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ON THE CONFORMAL INVARIANT AND ITS VANISHING OF A CODIM 3- QUATERNIONIC CARNOT-CARATHÉODORY STRUCTURE ON (4n + 3)-DIMENSIONAL MANIFOLDS

((4N+3) 次元多様体上の余次元 3 の四元数カルノー・カラテオドリー構造の共形不変量とその消滅)

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INTRODUCTION

H. Weyl has introduced the notion of conformal structure on Riemannian metrics on manifolds from the viewpoint of the Gauge theory. He constructed so-called Weyl conformal curvature tensor, which is a conformal invariant of Riemannian metrics and caputured the conformal flatness on manifolds apart from the metrics for the first time. When the Weyl curvature tensor vanishes, the Riemannian manifold is said to admit a conformally flat structure. The purpose of this note is to intoduce a geometric structure on a $(4n+3)$-manifold (called **quaternionic Carnot-Carathéodory structure**) and study a conformal invariance whose vanishing gives a uniformization.
The detail will appear elsewhere. First of all we must explain why dimension $(4n+3)$ comes out from the viewpoint of conformal structure. When the Weyl conformal curvature tensor of an n-dimensional Riemannian manifold M vanishes, M is said to be a conformally flat manifold, in which M is locally developed into the standard sphere S^n. The model space with standard conformally flat structure is the sphere S^n whose structure-preserving transformations consists of the group of conformal tranformations $\mathrm{Conf}(S^n)$. Let (M,ω) be a $(2n+1)$-dimensional contact manifold. By definition, the 1-form ω satisfies $\omega\wedge d\omega^n\neq 0$ so that it determines a contact subbundle ($2n$-dimensional oriented subbundle of TM) Null $\omega=\{X\in TM|\omega(X)=0\}$. If Null ω possesses a complex structure J, (Null ω, J) is called a CR-structure on M. (In addition, (ω,J) is said to be a **pseudo-Hermitian** structure.)

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There is no canonical way to choose a contact form ω which represents a CR-structure on M (that is, up to multiple of *positive functions*). The Levi form will be required to be positive definite, and hence $\omega = \lambda \cdot \omega'$ ($\lambda : M \to \mathbb{R}^+$) if and only if both ω' and ω provide the same CR-structure (keeping the complex struture J fixed). Then Chern and Moser have defined the fourth order tensor S from (ω, J) which is invariant under the CR-structure on M. From the viewpoint of Weyl conformal structure of Riemannian metrics, the conformal invariance of contact forms is stated as $\omega = \lambda \cdot \omega'$ if and only if $S(\omega, J) = S(\omega', J)$. (Incidentally, Bochner has defined the (Bochner) curvature tensor on Kähler manifolds as an analogue of Weyl conformal curvatutre tensor. The tensor description of Chern-Moser curvature tensor S coincides with that of Bochner curvature tensor.)

When the Chern-Moser curvature tensor of $(2n+1)$-dimensional CR manifold M vanishes, M is called **spherical** CR-manifold, and it is developed locally into the model geometry $(\mathrm{Aut}_{CR}(S^{2n+1}), S^{2n+1})$. Here $\mathrm{Aut}_{CR}(S^{2n+1})$ is the group of **Cauchy-Riemann** transformations of S^{2n+1}.

