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Sufficient conditions for Teichmüller modular groups to be of the second kind

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1 Introduction

We consider the action of the reduced Teichmüller modular group $\mathrm{Mod}^{\#}(R)$ for a hyperbolic Riemann surface R, which is a group of automorphisms of the reduced Teichmüller space $T^{\#}(R)$. If R is of analytically finite type, the reduced Teichmüller modular group is nothing but the ordinary Teichmüller modular group $\mathrm{Mod}(R)$, and it is well known that $\mathrm{Mod}(R)$ acts properly discontinuously on $T(R)$. However, if R is of infinite type, $\mathrm{Mod}^{\#}(R)$ does not act properly discontinuously on $T^{\#}(R)$, in general. On the basis of this fact, in [1], we have introduced new notions, the limit set and the region of discontinuity for a Teichmüller modular group as an analogy to the theory of the Kleinian groups acting on the Riemann sphere.

Definition 1 We say that a point p in $T^{\#}(R)$ is a *limit point* for a subgroup G of $\mathrm{Mod}^{\#}(R)$ if there exist a point $q \in T^{\#}(R)$ and a sequence $\{\chi_n\}$ of distinct elements of G such that $\lim_{n\to\infty} d_T(\chi_n(q), p) = 0$. The set of the limit points is called the *limit set* of G, and denoted by $\Lambda(G)$. The complement $T^{\#}(R) - \Lambda(G)$ of the limit set is denoted by $\Omega(G)$, and called the *region of discontinuity* of G.

Similarly, for a subgroup G of the ordinary modular group $\mathrm{Mod}(R)$, we can define $\Lambda(G)$ and $\Omega(G)$ in $T(R)$. For a Riemann surface R of analytically finite type, we have $\Lambda(\mathrm{Mod}(R)) = \Lambda(\mathrm{Mod}^{\#}(R)) = \emptyset$. On the other hand, for a Riemann surface R whose Fuchsian model is of the second kind, we always have $\Omega(\mathrm{Mod}(R)) = \emptyset$. This is the reason why we consider the reduced modular group $\mathrm{Mod}^{\#}(R)$, not the ordinary modular group $\mathrm{Mod}(R)$, for a Riemann surface R of infinite type.

Definition 2 *For a subgroup G of* $\mathrm{Mod}^{\#}(R)$*, we say that G is of the first kind if $\Omega(G) = \emptyset$, and otherwise of the second kind.*

In this note, we focus our attention on normal covering surfaces of Riemann surfaces, and consider sufficient conditions for Teichmüller modular groups to be of second kind.

2 Sufficient conditions

Throughout this note, we assume that a Riemann surface R has the non-abelian fundamental group. In [1] and [2], we have shown sufficient conditions of Riemann surfaces R for $\mathrm{Mod}^{\#}(R)$ to be of the first kind or of the second kind, which are stated below.

Definition 3 For a given $M > 0$, we say that a point p of R belongs to a subset R_M of R if there exists a non-trivial simple closed curve c_p containing p such that the hyperbolic length of c_p is less than M.

Definition 4 We say that R satisfies the *lower bound condition* if there exists an $\epsilon > 0$ such that R_ϵ consists only of cusp neighborhoods. Further we say that R satisfies the *upper bound condition* if there exist a constant $M > 0$ and a connected component R_M^* of R_M such that a homeomorphism of $\pi_1(R_M^*)$ to $\pi_1(R)$ that is induced by the inclusion map of R_M^* into R is surjective.

Theorem 1 ([1]) *If R does not satisfy the lower bound condition, then* $\mathrm{Mod}^{\#}(R)$ *is of the first kind.*

Theorem 2 ([1]) *If R satisfies the lower and upper bound conditions, then* $\mathrm{Mod}^{\#}(R)$ *is of the second kind.*

In connection with these results, we have stated the following conjecture in [1].

Conjecture If R satisfies the lower bound condition, then $\mathrm{Mod}^{\#}(R)$ is of the second kind. That is, considering Theorem 1, we conjecture that $\mathrm{Mod}^{\#}(R)$ is of the second kind if and only if R satisfies the lower bound condition.

In [2], we have proved a partial solution of this conjecture, giving a weaker condition than the upper bound condition.

Theorem 3 ([2]) *Let R be a Riemann surface that satisfies the following two conditions:*

1. *R satisfies the lower bound condition.*
2. *There exists a constant $M > 0$ such that, for any connected component V of the complement of R_M, V is either simply or doubly connected and $R - \overline{V}$ consists of finitely many connected components.*

Then $\mathrm{Mod}^{\#}(R)$ *is of the second kind.*

Remark 1 The upper and lower bound conditions and the conditions in Theorem 3 are quasiconformally invariant. Then these conditions are regarded as conditions for the Teichmüller space.

