(f) = \overline{\bigcup_{n \in \mathbb{N}} \bigcup_{x \in X} f_x^n(\text{ critical points of } f_x)}.$$

This is called the fiber post critical set of fibered rational map f. If $f : Y \to Y$ is hyperbolic along fiberes: that is, $P(f) \subset F(f)$, then f is semi-hyperbolic along fiberes with the constant $N = 1$.

2. In Corollary 6.7 of [Se3] O.Sester showed that any 'non-reccurent quadratic fibered polynomials' with connected fiberwise filled-in Julia sets are semi-hyperbolic.

3. Let $\{h_1, \ldots, h_m\}$ be non-constant rational functions on $\overline{\mathbb{C}}$. Let $f : Y \to Y$ be the fibered rational map in Example 1.6.1. By easy arguments we can show that $f : Y \to Y$ is semi-hyperbolic along fiberes if and only if G is semi-hyperbolic.

   In [S4], if G is a finitely generated rational semigroup, then a sufficient condition to be semi-hyperbolic for a point $z \in J(G)$ was given, which gives a generalization of R.Mañé's work([Ma]). Further in [S4], the following statement was shown: Assume that there exists an element of G with the degree at least two, that each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic and that $J(G) \neq \overline{\mathbb{C}}$. Then G is semi-hyperbolic if and only if all of the following conditions are satisfied.

   (a) for each $z \in J(G)$ there exists a neighborhood U of z in $\overline{\mathbb{C}}$ such that for any sequence $(g_n) \subset G$, any domain V in $\overline{\mathbb{C}}$ and any point $\zeta \in U$, we have that the sequence (g_n) does NOT converge to ζ locally uniformly on V

   (b) for each $j = 1, \ldots, m$ each $c \in C(f_j) \cap J(G)$ satisfies

   $$d(c, \ (G \cup \{id\})(f_j(c))) > 0$$

   From this fact it was shown in [S4] that if we assume that there exists an element of G with the degree at least two, that each element of Aut $\overline{\mathbb{C}} \cap G$(if this is not empty) is loxodromic, that there is no super attracting fixed point of any element of G in $J(G)$ and $F(G) \neq \emptyset$, then G is semi-hyperbolic.

   By this theorem we know that $G = \langle z^2 + 2, \ z^2 - 2 \rangle$ is semi-hyperbolic. This is NOT hyperbolic. See [S4].

We need some technical conditions.

finition 1.10 (Condition(C1)). Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f :$ $\to Y$ be a rational fibered over $g : X \to X$. We say that f satisfies the ndition (C1) if there exists a family $\{D_x\}_{x \in X}$ of topological disks with $\subset Y_x$, $x \in X$ such that the following conditions are satisfied:

1. for each $x \in X$ there exists a point $z_x \in Y_x$ and a positive number r_x such that $D_x = \tilde{B}(z_x, r_x)$,
2. $\overline{\bigcup_{x\in X} \bigcup_{n\geq 0} f_x^n(D_x)} \subset \tilde{F}(f)$,
3. for any $x \in X$, we have that $\mathrm{diam}(f_x^{(n)}(D_x)) \to 0$, as $n \to \infty$, and
4. $\inf_{x\in X} r_x > 0$.

Definition 1.11 (Condition(C2)). Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. We say that f satisfies the condition (C2) if for each $x_0 \in X$ there exists an open neighborhood O of x_0 and a family $\{D_x\}_{x\in O}$ of topological disks with $D_x \subset Y_x, x \in O$ such that the following conditions are satisfied:

1. for each $x \in O$ there exists a point $z_x \in Y_x$ and a positive number r_x such that $D_x = \tilde{B}(z_x, r_x)$,
2. $\overline{\bigcup_{x\in O} \bigcup_{n\geq 0} f_x^n(D_x)} \subset \tilde{F}(f)$,
3. for any $x \in O$, we have that $\mathrm{diam}(f_x^{(n)}(D_x)) \to 0$, as $n \to \infty$, and
4. $x \mapsto D_x$ is continuous in O.

Example 1.12.

1. Let $\{h_1, \dots h_m\}$ be non-constant rational functions on $\overline{\mathbb{C}}$ with $\deg(h_1) \geq 2$. Let $f : Y \to Y$ be the fibered rational map associated with the generator system $\{h_1, \dots, h_m\}$ of rational semigroup $G = \langle h_1, \dots, h_m \rangle$, which is described in Example 1.6.1. Suppose that f is semi-hyperbolic along fibers and that $\pi_{\overline{\mathbb{C}}}(\tilde{J}(f)) = J(G)$ is not equal to the Riemann sphere. Then we have that f satisfies the condition (C2). Actually, there exists an attracting fixed point a of some element of G in $F(G)$. Since G is semi-hyperbolic, we have that setting $D_x = D(a, \epsilon)$ for each $x \in \Sigma_m$ where ϵ is a positive number, f satisfies the condition (C2) with the family of disks $(D_x)_{x\in\Sigma_m}$. For more details, see Theorem 1.35 and Remark 5 in [S4].

2. Let $(\pi,\ Y = X \times \overline{\mathbb{C}},\ X)$ be a trivial $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$ satisfying that f_x is a polynomial mapping of degree at least two for each $x \in X$. Then setting $D_x = D$ where D is a small neighborhood of infinity for each $x \in X$, the fibered rational map f satisfies the condition (C2) with the family of disks $(D_x)_{x\in X}$.

We give the definition of 'conical' set in the Julia set.

Definition 1.13. (conical set for fibered rational maps) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$.

Let $N \in \mathbb{N}$ and $r > 0$. We denote by $\tilde{J}_{con}(f, N, r)$ the set of points $z \in \tilde{J}(f)$ satisfying that for any $\epsilon > 0$, there exists a positive integer n such that the element $U \in c(\tilde{B}(f^n(z), r),\ f^n|_{Y_{\pi(z)}})$ containing z satisfies the following conditions:

1. $\operatorname{diam} U < \epsilon$,
2. U is simply connected, and
3. $\deg(f^n : U \to \tilde{B}(f^n(z), r)) \leq N$.

We set $\tilde{J}_{con}(f, N) = \bigcup_{r>0} \tilde{J}_{con}(f, N, r)$ and $\tilde{J}_{con}(f) = \bigcup_{N \in \mathbb{N}} \tilde{J}_{con}(f, N)$.

Definition 1.14. (conical set for rational semigroups) Let G be a rational semigroup. Let $N \in \mathbb{N}$ and $r > 0$. We denote by $J_{con}(G, N, r)$ the set of points $z \in J(G)$ satisfying that for any $\epsilon > 0$, there exists an element $g \in G$ such that $g(z) \in J(G)$ and the element $U \in c(B(g(z), r),\ g)$ containing z satisfies the following conditions:

1. $\operatorname{diam} U < \epsilon$,
2. U is simply connected, and
3. $\deg(g : U \to B(g(z), r)) \leq N$.

We set $J_{con}(G, N) = \cup_{r>0} J_{con}(G, N, r)$ and $J_{con}(G) = \cup_{N \in \mathbb{N}} J_{con}(G, N)$.