When the curvature form vanishes respectively, the geometry appears as *Conformally flat*) structure (resp. *Spherical* CR structure). Thus the Klein's classical geometry implies that each geometry is viewed as the boundary geometry of real hyperbolic geometry and complex hyperbolic geometry. In fact, the real (resp. complex) hyperbolic space $\mathbb{H}_R^{n+1}$ (resp. $\mathbb{H}_C^{n+1}$) has a compactification on which the isometry group $\mathrm{PO}(n+1, 1)$ (resp. $(\mathrm{PU}(n+1, 1))$ extends to a smooth action $\Longrightarrow$ $(\mathrm{Conf}(S^n), S^n)$, $(\mathrm{Aut}_{CR}(S^{2n+1}), S^{2n+1})$. In this case, the action on the boundary is real analytic, well known as conformal, CR-transformation. (Note that the group $\mathrm{Conf}(S^n)$ is isomorphic to $\mathrm{PO}(n+1, 1) = \mathrm{Iso}(\mathbb{H}_R^{n+1})$ as a Lie group, while its action is viewed as conformal action, similarly for $\mathrm{Aut}_{CR}(S^{2n+1})$.) At this stage, as a compactification of rank 1-symmetric space of semisimple noncompact type, there is quaternionic hyperbolic space with isometry group $(\mathrm{PSp}(n+1, 1), \mathbb{H}_{\mathbf{H}}^{n+1})$. The action of the isometry group naturally extends to a smooth action on the boundary sphere S^{4n+3}. (In fact, it is characterized as a restriction of a quaternionic projectice transformation to the $(4n+3)$-sphere.) As $\mathrm{PSp}(n+1, 1)$ acts transitively on S^{4n+3}, we write its action $\mathrm{Aut}_{QC}(S^{4n+3})$, and so obtain a geometry $(\mathrm{Aut}_{QC}(S^{4n+3}), S^{4n+3})$. A manifold equipped locally with this geometry $(\mathrm{Aut}_{QC}(S^{4n+3}), S^{4n+3})$ is said to be a *Spherical* Q C-C manifold. (It used to be called pseudo quaternionic flat manifold in [10].)
In view of these, we study a geometric structure on a $(4n+3)$-manifold M and define a conformal equivalence of the geometric structure and

find a curvature tensor T which gives a conformal invariance for which the vanishing of T makes M uniformizable with respect to the spherical Q C-C geometry $(\mathrm{Aut}_{QC}(S^{4n+3}), S^{4n+3})$.

As a consequence, combined with the fact that the vanishing of Weyl conformal curvature tensor, Chern-Moser curvature tensor makes a conformal manifold (resp. CR-manifold) M a conformally flat manifold (resp. a spherical CR-manifold), this characterize the boundary behavoir of isometry groups on the real, complex, quaternionic hyperbolic geometry such as conformal, CR, quaternionic Carnot-Carathéodory transformation, and hence establish the Conformal geometry (Parabolic geometry) on the boundary of rank 1-symmetric space of noncompact semisimple type.

Contents

1. Preliminaries

Definition 1.1. *A quaternionic Carnot-Carathéodory structure on a* $(4n+3)$*-manifold* M *is a subbundle* B *given by an exact sequence:*

$$1 \to B \to TM \xrightarrow{\theta} L \to 1$$

satisfying the following conditions.

1. *There exists a open cover* $\{U_\alpha\}_{\alpha\in\Lambda}$ *of* M *such that if* $U_\alpha \cap U_\beta \neq \emptyset$, *then there is a smooth map* $\lambda_{\alpha\beta} = u_{\alpha\beta}\cdot a_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbf{H}^* = \mathbb{R}^+ \times S^3$ $(u_{\alpha\beta} \in \mathbb{R}^+, a_{\alpha\beta} \in S^3)$. $A^{(\alpha\beta)} \in \mathrm{SO}(3)$ *is a matrix given by* $\mathrm{Ad}_{\bar{a}_{\alpha\beta}}$ $(\mathrm{Ad}_{\bar{a}}(z) = \bar{a}za)$.

2. *L is a 3-dimensional vector bundle whose fiber is isomorphic to the Lie algebra* $\mathfrak{so}(3) = \mathrm{Im}\,\mathbf{H} = \mathbb{R}\boldsymbol{i} + \mathbb{R}\boldsymbol{j} + \mathbb{R}\boldsymbol{k}$*, where the gluing condition between* $L|U_\alpha$ *and* $L|U_\beta$ *is defined as:* $\begin{pmatrix} \xi_1^{(\alpha)} \\ \xi_2^{(\alpha)} \\ \xi_3^{(\alpha)} \end{pmatrix} = u_{\alpha\beta}^2 A^{(\alpha\beta)} \begin{pmatrix} \xi_1^{(\beta)} \\ \xi_2^{(\beta)} \\ \xi_3^{(\beta)} \end{pmatrix}$.
3. *B supports a quaternionic structure* $\{I^{(\alpha)}, J^{(\alpha)}, K^{(\alpha)}\}_{\alpha\in\Lambda}$*; there exists a triple of almost complex structures* $\{I^{(\alpha)}, J^{(\alpha)}, K^{(\alpha)}\}$ *defined on each* $B|U_\alpha$ *such that on* $B|U_\alpha \cap U_\beta$*:*

$$\begin{pmatrix} I^{(\beta)} \\ J^{(\beta)} \\ K^{(\beta)} \end{pmatrix} = {}^tA^{(\alpha\beta)} \cdot \begin{pmatrix} I^{(\alpha)} \\ J^{(\alpha)} \\ K^{(\alpha)} \end{pmatrix}. \tag{1.1}$$