3 Normal covering surfaces

Throughout this section, let $\tilde{R}$ be a normal covering surface of a Riemann surface R which is not a universal cover. First, we have given a sufficient condition for $\mathrm{Mod}^{\#}(\tilde{R})$ to be of the second kind.

Proposition 1 ([1]) *If R is of analytically finite type, then $\tilde{R}$ satisfies the upper and lower bound conditions. Thus, by Theorem 2,* $\mathrm{Mod}^{\#}(\tilde{R})$ *is of the second kind.*

A simple example of this proposition is as follows.

Example 1 *Let*

$$\tilde{R} = \mathbf{C} - \{n \mid n \in \mathbf{Z}\}.$$

Then $\mathrm{Mod}^{\#}(\tilde{R})$ *is of the second kind. Indeed, $\tilde{R}$ is a normal covering surface of $\mathbf{C} - \{0, 1\}$, which is an analytically finite Riemann surface.*

In connection with Proposition 1, we have the following.

Proposition 2 ([3]) *If $\tilde{R}$ satisfies the lower and upper bound conditions, then R also satisfies these conditions.*

Remark 2 In [3], we have considered a weakened upper bound condition. The same proof can be applied also for Riemann surfaces that satisfy the (generalized) upper bound condition in Definition 4.

From Theorem 2 and Proposition 2, we have a condition of a normal covering surface $\tilde{R}$ for $\mathrm{Mod}^{\#}(R)$ to be of the second kind.

Proposition 3 *If $\tilde{R}$ satisfies the lower and upper bound conditions, then* $\mathrm{Mod}^{\#}(\tilde{R})$ *and* $\mathrm{Mod}^{\#}(R)$ *are of the second kind.*

If $\tilde{R}$ does not satisfy the upper bound condition, then Proposition 3 is not necessarily true.

Example 2 Let

$$\tilde{R} = \mathbf{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbf{Z}} \left\{ \frac{m}{n} \pm n^2\sqrt{-1} \right\},$$

and $R = \tilde{R}/\langle f \rangle$, where $f(z) = z + 1$. Although $\tilde{R}$ does not satisfy the upper bound condition, $\tilde{R}$ satisfies the conditions in Theorem 3. Then $\mathrm{Mod}^{\#}(\tilde{R})$ is of the second kind. On the other hand, R satisfies the upper bound condition, but does not satisfy the lower bound condition. Then, from Theorem 1, $\mathrm{Mod}^{\#}(R)$ is of the first kind.

That $\tilde{R}$ satisfies the upper bound condition is not a necessary condition for both $\mathrm{Mod}^{\#}(\tilde{R})$ and $\mathrm{Mod}^{\#}(R)$ to be of the second kind.

Example 3 Let

$$\tilde{R} = \mathbf{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbf{Z}} \left\{ \frac{m}{n} + (2n+1)\sqrt{-1} \right\},$$

and $R = \tilde{R}/\langle f \rangle$, where $f(z) = z + 1$. Although $\tilde{R}$ does not satisfy the upper bound condition, $\tilde{R}$ satisfies the conditions in Theorem 3. Then $\mathrm{Mod}^{\#}(\tilde{R})$ is of the second kind. Further, R satisfies the lower and upper bound conditions. Then, by Theorem 2, $\mathrm{Mod}^{\#}(R)$ is also of the second kind.

In the last of this note, we give a following problem.

Problem If $\mathrm{Mod}^{\#}(\tilde{R})$ is of the first kind, then so is $\mathrm{Mod}^{\#}(R)$.

It is clear that, if $\tilde{R}$ does not satisfy the lower bound condition, then neither does R. Hence, from Theorem 1, if $\tilde{R}$ does not satisfy the lower bound condition, then $\mathrm{Mod}^{\#}(\tilde{R})$ and $\mathrm{Mod}^{\#}(R)$ are of the first kind. Further, if the conjecture, which is stated in the previous section, is true, then the problem is solved affirmatively.

References

[1] E. Fujikawa, Limit sets and regions of discontinuity of Teichmüller modular groups, preprint.

[2] E. Fujikawa, Teichmüller modular groups with non-empty regions of discontinuity, preprint.

[3] E. Fujikawa, The order of conformal automorphisms of Riemann surfaces of infinite type, preprint.