Definition 1.15. (good points for fibered rational maps) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. We set

$$\tilde{J}_{good}(f) = \{z \in \tilde{J}(f) \mid \limsup_{n\to\infty} d(f^n(z), UH(f)) > 0\}.$$

Definition 1.16. (good points for finitely generated rational semigroups) Let $\langle h_1, \dots, h_m \rangle$ be a rational semigroup. Let $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the fibered rational map associated with the generator system $\{h_1, \dots, h_m\}$. Then we set $J_{good}(G) = \pi_{\overline{\mathbb{C}}}(\tilde{J}_{good}(f))$. Note that this definition does not depend on the choice of any generator system of G which consists of finitely many elements.

2 Results on Fibered Rational Maps

In this section we state some results on dynamics of fibered rational maps which are deduced by semi-hyperbolicity, except Theorem 2.6 . The proofs are given in §4.

Theorem 2.1. *(**measure zero**) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose all of the following conditions:*

1. *f satisfies the condition (C1),*

2. *for each $x \in X$, the boudary of $\hat{J}_x(f) \cap UH(f)$ in Y_x does not separate points in Y_x,*

3. *$\tilde{J}(f) \setminus \bigcup_{n \in N} f^{-n}(UH(f)) \subset \tilde{J}_{good}(f)$ and*

4. *for each $z \in \tilde{J}(f) \cap UH(f)$ and each open neighborhood V of z in $Y_{\pi(z)}$ we have that the diameter of $f^n_{\pi(z)}(V)$ does not tend to zero as $n \to \infty$.*

Then $\tilde{J}(f) = \bigcup_{x \in X} J_x$ and for each $x \in X$, the 2-dimensional Lebesgue measure of $J_x \setminus \bigcup_{n \in N} f^{-n}(UH(f))$ is equal to zero.

Definition 2.2. Let (Y, d) be a metric space. Let k be a constant with $0 < k < 1$. Let J be a subset of Y. We say that J is k-porous if for any $x \in J$ and any positive number r there exist a ball in $\{y \in Y \mid d(y, x) < r\} \setminus J$ with the radius at least kr.

Remark 3. If Y is the Euclidean space $\mathbb{R}^n$ and d is the Euclidean metric, the Box dimension of any k-porous bouded set J in $\mathbb{R}^n$ is less than $n - c(k, n)$, where $c(k, n)$ is a positive constant which depends only on k and n ([PR]).

Theorem 2.3. (**porosity**) *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose that f satisfies the condition (C1) and that f is semi-hyperbolic. Then there exists a constant k with $0 < k < 1$ such that J_x is k-porous in Y_x for each $x \in X$. In particular, there exists a constant $0 \le c < 2$ such that for each $x \in X$,*

$$\dim_H(J_x) \le \dim_B(J_x) \le c,$$

where $\dim_H$ denotes the Hausdorff dimension and $\dim_B$ denotes the Box dimension with respect to the metric on Y_x induced by ω_x (ω_x is the form in the remark in Definition 1.1).

Theorem 2.4. (**a rigidity**) *Let (π, Y, X) and $(\tilde{\pi}, \tilde{Y}, \tilde{X})$ be two $\overline{\mathbb{C}}$-bundles. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$ and $\tilde{f} : \tilde{Y} \to \tilde{Y}$ a fibered rational map over $\tilde{g} : \tilde{X} \to \tilde{X}$. Let $u : Y \to \tilde{Y}$ be a homeomorphism which is a bundle conjugacy between f and $\tilde{f}$:i.e. u satisfies that $\tilde{\pi} \circ u = v \circ \pi$ for some homeomorphism $v : X \to X$ and $\tilde{f} \circ u = u \circ f$. Suppose that f is semi-hyperbolic along fiberes and satisfies the condition (C1). For each $w \in X$, let $u_w : Y_w \to \tilde{Y}_{v(w)}$ be the restriction of u. Let $x \in X$ be a point. Then if u_x is K-quasiconformal on F_x, for each $a \in \overline{\bigcup_{n \in \mathbb{Z}} \{g^n(x)\}}$ we have that $u_a : Y_a \to \tilde{Y}_{v(a)}$ is K-quasiconformal on the whole Y_a.*

Definition 2.5. Let C be a positive number. Let K be a closed subset of $\overline{\mathbb{C}}$. We say that K is C-uniformly perfect if for any doubly connected domain A in $\overline{\mathbb{C}}$ satisfying that A separates K i.e. both two connected components of $\overline{\mathbb{C}} \setminus A$ have non-empty intersections with K, mod A (the modulus of A. For the definition, see [LV])is less than C.

Remark 4. Uniform perfectness implies many good properties ([BP],[Po],[Su This term was introduced in [Po]. In [Su], there is a survey on uniform perfectness.

Theorem 2.6. (**uniform perfectness**) *Let (π, Y, X) be a $\overline{\mathbb{C}}$-buundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$ satisfying that $d(x) \geq 2$ for any $x \in X$. Then we have the following.*

1. *There exists a positive constant C such that for any $x \in X$, we have that J_x and $\hat{J}_x$ are C-uniformly perfect.*

2. *Suppose further $f(\tilde{F}(f)) \subset \tilde{F}(f)$ (for example, assume that $g : X \to X$ is an open map). If a point $z \in Y$ satisfies that $f^n_{\pi(z)}(z) = z$ and $(f^n_{\pi(z)})'(z) = 0$ for some $n \in \mathbb{N}$ and $z \in \hat{J}_{\pi(z)}$, then z belongs to the interior of $\hat{J}_{\pi(z)}$ with respect to the topology of $Y_{\pi(z)}$.*

3 Results on Rational Semigroups

In this section we state some results on dynamics of semigroups generated by rational functions on the Riemann sphere. The proofs are given in §4.

Definition 3.1. Let $G = \langle h_1, h_2, \ldots, h_m \rangle$ be a finitely generated rational semigroup. Let U be an open set in $\overline{\mathbb{C}}$. We say that G satisfies the *open set condition* with U with respect to the generator systems $\{h_1, h_2, \ldots, h_m\}$ if for each $j = 1, \ldots, m$, $h_j^{-1}(U) \subset U$ and $\{h_j^{-1}(U)\}_{j=1,\ldots,m}$ are mutually disjoint.

Theorem 3.2. (**porosity**) *Let $G = \langle h_1, \ldots, h_m \rangle$ be a rational semigroup with an element of degree at least two. Suppose all of the following conditions;*

1. *G satisfies the open set condition with an open set U with respect to the generator system $\{f_1, \ldots, f_m\}$,*

2. *$\sharp(UH(G) \cap J(G)) < \infty$ and*

3. *$UH(G) \cap J(G) \subset U$.*

Then we have that $J(G) = \overline{U}$ or that $J(G)$ is porous(and so the Box dimension of $J(G)$ is strictly less than 2). Moreover, the fibered rational map $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ associated with the generator system $\{h_1, \ldots, h_m\}$ satisfies that

$$\tilde{J}(f) = \bigcup_{x \in \Sigma_m} J_x.$$

Definition 3.3. Let G be a rational semigroup and δ a non-negative number. We say that a Borel probability measure μ on $\overline{\mathbb{C}}$ is δ-subconformal if for each $g \in G$ and for each Borel measurable set A

$$\mu(g(A)) \leq \int_A \|g'(z)\|^\delta d\mu,$$

where we denote by $\|\cdot\|$ the norm of the derivative with respect to the spherical metric. For each $x \in \overline{\mathbb{C}}$ and each real number s we set

$$S(s,x) = \sum_{g\in G}\sum_{g(y)=x} \|g'(y)\|^{-s}$$

counting multiplicities and

$$S(x) = \inf\{s \mid S(s,x) < \infty\}.$$

If there is not s such that $S(s,x) < \infty$, then we set $S(x) = \infty$.Also we set

$$s_0(G) = \inf\{S(x)\},\ s(G) = \inf\{\delta \mid \exists\mu : \delta\text{-subconformal measure}\}$$

We have an estimate on $s_0(G)$ when G satisfies the open set condition.