4. *When the projection θ is viewed as L-valued 1-form,*

$$\theta \wedge \theta \wedge \theta \wedge \overbrace{d\theta \wedge \cdots \wedge d\theta}^{2n-times} \neq 0 \ \textit{in}\ \mathbb{R} \subset \mathbf{H} = \Gamma(M, \Omega^{4n+3}(L)).$$

Moreover, following the idea of Chern-Moser, Webster to pseudo-Hermitian structure, we require the following: Locally θ is described as

$$\theta|U_\alpha = \omega_1^{(\alpha)} \cdot \xi_1^{(\alpha)} + \omega_2^{(\alpha)} \cdot \xi_2^{(\alpha)} + \omega_3^{(\alpha)} \cdot \xi_3^{(\alpha)}. \tag{1.2}$$

We obtain an Im $\mathbf{H}$-valued 1-form: $\omega^{(\alpha)} = \omega_1^{(\alpha)}\boldsymbol{i} + \omega_2^{(\alpha)}\boldsymbol{j} + \omega_3^{(\alpha)}\boldsymbol{k}$.
(for brevity, omit α in $\omega^{(\alpha)}$, $\omega_1^{(\alpha)}$, $I^{(\alpha)}$.)
Suppose that B supports a positive definite bilinera form and choose the orthonormal basis $\{e_i\}_{i=1,\cdots,4n}$ of B. Let $\theta^i(e_j) = \delta_{ij}$ and choose locally 1-forms $\{\theta^i\}_{i=1,\cdots,4n}$ such as the frame $\{\omega_1, \omega_2, \omega_3, \theta^1, \cdots, \theta^{4n}\}$ becomes a coframe field of M. such that

As usual, a triple of almost complex strutures $\{I, J, K\}$ is represented by the matrix: $Ie_i = I_{ij}e_j,\ Je_i = J_{ij}e_j,\ Ke_i = K_{ij}e_j$.
We require the differential of ω satisfies the following equation:

$$d\omega + \omega \wedge \omega \equiv (I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k})\theta^i \wedge \theta^j \mod \omega_1, \omega_2, \omega_3. \tag{1.3}$$

In order to find a curvature, we assume that there are 1-forms φ_j^i, τ_a^i $(i,j=1,\cdots,4n; a=1,2,3)$ such that:

$$d\theta^i = \theta^j \wedge \varphi_j^i + \sum_a \omega_a \wedge \tau_a^i. \tag{1.4}$$

Using φ_j^i, the covariant derivative $\nabla : \Gamma(B) \longrightarrow \Gamma(B \otimes T^*M)$ is defined as follows.

$$\nabla e_i = \sum_{j=1}^{4n} \varphi_i^j e_j. \tag{1.5}$$

Corresponding to the quaternionic Kähler structure, we require the following:

$$\nabla d\omega_1 \equiv 0, \quad \nabla d\omega_2 \equiv 0, \quad \nabla d\omega_3 \equiv 0 \ \text{ on } B \ \bmod \omega_1, \omega_2, \omega_3. \tag{1.6}$$

Moreover, to be completely integrable, the torsion forms τ_a^i $(a=1,2,3)$ will satisfy:

$$\begin{aligned} \tau_1^i &\equiv 0 \bmod \theta^k, \ \omega_1 \ (k=1,\cdots,4n), \\ \tau_2^i &\equiv 0 \bmod \theta^k, \ \omega_2 \ (k=1,\cdots,4n), \\ \tau_3^i &\equiv 0 \bmod \theta^k, \ \omega_3 \ (k=1,\cdots,4n). \end{aligned} \tag{1.7}$$

Remark 1.2. *Recall the definition of* ∇ :
$\nabla_Z d\omega_1(e_i, e_j) = \nabla_Z(d\omega_1(e_i, e_j)) - d\omega_1(\nabla_Z e_i, e_j) - d\omega_1(e_i, \nabla_Z e_j)$. *Since* $\omega_1(e_i, e_j) = I_{ij}$, *we get*

$$\nabla_Z d\omega_1(e_i, e_j) = (dI_{ij} - \varphi_i^\sigma I_{\sigma j} - I_{i\sigma}\varphi_j^\sigma).(Z) \quad (Z \in TM).$$

Hence, (1.6) *is equivalent to the below:*

$$\begin{aligned} dI_{ij} - \varphi_i^\sigma I_{\sigma j} - I_{i\sigma}\varphi_j^\sigma | B &= 0, \\ dJ_{ij} - \varphi_i^\sigma J_{\sigma j} - J_{i\sigma}\varphi_j^\sigma | B &= 0, \\ dK_{ij} - \varphi_i^\sigma K_{\sigma j} - K_{i\sigma}\varphi_j^\sigma | B &= 0. \end{aligned} \tag{1.8}$$

2. G-STRUCTURE

Let G be the subgroup of $\mathrm{GL}(4n+3, \mathbb{R})$ consisting of matrices;

$$\left(\begin{array}{c|ccc} & v_1^1 & \cdots & v_1^{4n} \\ u^2 \cdot A & v_2^1 & \cdots & v_2^{4n} \\ & v_3^1 & \cdots & v_3^{4n} \\ \hline 0 & & u \cdot U & \end{array}\right), \tag{2.1}$$

where $u \cdot U = U' \cdot (u \cdot a) = U' \cdot \lambda \in \mathrm{Sp}(n) \cdot \mathbf{H}^*$, $u^2 \cdot A = \bar{\lambda} \cdot \bar{\lambda}$, $(v_a^1, \cdots, v_a^{4n}) \in \mathbb{R}^{4n}$.

Recall a G-structure on M is a reduction of the structure group of TM to G. Let $G \to P \to M$ be the principal bundle of the G-structure consisting of coframe fields

$$\{\omega_1, \omega_2, \omega_3, \theta^1, \cdots, \theta^{4n}\}.$$

If $\mathrm{Aut}(M)$ is the group of G-automorphisms, then the Lie algebra $\mathfrak{g}$ of G is
$\mathbb{H}^n \times \mathbb{H}^n \times \mathbb{H}^n \times \mathfrak{sp}(n) + \mathfrak{sp}(1) + \mathbb{R}$, and $\mathbb{H}^n$ is of **infinite type**, $\mathfrak{sp}(n) + \mathfrak{sp}(1) + \mathbb{R}$ is of order 2. Thus, $\mathfrak{g}$ has no element of Rank 1. Especially $\mathfrak{g}$ is **elliptic**. Fron the theory of G-structure, $\mathrm{Aut}(M)$ is a finite dimensional Lie group. We call it the group of quaternionic Carnot-Carathéodory transformations. If an element f belongs to $\mathrm{Aut}(M)$, using a coframe field $\{\omega_1, \omega_2, \omega_3, \theta^1, \cdots, \theta^{4n}\}$,

$$\begin{aligned} f^*(\omega_1, \omega_2, \omega_3) &= u^2(\omega_1, \omega_2, \omega_3)A \\ f^*\theta^i &= u\theta^k U_k^i + \sum_a \omega_a v_a^i \quad (\text{some } v_a^i \in \mathbb{R}) \end{aligned} \tag{2.2}$$

If we put the above form to be a $\mathrm{Im}\mathbf{H}$-valued 1-form $\omega = \omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k}$,

$$f^*\omega = \bar{\lambda} \cdot \omega \cdot \lambda \; (= u^2 \bar{a} \cdot \omega \cdot a).$$

The problem in question is to find *local invariants* under a *quaternionic Carnot-Carathéodory* transformation f.