Proposition 3.4. *Let $G = \langle h_1, \ldots h_m\rangle$ be a rational semigroup. When $m = 1$, we assume that h_1 is neither identity nor an elliptic Möbius transformation. Suppose G satisfies the open set condition with an open set U with respect to the generator system $\{h_1, \ldots, h_m\}$. Suppose also that $J(G) \neq \overline{U}$. Then there exists an open set V included in $U \cap F(G)$ such that for almost $x \in V$ with respect to the 2-dimensional Lebesgue measure, we have*

$$S(2,x) < \infty.$$

In particular, $s_0(G) \leq 2$.

Theorem 3.5. (**Hausdorff dimension**) *Let $G = \langle h_1, \ldots h_m\rangle$ be a rational semigroup. Under the same assumption as that of Theorem 3.2, we have that*

$$\dim_H(J(G)) \leq s(G) \leq s_0(G),$$

where $\dim_H$ denotes the Hausdorff dimension with respect to the spherical metric in $\overline{\mathbb{C}}$.

Example 3.6. Let $h_1(z) = z^2 + 2$, $h_2(z) = z^2 - 2$ and $U = \{|z| < 2.\}$. Then we have $h_1^{-1}(U) \cup h_2^{-1}(U) \subset U$ and $h_1^{-1}(U) \cap h_2^{-1}(U) = \emptyset$. Let h_3 be a polynomial which is conjugate to h_4^n by an affine map α, where $h_4(z) = z^2 + \frac{1}{4}$ and $n \in \mathbb{N}$ is a number large enough. Taking α appropriately, we have $J(h_3) \subset U \setminus (h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}))$. Taking n large enough, we have $h_3^{-1}(U) \subset U \setminus (h_1^{-1}(\overline{U}) \cup h_2^{-1}(\overline{U}))$. Then $G = \langle h_1, h_2, h_3\rangle$ satisfies the conditions in the assumption of Theorem 3.2. In this case $UH(G) \cap J(G)$ is the parabolic fixed point of h_3. By Theorem 3.2, we get that $J(G)$ is porous and in particular, the Box dimension is strictly less than 2.

4 Tools and Proofs

4.1 Tools

To show theorems in §2 and §3 , we need the followings. For the research on semi-hyperbolicity of usual dynamics of rational functions, see [CJY] and [Ma].

Notations.

1. Let X be a compact set in $\overline{\mathbb{C}}$ and z be a point in $\overline{\mathbb{C}} \setminus X$. Then we set
$$\mathrm{Dist}(X, z) = \max_{y \in X} d(y, z) / \min_{y \in X} d(y, z).$$
2. For two positive numbers A and B, $A \asymp B$ means $K^{-1} \leq A/B \leq K$ for some constant K independent of A and B.

Lemma 4.1 ([CJY]). *(**distortion lemma for proper maps**) For any positive integer N and real number r with $0 < r < 1$, there exists a constant $C = C(N, r)$ such that if $f : D(0,1) \to D(0,1)$ is a proper holomorphic map with $\deg(f) = N$ and $f(0) = 0$, then*
$$D(f(z_0), C) \subset f(D(z_0, r)) \subset D(f(z_0), r)$$
for any $z_0 \in D(0,1)$. Here we can take $C = C(N, r)$ independent of f.

The following is a generalized distortion lemma for proper maps.

Lemma 4.2 ([S4],[S6]). *Let V be a domain in $\overline{\mathbb{C}}$, K a continuum in $\overline{\mathbb{C}}$ with $diam_S K = a$. Assume $V \subset \overline{\mathbb{C}} \setminus K$. Let $f : V \to D(0,1)$ be a proper holomorphic map of degree N. Then there exists a constant $r(N, a)$ depending only on N and a such that for each r with $0 < r \leq r(N, a)$, there exists a constant $C = C(N, r)$ depending only on N and r satisfying that for each connected component U of $f^{-1}(D(0, r))$,*
$$diam_S\ U \leq C,$$
where we denote by $diam_S$ the spherical diameter. Also we have $C(N, r) \to 0$ as $r \to 0$.

The following lemma is a slightly modified version of Lemma 2.15 in [S4].

Lemma 4.3 ([S4]). *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Assume f satisfies the condition (C1). Assume $z_0 \in SH_N(f)$ for some $N \in \mathbb{N}$. b Then there exists a positive number δ_0 such that for each δ with $0 < \delta < \delta_0$ there exists a neighborhood U of $x_0 := \pi(z_0)$ in X satisfying that for each $n \in \mathbb{N}$, each $x \in U$ and each $x_n \in p^{-n}(x)$, we have that each element of $c(i_x i_{x_0}^{-1} \tilde{B}(z_0, \delta),\ f_{x_n}^n)$ is simply connected.*

The following theorem says about what happens if there exists a non-constant limit function on a component of a fiber-Fatou set. This is the key to state other results.

Theorem 4.4 ([S4]). *(**Key theorem I**) Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Assume f satisfies the condition (C1). Let $z \in Y$ be a point with $z \in F_{\pi(z)}$. Let (i_x) be a local parametrization. Let U be a connected open neighborhood of $i_{\pi(z)}^{-1}(z)$ in $\overline{\mathbb{C}}$. Suppose that there exists a sequence (n_j) of $\mathbb{N}$ such that $R_j := i_{\pi f^{n_j}(z)}^{-1} \circ f_{\pi(z)}^{n_j} \circ i_{\pi(z)}$ converges to a non-constant map ϕ uniformly on U as $j \to \infty$. Further suppose $f_{\pi(z)}^{n_j}(z)$ converges to a point $z_0 \in Y$. Let $S_{i,j} = f_{g^{n_i}\pi(z)}^{n_j - n_i}$ for $1 \leq i \leq j$. We set*

$$V = \{a \in Y_{\pi(z_0)} \mid \exists \epsilon > 0, \ \lim_{i \to \infty} \sup_{j > i} \sup_{\xi \in \tilde{B}(a, \epsilon)} d(S_{i,j} \circ \varphi(\xi), \ \xi) = 0\},$$

where φ is a map from $Y_{\pi(z_0)}$ onto $Y_{g^{n_i}\pi(z)}$ defined by the local parametrization around $\pi(z_0)$. Then V is a non-empty open proper subset of $Y_{\pi(z_0)}$ and we have that

$$\partial V \subset \hat{J}_{\pi(z_0)}(f) \cap UH(f).$$

Corollary 4.5. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Assume f satisfies the condition (C1). Assume also that for each $x \in X$, the boundary of $\hat{J}_x(f) \cap UH(f)$ in Y_x does not separate points in Y_x. Then for each $z \in Y$ with $z \in F_{\pi(z)}$, we have that $\operatorname{diam} f_{\pi(z)}^n(W) \to 0$ as $n \to \infty$ for each open connected neighborhood W of z in $Y_{\pi(z)}$ and that $d(f_{\pi(z)}^n(z), UH(f)) \to 0$ as $n \to \infty$.*

4.2 Proofs of results on fibered rational maps

We start with the following.

Proposition 4.6. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose that $\hat{J}_x$ has no interior points for each $x \in X$. Then the two dimensional Lebesgue measure of $\tilde{J}_{con}(f) \cap J_x$ is equal to zero for each $x \in X$.*

Proof. Fix $N \in \mathbb{N}$. Suppose that there exists a point $x \in X$ such that $\tilde{J}_{con}(f, N) \cap J_x$ (this is an open set in J_x) has positive measure. Then there exists a Lebesgue density point $y \in \tilde{J}_{con}(f, N) \cap J_x$. Let $y_m = f_x^m(y)$ and $x_m = g^m(x)$ for any $m \in \mathbb{N}$. Let $\delta > 0$ be a number such that $y \in \tilde{J}_{con}(f, N, \delta)$. Let U_m, U'_m be the elements of $c(\tilde{B}(y_m, \delta/2), \ f_x^m)$, $c(\tilde{B}(y_m, \delta), \ f_x^m)$ containing y respectively. Since $y \in \tilde{J}_{con}(f, N, \delta)$, there exists a subsequence (n) in $\mathbb{N}$ with $n \to \infty$ such that U'_n is simply connected, $\deg(f_x^n : U'_n \to \tilde{B}(y_n, \delta)) \leq N$

for each n and $\operatorname{diam} U'_n \to 0$ as $n \to \infty$. By Corollary 1.9 in [S4] for any local parametrization i_x,

$$\lim_{n\to\infty} \frac{m(i_x^{-1}(U_n \cap J_x))}{m(i_x^{-1}(U_n))} = 1, \tag{1}$$

where m denotes the spherical measure of $\overline{\mathbb{C}}$. Using an argument in the proof of Theorem 4.4 in [S4], from (1) we can show that

$$\lim_{n\to\infty} \frac{m(i_{x_n}^{-1}(\tilde{B}(y_n, \delta/2) \cap F_{x_n}))}{m(i_{x_n}^{-1}(\tilde{B}(y_n, \delta/2)))} = 0, \tag{2}$$

where i_{x_n} denotes a local parametrization. There exists a subsequence (n_j) of (n), a point $y_\infty \in Y$ and a point $x_\infty \in X$ such that $y_{n_j} \to y_\infty$ and $x_{n_j} \to x_\infty$ as $j \to \infty$. By (2) we have that $\tilde{B}(y_\infty,\ \delta/2) \subset \hat{J}_{x_\infty}$. On the other hand, by the assumption we have that for any $a \in X$, $\hat{J}_a$ has no interior point. This is a contradiction.

□

Proposition 4.7. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose f satisfies the condition (C1). Then we have the following.*

1.

$$J_{good}(f) \cap \bigcup_{x\in X} J_x \subset \tilde{J}_{con}(f).$$

2. If we assume further that for each $x \in X$, the boundary of $\hat{J}_x(f) \cap UH(f)$ in Y_x does not separate points in Y_x, then

$$J_{good}(f) \subset \tilde{J}_{con}(f) \cap \bigcup_{x\in X} J_x.$$

Proof. First we will show the first statement. Let $z \in \bigcup_{x\in X} J_x$ be a point satisfying that $\limsup_{n\to\infty} d(f^n(z), UH(f)) > 0$. For each $m \in \mathbb{N}$ let $z_m = f^m(z)$ and $x_m = \pi f^m(z)$. For each $m \in \mathbb{N}$ and each $r > 0$ let $U_m(r), U'_m(r)$ be the elements of $c(\tilde{B}(z_m, r/2),\ f^m_{\pi(z)}), c(\tilde{B}(z_m, r),\ f^m_{\pi(z)})$ containing z respectively. There exists a positive number δ, positive integer N and a sequence (n) in $\mathbb{N}$ such that $\deg(f^n_{\pi(z)} : U'_n(\delta) \to \tilde{B}(z_n, \delta)) \le N$. By Lemma 4.3, taking δ small enough we can assume that $U'_n(\delta)$ is simply connected.

Suppose that diam $(U_n(\delta))$ does not tend to zero as $n \to \infty$ in (n). Then by distortion lemma for proper maps there exists a subsequence (n_j) of (n) with $n_j \to \infty$ and a positive number r such that $U_{n_j}(\delta) \supset \tilde{B}(z, r)$ for each j. Hence

$$f^{n_j}(\tilde{B}(z, r)) \subset \tilde{B}(f_{n_j}(z), \delta) \tag{3}$$

for each j. By condition (C1), if we take δ small enough (3) contradicts to that $z \in \cup_{x \in X} J_x$. Hence we get that diam $U_n(\delta) \to 0$ as $n \to \infty$ in (n). Hence we get that $z \in \tilde{J}_{con}(f)$.

The second statement follows from Corollary 4.5 and the first statement. □

Corollary 4.8. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Suppose that $\tilde{J}(f) = \bigcup_{x \in X} J_x$ and that f satisfies the condition (C1). Then for each $x \in X$ we have that*

$$d(f_x^n(y), UH(f)) \to 0, \quad \textit{as } n \to \infty,$$

for almost every $y \in J_x$ with respect to the Lebesgue measure in Y_x.

Proof. By condition (C1) $\hat{J}_x = J_x$ has no interior points for each $x \in X$. By Proposition 4.6 and Proposition 4.7, we get the statement. □

Proof. of **Theorem 2.1.** Suppose that there exists a point $z \in \tilde{J}(f)$ satisfying that $z \in F_{\pi(z)}$. By Corollary 4.5, For each open connected neighborhood W of z in $F_{\pi(z)}$ we have diam $f^n(W) \to 0$ and $d(f^n(z), UH(f)) \to 0$ as $n \to \infty$. But by condition 3 and 4 in the assumption of our theorem, it causes a contradiction. Hence we have shown that $\tilde{J}(f) = \bigcup_{x \in X} J_x$. By Corollary 4.8 we get that the 2-dimensional Lebesgue measure of $J_x \setminus \bigcup_{n \in \mathbb{N}} f^{-n}(UH(f))$ is equal to zero. □

Proof. of **Theorem 2.3.** For any $y' \in \bigcup_{x \in X} J_x$ and $r > 0$, we set

$$h(y', r) = \sup\{s \mid \exists y'' \in J_{\pi(y')},\ \tilde{B}(y'', s) \subset F_{\pi(y')}\}$$

and $h(r) = \inf\{h(y', r) \mid y' \in \bigcup_{x \in X} J_x\}$. By Theorem 2.1, we have $\tilde{J}(f) = \bigcup_{x \in X} J_x$. By the condition (C1) we have int$J_x = \emptyset$ for any $x \in X$. Hence we get that $h(r) > 0$ for any $r > 0$.

Since f is semi-hyperbolic and satisfies the condition (C1), by Lemma 4.3 we have that there exists a positive number δ_1 and a number $N \in \mathbb{N}$ such that for any $y' \in \tilde{J}(f)$, $0 < \delta \leq \delta_1$, $n \in \mathbb{N}$ and any component V of $(f^n)^{-1}(\tilde{B}(y', 2\delta))$, V is simply connected and $\deg(f^n : V \to \tilde{B}(y', 2\delta)) \leq N$.