Theorem 2.1. *Let ω be a* Im $\mathbf{H}$*-valued* 1*-form representing a quaternionic Carnot-Carathéodory structure on a* $(4n+3)$*-manifold M. There exists a fourth order curvature tensor $T = (T^i_{jk\ell})$ $(n \geq 1)$ such that if $\omega' = \bar{\lambda} \cdot \omega \cdot \lambda$ for any function $\lambda : M \to \mathbf{H}^*$, then it satisfies* **conformal invariant**: $T(\omega) = T(\omega')$.

Theorem 2.2. *Let M be a* $(4n+3)$*-dimensional quaternionic Carnot-Carathéodory manifold*
$(n \geq 1)$. *If the curvature tensor T vanishes everywhere on M, then M is uniformizable over S^{4n+3} with respect to* $\mathrm{PSp}(n+1, 1)$.

Recall that the complex contact manifold has the relation of the first Chern classes concerning holomorphic subbundles. We have a similar relation in this case. We introduce the notion of "Quaternionic" vector bundle, and obtained the following.

Theorem 2.3. *Let M be a* $(4n+3)$*-dimensional quaternionic Carnot-Carathéodory manifold*

$(n \geq 1)$. *There is a relation of the first Pontrjagin classes between* TM *and the subbundle* L :

$$2p_1(M) = (n+2)p_1(L).$$

Using this,

Corollary 2.4. *The necessary and sufficient condition for* M *to admit a global* $\mathrm{Im}\,\mathbf{H}(=\mathfrak{sp}(1))$*-valued* 1*-form* ω $(\omega|U_\alpha = \bar{\lambda}_\alpha \cdot \omega^{(\alpha)} \cdot \lambda_\alpha)$ *is* $2p_1(M) = 0$.

Remark 2.5.
(1) *Denote by* $\mathcal{R}(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1))$ *the space of all curvature tensors whose holonomy group is* $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ $(n > 1)$. $\mathcal{R}(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1))$ *is decomposed into*

$$\mathcal{R}_0(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)) \oplus \mathcal{R}_{\mathbf{HP}}(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)).$$

Here,

1. $\mathcal{R}_{\mathbf{HP}}(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)) = \mathbb{R}\cdot R_{\mathbf{HP}}$ ($R_{\mathbf{HP}}$ *is the quaternionic curvature tensor of the quaternionic projective space* $\mathbb{HP}^n$).
2. $\mathcal{R}_0(\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)) = \{R \mid R$ *is a curvature tensor with zero Ricci tensor*$\}$

According to this decomposition, a curvature tensor is described as

$$R = W_0 + c\cdot R_{\mathbf{HP}}.$$

In this case, the component W_0 *is the Weyl curvature tensor. The curvature tensor* T *of a* $(4n+3)$*-dimensional quaternionic Carnot-Carathéodory manifold* $(n > 1)$ *has the same formula as that of Weyl curvature tensor* W_0. $(c = 1)$.

(2) *The curvature tensor* T *of a* 7*-dimensional quaternionic Carnot-Carathéodory manifold*
$(n = 1)$ *coincides with the Weyl curvature tensor* $W \in \mathcal{R}_0(\mathrm{SO}(4))$.

(3) *There is the similar result to the case* $\dim M = 3$. *Since* $T = 0$ (B *is empty*) *in this case,* T *replaces the Weyl-Schouten curvature tensor* S-W. *If it vanishes, then* M *will be a* 3*-dimensional conformally flat manifold.*

3. Calculation and Equation

We start with the folloing.

$$(3.1) \qquad d\omega + \omega\wedge\omega = (I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k})\theta^i \wedge \theta^j.$$

This is equivalent to that

$$
(3.2) \qquad \begin{aligned} d\omega_1 + 2\omega_2 \wedge \omega_3 &= I_{ij}\theta^i \wedge \theta^j, \\ d\omega_2 + 2\omega_3 \wedge \omega_1 &= J_{ij}\theta^i \wedge \theta^j, \\ d\omega_3 + 2\omega_1 \wedge \omega_2 &= K_{ij}\theta^i \wedge \theta^j. \end{aligned}
$$