Let $y \in \tilde{J}(f)$ and $r > 0$. We set $B_n = f^n(\tilde{B}(y, r))$ and $y_n = f^n(y)$ for each $n \in \mathbb{N}$. Since $y \in J_{\pi(y)}$, we have that there exists the smallest positive integer n_0 such that diam $B_{n_0+1} > \delta_1$. Then there exists a constant l_0 such that $l_0\delta_2 <$diam B_{n_0}. By Corollary 2.3 in [Y], there exists a constant K depending only on N and a ball $\tilde{B}(y_{n_0}, r_0) \subset B_{n_0}$ with $r_0 \geq$diam $B_{n_0}/K \geq \frac{l_0\delta_2}{K}$ such that the component of $(f^{n_0})^{-1}(\tilde{B}(y_{n_0}, r_0))$ containing y is a subset of $\tilde{B}(y, r)$. There exists a ball $\tilde{B}(y', \frac{2}{3}h(r_0))$ included in $\tilde{B}(y_{n_0}, r_0) \cap F_{\pi(y_{n_0})}$.

Let D_0 be a component of $(f^{n_0})^{-1}(\tilde{B}(y', \frac{1}{2}h(r_0)))$ contained in $\tilde{B}(y, r)$. We have that $D_0 \subset F_{\pi(y)}$. Let $y'' \in D_0 \cap (f^{n_0})^{-1}(y')$ be a point. Then by Corollary 1.8 and 1.9 in [S4], Dist $(\partial D_0, y'') \leq M$ for some M depending only on N and diam $D_0 \asymp r$. Hence there exists a constant $0 < k < 1$ which does not depend on y and r such that $B(y'', kr) \subset D_0 \subset F_{\pi(y)}$.

□

4.3 Proofs of results on rational semigroups

— **Notation** Throughout this subsection, for a generator system $\{h_1, \dots h_m\}$ let $f : \Sigma_m \times \overline{\mathbb{C}} \to \Sigma_m \times \overline{\mathbb{C}}$ be the fibered rational map over the shift map $\sigma : \Sigma_m \to \Sigma_m$, where $\Sigma_m = \{1, \dots, m\}^{\mathbb{N}}$, associated with the generator system $\{h_1, \dots, h_m\}$. We set $q_x^{(n)}(y) = \pi_{\overline{\mathbb{C}}}(f_x^n(y))$ for any $(x, y) \in \Sigma_m \times \overline{\mathbb{C}}$.

Lemma 4.9. *Let E be a finite subset of $\overline{\mathbb{C}}$. Let $\langle h_1, \dots, h_m \rangle$ be a rational semigroup. Then for any number $M > 0$ there exists a positive integer n_0 such that for any $(n, x, y) \in \mathbb{N} \times \Sigma_m \times E$ with $n \geq n_0$ which satisfies all of the following conditions:*

1. *$q_x^{(j)}(y) \in E$ for $j = 0, \dots, n$*
2. *$(q_x^{(n)})'(y) \neq 0$ and*
3. *for any $i \in \mathbb{N}$ and $j \in \mathbb{N}$ with $i + j \leq n$, if $q_{\sigma^i(x)}^{(j)}(q_x^{(i)}(y)) = q_x^{(i)}(y)$ then $|(q_{\sigma^i(x)}^{(j)})'(q_x^{(i)}(y))| > 1$,*

we have that $|(q_x^{(n)})'(y)| > M$.

Proof. This lemma can be shown by induction on $\sharp E$ using the same method as that in Lemma 1.32 in [S4]. □

Lemma 4.10. *Let $\langle h_1, \dots, h_m \rangle$ be a finetely generated rational semigroup. Suppose $\sharp(UH(G) \cap J(G)) < \infty$ and $UH(G) \cap J(G) \neq \emptyset$. Then for each $z \in UH(G) \cap J(G)$ there exists an element $g \in G$, an element $h \in G$ and a point $w \in UH(G) \cap J(G)$ such that $h(w) = z$, $g(w) = w$ and $|g'(w)| \leq 1$.*

Proof. Suppose that there exists a point $z \in UH(G) \cap J(G)$ for which there exists no (g, h, w) in the conclusion of our lemma. Then by Lemma 4.9 and the Koebe distortion theorem, we can easily see that for arbitrarily small $\epsilon > 0$ there exists a positive number δ and a positive constant N such that if a point $w_0 \in UH(G) \cap J(G)$ and an element $g_0 \in G$ satisfy $g_0(w_0) = z$ then the diameter of the component V of $g_0^{-1}(B(z, \delta))$ containing w_0 is less than ϵ and $\deg(g_0 : V \to B(z, \delta)) \leq N$. Then taking ϵ small enough, since G is finitely generated and $\sharp(UH(G) \cap J(G)) < \infty$ we can easily deduce that there exists a positive constant N' such that for any element $g_1 \in G$ and any component W of $g_1^{-1}(V)$, we have that $\deg(g_1 : W \to V) \leq N'$. This implies that $z \in SH_{N+N'}(G)$ and this contradicts to that $z \in UH(G)$. □

Lemma 4.11. *Under the assumpstion of Theorem 3.2, there exists a disk D in $F(G)$ such that*

1. $\overline{\bigcup_{g\in G} g(D)} \subset F(G)$ *and*

2. *diam* $q_x^{(n)}(D) \to 0$ *as* $n \to \infty$ *uniformly on* $x \in \Sigma_m$

In particular, the fibered rational map f satisfies the condition (C2).

Proof. Let $h \in G$ be an element of degree at least two. Since $\emptyset \neq \overline{\mathbb{C}} \setminus \overline{U} \subset F(G)$ and $UH(G) \cap J(G) \subset U$, we have that there exists an attracting periodic point z_0 in $F(G) \setminus \overline{U}$. Since $UH(G) \cap J(G) \subset U$ again, it follows that there exists a disk D around z_0 such that $\overline{\bigcup_{g\in G} g(D)} \subset F(G)$. By Lemma 1.30 in [S4], the statement of our lemma follows. □

Lemma 4.12. *Under the assumption of Theorem 3.2, if $UH(G) \cap J(G) \neq \emptyset$ then for each point $z \in UH(G) \cap J(G)$ there exists the unique element $h \in G$ satisfying that $h^n(z) = z$ for each $n \in \mathbb{N}$. Further we have that z is a parabolic fixed point of h.*

Proof. By Lemma 4.10 and the open set condition, there exists the unique element $h \in G$ with $h^n(z) = z$ for each $n \in \mathbb{N}$. Further we must have $|h'(z)| \leq 1$.

If $\deg(h) = 1$, then by Lemma 4.11 it follows that z is a repelling fixed point of h. This is a contradiction. If $\deg(h) \geq 2$, then since we are assuming that $\sharp(UH(G) \cap J(G)) < \infty$ we have that z is an attracting or parabolic fixed point of h. Suppose z is an attracting fixed point of h. Then there exists an open neighborhood V of z in U such that $h(V) \subset V$. Let $x \in \Sigma_m$ be the point such that $h_{x_n} \circ \cdots \circ h_{x_1} = h$ for each n, where $x = (x_1, x_2, \dots)$. Then by the open set condition for any $x' \in \Sigma_m \setminus \{x\}$ and any $n \in \mathbb{N}$ we have that $h_{x'_n} \circ \cdots h_{x'_1}(V) \subset \overline{\mathbb{C}} \setminus U$. Hence we have that G is normal in V and this is a contradiction. □

Lemma 4.13. *Under the assumption of Theorem 3.2, we have that for each $(x, y) \in \pi_{\overline{\mathbb{C}}}^{-1}(G^{-1}(J(G) \setminus UH(G)))$, $\limsup_{n\to\infty} d(q_x^{(n)}(y), UH(G)) > 0$.*

Proof. Let (x, y) be a point in $\pi_{\overline{\mathbb{C}}}^{-1} G^{-1}(J(G) \setminus UH(G))$. Then $q_x^{(n)}(y) \in J(G) \setminus UH(G)$ for each $n \in \mathbb{N}$.

Assume that $\lim_{n\to\infty} d(q_x^{(n)}(y), UH(G)) = 0$. We will deduce a contradiction. For each $z \in UH(G) \cap J(G)$, let g_z be the element of G in the statement of Lemma 4.12 Let $H = \{g_z \mid z \in UH(G) \cap J(G)\}$. Then we have $\sharp(H) < \infty$. Let $\epsilon > 0$ be a small number such that if a point $z \in UH(G) \cap J(G)$ and an element $h \in H$ satisfy $h(z) = z$, then

$$h(B(z, \epsilon)) \subset U. \tag{4}$$

Let A_ϵ be the ϵ-neighborhood of $UH(G) \cap J(G)$ in $\overline{\mathbb{C}}$. Then there exists a number $n_0 \in \mathbb{N}$ such that $q_x^{(n)}(y) \in A_\epsilon$ for each $n \geq n_0$.

For each $n \geq n_0$, let $z_n \in UH(G) \cap J(G)$ be the unique point such that $d(z_n, q_x^{(n)}(y)) < \epsilon$. Since $g(UH(G)) \subset UH(G)$ for each $g \in G$ we may assume that

$$q_{\sigma^n(x)}^{(1)}(z_n) = z_{n+1}$$

for each $n \geq n_0$.

Since $\sharp(UH(G) \cap J(G)) < \infty$, there exists a positive integer $n_1 \geq n_0$ and $l \in \mathbb{N}$ such that $z_{n_1+l} = z_{n_1}$. Let $g_1 \in G$ be the unique element such that $g_1(z_{n_1}) = z_{n_1}$. Let $w \in \{1, \dots, m\}^l$ be the word such that $h_{w_l} \circ \cdots \circ h_{w_1} = g_1$. Then by (4) and the open set condition we have that $\sigma^{n_1}(x) = w^\infty$. Since we are assuming $d(q_x^{(n)}(y),\ UH(G)) \to 0$ as $n \to \infty$, by $z_{n_1+l} = z_{n_1}$ we get that $g_1^k(q_x^{(n_1)}(y)) \to z_{n_1}$ as $k \to \infty$. Hence by Lemma 4.12 we must have that z_{n_1} is a parabolic fixed point of g_1 and $q_x^{(n_2)}(y)$ belongs to $W \cap \mathcal{P}$, where W is a small neighborhood of z_{n_1} in U, $\mathcal{P}$ is the union of attracting petals of g_1 at z_{n_1} and n_2 is a large positive number with $n_2 \geq n_1$. Then there exists an open neighborhood V of y such that $q_x^{(n_2)}(V) \subset W \cap \mathcal{P}$. Taking W so small and n_2 so large we may assume that $g_1^s(q_x^{(n_2)}(V)) \subset W \cap \mathcal{P}$ for any $s \in \mathbb{N}$. Since $h_j^{-1}(U) \subset U$ for each $j = 1, \dots, m$, we get $q_x^{(n)}(V) \subset U$ for each $n \in \mathbb{N}$. By the open set condition, for any $x' \in \Sigma_m \setminus \{x\}$ we have that $q_{x'}^{(n)}(V) \subset \overline{\mathbb{C}} \setminus U$ for each $n \in \mathbb{N}$. Hence we get that G is normal in V and this contradicts to that $y \in J(G)$.

□

Now we will give a proof of Theorem 3.2.

Proof. of **Theorem 3.2.** Suppose $J(G) \neq \overline{U}$. Then by Proposition 4.3 in [S4], we have $\mathrm{int}J(G) = \emptyset$. For any $y' \in J(G)$ and $r > 0$, we set

$$h(y', r) = \sup\{s \mid \exists y'' \in \overline{\mathbb{C}},\ B(y'', s) \subset F(G) \cap B(y', r) \cap U\}$$

and $h(r) = \inf\{h(y', r) \mid y' \in J(G)\}$. Then since $\mathrm{int}J(G) = \emptyset$, we have $h(r) > 0$ for any $r > 0$.

Let $\delta_0 > 0$ be a small number. Let B be the δ_0-neighborhood of $UH(G) \cap J(G)$ in $\overline{\mathbb{C}}$. By Lemma 4.3 and Lemma 4.11, we have that there exists a positive number δ_1 and a number $N \in \mathbb{N}$ such that for any $y' \in J(G) \setminus B$, $0 < \delta \leq \delta_1$ and any component V of $g^{-1}(B(y', 2\delta))$, V is simply connected and $\deg(g : V \to B(y', 2\delta)) \leq N$. By Lemma 4.13 and Theorem 2.1, we have

$$\tilde{J}(f) = \bigcup_{x \in \Sigma_m} J_x. \tag{5}$$

Let $y \in J(G)$ be a point. Since $\pi_{\overline{\mathbb{C}}} \tilde{J}(f) = J(G)$ (Proposition 3.2 in [S5]), by (5) we have that there exists a point $x \in \Sigma_m$ such that $y \in J_x$.

Let $\delta_2 = \min\{\delta_0, \delta_1\}$. Let r be a positive number. We set $B_n = q_x^{(n)}(B(y,r)$ and $y_n = q_x^{(n)}(y)$ for each $n \in \mathbb{N}$. Since $y \in J_x$, we have that there exists the smallest positive integer n_0 such that diam $B_{n_0+1} > \delta_2$. Then there exists a constant l_0 such that $l_0\delta_2 <$diam B_{n_0}.

Case 1. $y_{n_0} \in J(G) \setminus B$.

By Corollary 2.3 in [Y], there exists a constant K depending only on N and a ball $B(y_{n_0}, r_0) \subset B_{n_0}$ with $r_0 \geq$diam $B_{n_0}/K \geq \frac{l_0\delta_2}{K}$ such that the component of $(q_x^{(n_0)})^{-1}(B(y_{n_0}, r_0))$ containing y is a subset of $B(y,r)$. There exists a ball $B(y', \frac{2}{3}h(r_0))$ included in $B(y_{n_0}, r_0) \cap F(G) \cap U$.

Let D_0 be a component of $(q_x^{(n_0)})^{-1}(B(y', \frac{1}{2}h(r_0)))$ contained in $B(y,r)$. By the open set condition, we have $g^{-1}(U \cap F(G)) \subset U \cap F(G)$ for each $g \in G$. Hence we have $D_0 \subset F(G) \cap U$. Let $y'' \in D_0 \cap (q_x^{(n_0)})^{-1}(y')$ be a point. Then by Corollary 1.8 and 1.9 in [S4], Dist $(\partial D_0, y'') \leq M$ for some M depending only on N and diam $D_0 \asymp r$. Hence there exists a constant $0 < k < 1$ which does not depend on y and r such that $B(y'', kr) \subset D_0 \subset F(G) \cap B(y,r)$.

Case 2. $y_{n_0} \in B$.