Differentiate the above equation $d\omega_1 + 2\omega_2 \wedge \omega_3 = I_{ij}\theta^i \wedge \theta^j$ and substitute (1.4), we have

$$
\begin{aligned}
&(dI_{ij} - \varphi_i^\sigma I_{\sigma j} - I_{i\sigma}\varphi_j^\sigma) \wedge \theta^i \wedge \theta^j + \omega_1 \wedge (I_{ij}\tau_1^i \wedge \theta^j + I_{ij}\theta^i \wedge \tau_1^j) + \\
&+ \omega_2 \wedge (I_{ij}\tau_2^i \wedge \theta^j + I_{ij}\theta^i \wedge \tau_2^j + 2K_{ij}\theta^i \wedge \theta^j) + \\
&+ \omega_3 \wedge (I_{ij}\tau_3^i \wedge \theta^j + I_{ij}\theta^i \wedge \tau_3^j - 2J_{ij}\theta^i \wedge \theta^j) = 0,
\end{aligned}
$$

which reduces to the following (simlarly for the rest fo others of (3.2)):

$$
(3.3) \qquad \begin{aligned}
&(dI_{ij} - \varphi_i^\sigma I_{\sigma j} - I_{i\sigma}\varphi_j^\sigma) \wedge \theta^i \wedge \theta^j + 2\omega_1 \wedge I_{ij}\theta^i \wedge \tau_1^j + \\
&+ 2\omega_2 \wedge (I_{ij}\theta^i \wedge \tau_2^j + K_{ij}\theta^i \wedge \theta^j) + \\
&+ 2\omega_3 \wedge (I_{ij}\theta^i \wedge \tau_3^j - J_{ij}\theta^i \wedge \theta^j) = 0.
\end{aligned}
$$

4. Three Reeb fields

Since it is easy to see that

$$
d\omega_a(X,Y) = \delta_{ij}\theta^i \cdot \theta^j(I_aX, Y) \quad (X, Y \in B),
$$

$d\omega_a|B \times B \to R$ is nondegenerate, and $d\omega_a|B$ is I_a-Hermitian ($a = 1,2,3,\ I_1 = I, I_2 = J, I_3 = K$):

Proposition 4.1.

$$
(4.1) \quad d\omega_a(I_aX, I_aY) = \delta_{ij}\theta^i \cdot \theta^j(I_aX, Y) = d\omega_a(X,Y) \quad (X, Y \in B).
$$

Using this,

Proposition 4.2. *There exist nonzero vector fields $\{\xi_1, \xi_2, \xi_3\}$ everywhere on M such that*

$$
(4.2) \qquad \begin{aligned} \omega_a(\xi_b) &= \delta_{ab}, \\ d\omega(\xi_a, X) &= 0 \ \forall \ X \in B. \end{aligned}
$$

Put $E = \{\xi_1, \xi_2, \xi_3\}$. Then each element of E satisfies that $[\xi_2, \xi_3] = 2\xi_1$, $[\xi_3, \xi_1] = 2\xi_2$, $[\xi_1, \xi_2] = 2\xi_3$.

Corollary 4.3. *E is completely integrable. Each leaf of E is locally isomorphic to the Lie group* SO(3).

5. Locally quaternionic Kähler structure

Let $\mathcal{E}$ be the local group of transformations generated by $E = \{\xi_1, \xi_2, \xi_3\}$. If we note that $\mathcal{E}$ acts properly and freely on a sufficiently small neighborhood U, then it induces a local principal fibration:

$$\mathcal{E} \to U \xrightarrow{\pi} U/\mathcal{E}. \tag{5.1}$$

Define a Riemannian metric on $U/\mathcal{E}$:

$$\hat{g} = \sum_{i=1}^{4n} \hat{\theta}^i \cdot \hat{\theta}^i. \tag{5.2}$$

And $\hat{\omega}^i_j$ denotes the Levi-Civita connection with respect to $\hat{g}$; for the orthonormal basis $\hat{e}_i = \pi_* e_i$ $(i = 1, \cdots, 4n)$, by definition $\hat{\nabla}\hat{e}_i = \hat{\omega}^j_i \hat{e}_j$. Choose a neighborhood $V_i \subset U/\mathcal{E}$ and let $s_i : V_i \to U$ be a section of the principal bundle $U \xrightarrow{\pi} U/\mathcal{E}$. For $\hat{x} \in V_i$ and $X_{s_i(\hat{x})} \in B_{s_i(\hat{x})}$, define automorphisms $\hat{I}_i, \hat{J}_i, \hat{K}_i$ on V_i