By Lemma 4.12 and that $UH(G) \cap J(G) \subset U$, taking δ_0 small enough and using the method in pp286-287 in [Y] we can show that there exists a ball $B(y'', k'r)$ in $B(y,r) \cap F(G)$ where k' is a constant with $0 < k' < 1$ which does not depend on y and r.

□

Now we will show Proposition 3.4.

Proof. of **Proposition 3.4.** By the open set condition, we have $J(G) \subset \overline{U}$. We will show the following.

Claim 1: There exists an open set V' included in $U \cap F(G)$ such that $h^{-1}(V') \cap V' = \emptyset$ for each $h \in G$.

Before showing this claim, we remark that we can easily show the following claim.

Claim 2: If there exists a point $z \in U \cap F(G)$ such that $z \in \overline{\mathbb{C}} \setminus \overline{G(z)}$, then the claim 1 holds with an small open neighborhood V' of z.

To show the claim 1, by the open set condition we have

$$\bigcup_{j=1}^{m} h_j^{-1}(U \cap F(G)) \subset U \cap F(G). \tag{6}$$

Suppose the equality does not hold in (6). Then there exists a point $z \in U \cap F(G)$ such that $h_j(z) \in \overline{\mathbb{C}} \setminus U$ for each $j = 1, \ldots, m$. Hence by the open set condition, we get that $z \in \overline{\mathbb{C}} \setminus \overline{G(z)}$. By the claim 2, the claim 1 holds.

Hence we may assume that

$$\bigcup_{j=1}^{m} h_j^{-1}(U \cap F(G)) = U \cap F(G). \tag{7}$$

Let $\alpha : U \cap F(G) \to U \cap F(G)$ be the map defined as: $\alpha(z) = h_j(z)$ if $z \in h_j^{-1}(U \cap F(G))$. This is well defined by (7) and the open set condition.

Let $z \in U \cap F(G)$ be a point. If $z \in \overline{\mathbb{C}} \setminus \overline{G(z)}$, then by the claim 2 we have the claim 1. Hence we may assume $z \in \overline{G(z)}$ i.e.

$$z \in \overline{\bigcup_{n=0}^{\infty} \{\alpha^n(z)\}}. \tag{8}$$

Let W be the connected component of $U \cap F(G)$ containing z. By (8) there exists the smallest positive integer n with $\alpha^n(W) \subset W$. By (8) and the open set condition, we have one of the following cases 1 and 2.

Case 1: W is included in an attracting basin of an element $g \in G$, z is the attracting fixed point in the basin and $g|_W = \alpha^n|_W$.

Case 2: W is included in a Siegel disk or a Herman ring of an element $g \in G$ of degree at least 2 and $g|_W = \alpha^n|_W$.

If we have the case 1, then there exists an open set V' included in W with $\alpha^{-l}(V') \cap V' = \emptyset$ for each $l \in \mathbb{N}$ i.e. $h^{-1}(V') \cap V' = \emptyset$ for each $h \in G$.

If we have the case 2, then taking V' in a connected component A of $\alpha^{-n}(W)$ with $A \cap W = \emptyset$, we have $\alpha^{-l}(V') \cap V' = \emptyset$ for each $l \in \mathbb{N}$ i.e. $h^{-1}(V') \cap V' = \emptyset$ for each $h \in G$.

Hence we have shown the claim 1. Let V' be an open set included in $U \cap F(G)$ such that $h^{-1}(V') \cap V' = \emptyset$ for each $h \in G$. Then by the open set condition we have $g^{-1}(V') \cap h^{-1}(V') = \emptyset$, if $g, h \in G$ and $g \neq h$. Further the post critical set of G

$$P(G) := \overline{\bigcup_{g \in G} \{\text{critical values of } g\}}$$

does not accumulate in V'. Let V be an open disk included in $V' \setminus P(G)$. Then we have that

$$\int_V \sum_{h \in G} \sum_{\alpha} \|\alpha'(z)\|^2 \, dm(z) < \infty,$$

where α runs over all well-defined inverse branches of h on V. Hence for almost every $x \in V$ with respect to the Lebesgue measure, we have $S(2, x) < \infty$. □

Now we will show Theorem 3.5. we need some lemmas.

Lemma 4.14. *Let G be a rational semigroup. Assume that $\infty \in F(G)$ and for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. Let A be a subset of $J(G)$. Suppose that there exist positive constants a_1 , a_2 and c with $0 < c < 1$ such that for each $x \in A$, there exist two sequences (r_n) and (R_n) of positive real numbers and a sequence (g_n) of elements of G satisfying all of the following conditions:*

1. *$r_n \to 0$ and for each n, $0 < \frac{r_n}{R_n} < c$ and $g_n(x) \in J(G)$.*
2. *for each n, $g_n(D(x, R_n)) \subset D(g_n(x), a_1)$.*
3. *for each n $g_n(D(x, r_n)) \supset D(g_n(x), a_2)$.*

Then

$$dim_H(A) \leq s(G).$$

Proof. We may assume that $\sharp(J(G)) \geq 3$. Let $\delta \geq s(G)$ be a number and μ a $\delta-$subconformal measure. By the method in the proof of Lemma 5.5 in [S4], we can show that there exists a constant $c' > 0$ not depending on $n \in \mathbb{N}$ and $x \in A$ such that

$$\frac{\mu(D(x, r_n))}{r_n^\delta} \geq c'.$$

From this and Theorem 7.2 in [Pe], we get $\dim_H A \leq \delta$. □

Proposition 4.15. *Let G be a rational semigroup. Assume that $F(G) \neq \emptyset$ and that for each $x \in E(G)$, there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$. Then we have*

$$dim_H(J_{con}(G)) \leq s(G).$$

Proof. We have only to show the following:

Claim: For fixed $N \in \mathbb{N}$ and $r > 0$, $\dim_H(J_{con}(G, N, r)) \leq s(G)$.

We will show this. We can assume $\infty \in F(G)$. Let $x \in J_{con}(G, N, r)$ be a point. Then there exists a sequence (g_n) in G such that for each $n \in \mathbb{N}$ we have $g_n \in J(G)$,

$$\deg(g : V_n(r) \to D(g_n(x), r) \leq N$$

and $V_n(r)$ is simply connected and diam $V_n(r) \to 0$ as $n \to \infty$, where $V_n(r)$ is tne element of $c(D(g_n(x), r),\ g_n)$ containing x. Let $\varphi_n : D(0,1) \to V_n(r)$ be the Riemann map such that $\varphi_n(0) = x$. By the Koebe distortion theorem we have for each n,

$$V_n(r) \supset D(x, \frac{1}{4}|\varphi_n'(0)|).$$

By Lemma 4.1 and the Koebe distortion theorem, there exists an $\epsilon > 0$ such that for each $n \in \mathbb{N}$,

$$V_n(\epsilon r) \subset D(x, \frac{1}{8}|\varphi_n'(0)|).$$

Since diam $V_n(r) \to 0$ as $n \to \infty$, we have $|\varphi_n'(0)| \to 0$ as $n \to \infty$. Applying Lemma 4.14, we obtain the claim. □

Now we will show the following theorem.

Theorem 4.16. *Let $G = \langle h_1, \dots , h_m \rangle$ be a finitely generated rational semigroup with $F(G) \neq \emptyset$. Let $f : Y \to Y$ be the fibered rational map associated with the generator system $\{h_1, \dots , h_m\}$, where $Y = \Sigma_m \times \overline{\mathbb{C}}$. Suppose that f satisfies the condition (C1) and that for each $x \in \Sigma_m$, the boundary of $\hat{J}_x(f) \cap UH(f)$ in Y_x does not separate points in Y_x. Then we have $J_{good}(G) \subset J_{con}(G)$ and*

$$\dim_H(J_{good}(G)) \leq s(G) \leq s_0(G).$$

Proof. We may assume $\sharp(J(G)) \geq 3$. First we will show the following:

Claim: If $E(G) \neq \emptyset$, then for each $x \in E(G)$ there exists an element $g \in G$ such that $g(x) = x$ and $|g'(x)| < 1$.

If there exists an element $h \in G$ with $\deg(h) \geq 2$, then this claim is trivial. Suppose that each element of G is of degree 1. By Lemma2.3 in [S5], we have $\sharp(E(G)) \leq 2$. Since f satisfies the condition (C1), for each i, h_i is loxodromic. Since $h_i(E(G)) = E(G)$ for each i, we must have that each $x \in E(G)$ is fixed by h_i for each i. Let $x \in E(G)$ be a point. Suppose $|h_i'(x)| > 1$ for each i. Then we get $J(G) = \{x\}$ and this is a contradiction since we are assuming that $\sharp(J(G)) \geq 3$. Hence $|h_i'(x)| < 1$ for some i. Hence the claim holds.

The statement of our theorem follows from the claim, the second statement in Proposition 4.7, Proposition 4.15 and Theorem 4.2 in [S2]. □

Now we will show Theorem 3.5.

Proof. of **Theorem 3.5.** This follows from Lemma 4.11, Lemma 4.13 and Theorem 4.16. □

References

[Bo] D.Boyd, *An invariant measure for finitely generated rational semigroups*, Complex Variables, **39**, (1999),No.3, 229-254.

[BBR] R.Brück, M.Büger and S.Reitz, *Random iterations of polynomials of the form $z^2 + c_n$: Connectedness of Julia sets*, Ergod.Th. and Dynam.Sys., **19**, (1999), No.5, 1221-1231.

[BP] A.F.Beardon and Ch.Pommerenke, *The Poincaré metric of plane domains,* J.London Math.Soc., (2)**18**(1978), 475-483.

[CJY] L.Carleson, P.W.Jones and J.-C.Yoccoz, *Julia and John,* Bol.Soc.Bras.Mat.**25**, **N.1** 1994, 1-30.

[D] M.Dabija, *Böttcher divisors.* Preprint.

[FS] J.E.Fornaess and N.Sibony, *Random iterations of rational functions,* Ergod.Th.and Dynam.Sys.**11**(1991), 681-708.

[GR] Z.Gong and F.Ren, A random dynamical system formed by infinitely many functions, Journal of Fudan University, **35**, 1996, 387-392.

[H1] S.-M.Heinemann, *Julia sets for holomorphic endomorphisms of* $\mathbb{C}^n$. , Ergod.Th. and Dyn.Sys. **16**(1996), 1275-1296.

[H2] S.-M.Heinemann, *Julia sets of skew products in* $\mathbb{C}^2$, Kyushu J.Math.**52**(1998), 299-329.

[HM1] A.Hinkkanen and G.J.Martin, *The Dynamics of Semigroups of Rational Functions I,* Proc.London Math.Soc. (3)**73**(1996), 358-384.

[HM2] A.Hinkkanen and G.J.Martin, *Julia Sets of Rational Semigroups* , Math.Z. **222**, 1996, no.2, 161-169.

[HM3] A.Hinkkanen and G.J.Martin, *Some Properties of Semigroups of Rational Functions,* XVIth Rolf Nevanlinna Colloquium(Joensuu,1995), 53-58, de Gruyter, Berlin, 1996.

[J1] M.Jonsson, *Dynamics of polynomial skew products on* $\mathbb{C}^2$, Math.Ann,**314**(1999), pp403-447.

[J2] M.Jonsson, *Ergodic properties of fibered rational maps* , Ark.Mat., 38 (2000), pp 281-317.

[KS] H.Kriete and H.Sumi, *Semihyperbolic transcendental semigroups,* J.Math.Kyoto.Univ., Vol.40, No.2, 2000, pp205-216.

[LV] O.Lehto and K.I.Virtanen, *Quasiconformal Mappings in the plane,* Springer-Verlag, 1973.

[Ma] R.Mañé, *On a Theorem of Fatou,* Bol.Soc.Bras.Mat., Vol.24, N. 1, 1-11. 1992.

[Pe] Y.Pesin, *Dimension theory in dynamical systems,* Chicago Lectures in Mathematics, 1997.

[Po] Ch.Pommerenke, *Uniformly perfect sets and the Poincaré metric,* Arch.Math., **32**(1979), 192-199.

[PR] F.Przytycki and S.Rohde, *Porosity of Collet-Eckmann Julia sets,* Fund.Math., **155**, (1998), 189-199.

[Se1] O.Sester, Thesis: *Etude Dynamique Des Polynômes Fibrés,* Universite De Paris-Sud, Centre d'Orsey, 1997.

[Se2] O.Sester, *Hyperbolicité des polynômes fibrés,* Bull.Sc.Math.France, **127**, (1999), No.3, 393-428.

[Se3] O.Sester, *Combinatorial configurations of fibered polynomials,* Ergod.Th.and Dynam.Sys., (2001), **21**, 915-955.

[St1] R.Stankewitz, *Completely invariant Julia sets of polynomial semigroups,* Proc.Amer.Math.Soc., **127**, (1999), No.10, 2889-2898.

[St2] R.Stankewitz, *Completely invariant sets of normality for rational semigroups*, Complex Variables, Vol 40.(2000), 199-210.

[St3] R.Stankewitz, *Uniformly perfect sets, rational semigroups, Kleinian groups and IFS's* , Proc.Amer.Math.Soc. **128**, (2000), No.9, 2569-2575.

[Su] T.Sugawa, *Various Domain Constants Related to Uniform Perfectness,* Complex Variables, Vol.36, pp-311-345, 1998.

[S1] H.Sumi, *On Dynamics of Hyperbolic Rational Semigroups,* Journal of Mathematics of Kyoto University, Vol.37, No.4, 1997, 717-733.

[S2] H.Sumi, *On Hausdorff Dimension of Julia Sets of Hyperbolic Rational Semigroups,* Kodai.Math.J. Vol.21, No.1, pp10-28, 1998.

[S3] H.Sumi, *Semi-hyperbolic fibered rational maps and rational semigroups,* preprint.

[S4] H.Sumi, *Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products*, Ergod. Th. and Dynam. Sys.(2001), **21**, 563-603.

[S5] H.Sumi, *Skew product maps related to finitely generated rational semigroups,* Nonlinearity, **13**, (2000), 995-1019.

[S6] H.Sumi, *A correction to the proof of a lemma in 'Dynamics of sub-hyperbolic and semi-hyperbolic rational semigroups and skew products'*, Ergod.Th.and Dynam.Sys., (2001), **21**, 1275-1276.

[S7] H.Sumi, *Topics in dynamics of rational semigroups and fibered rational maps*, in 'Studies on complex dynamics and related topics', RIMS Kokyuuroku 1220, (2001), pp78-123.

[Y] Y.Yin, *Geometry and dimension of Julia sets,* in 'The Mandelbrot Set, Theme and Variations', edited by Tan Lei, pp281-289, London Math.Soc. LNS 274, 2000.