$$\begin{aligned}
(\hat{I}_i)_{\hat{x}}(\pi_*(X_{s_i(\hat{x})})) &= \pi_* I_{s_i(\hat{x})} X_{s_i(\hat{x})},\\
(\hat{J}_i)_{\hat{x}}(\pi_*(X_{s_i(\hat{x})})) &= \pi_* J_{s_i(\hat{x})} X_{s_i(\hat{x})},\\
(\hat{K}_i)_{\hat{x}}(\pi_*(X_{s_i(\hat{x})})) &= \pi_* K_{s_i(\hat{x})} X_{s_i(\hat{x})}
\end{aligned} \tag{5.3}$$

As $\pi_* : B_{s_i(\hat{x})} \to T_{\hat{x}}(U/\mathcal{E})$ is isomorphic, $\hat{I}_i, \hat{J}_i, \hat{K}_i$ are well-defined complex structures on V_i. Passing to all cover $\{V_i\}_{i\in\Lambda}$ in $U/\mathcal{E}$, if we do this process, we get a family $\{\hat{I}_i, \hat{J}_i, \hat{K}_i\}_{i\in\Lambda}$. Moreover,

Proposition 5.1. *The family $\{\hat{I}_i, \hat{J}_i, \hat{K}_i\}_{i\in\Lambda}$ is a quaternionic structure on $U/\mathcal{E}$.*

Then, it is shown that $\hat{\nabla}$ satisfies the quaternionic Kähler condition. $(\hat{I}_i = \hat{I}, \hat{J}_i = \hat{J}, \hat{K}_i = \hat{K})$:

$$\hat{\nabla}\begin{pmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{pmatrix} = 2\begin{pmatrix} 0 & s^*\omega_3 & -s^*\omega_2 \\ -s^*\omega_3 & 0 & s^*\omega_1 \\ s^*\omega_2 & -s^*\omega_1 & 0 \end{pmatrix}\begin{pmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{pmatrix}. \tag{5.4}$$

From this,

Proposition 5.2. $(U/\mathcal{E}, \hat{g}, \{\hat{I}_i, \hat{J}_i, \hat{K}_i\}_{i\in\Lambda})$ *is a quaternionic Kähler manifold*
($\dim U/\mathcal{E} > 4$).

Corollary 5.3. *The Ricci tensor satisfies: $\hat{R}_{j\ell} = 4(n+2)\delta_{j\ell}$. In particular, $(U/\mathcal{E}, \hat{g})$ is a Einstein manifold.* $(n \geq 1)$.

Remark 5.4. *If we define the fourth order tensor $R^i_{jk\ell}$ by the following equation*

$$d\omega^i_j - \omega^\sigma_j \wedge \omega^i_\sigma \equiv \frac{1}{2} R^i_{jk\ell}\theta^k \wedge \theta^\ell \quad mod\, \omega_1, \omega_2, \omega_3, \tag{5.5}$$

then the curvature tensor $\hat{R}^i_{jk\ell}$ of $U/\mathcal{E}$ satisfies

$$R^i_{jk\ell} = \pi^* \hat{R}^i_{jk\ell}.$$

6. Existence of conformal curvature T

We give a sketch of existence to our curvature tensor stated in Theorem 2.1. Let $d\omega + \omega \wedge \omega = (I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k})\theta^i \wedge \theta^j$ be as before. If $f \in \mathrm{Aut}(M)$, then $f^*\omega = \bar{\lambda} \cdot \omega \cdot \lambda$. Letting $\omega' = f^*\omega$, choose w^i_a $(a = 1, 2, 3)$ such as $U^i_k w^k_a = v^i_a$. Substitute (2.2) into the above equation;

$$d\omega' + \omega' \wedge \omega' = (I_{ij}\boldsymbol{i} + J_{ij}\boldsymbol{j} + K_{ij}\boldsymbol{k})(u^2 U^i_k U^j_\ell \theta^k \wedge \theta^\ell + \sum_a \omega_a \wedge 2u w^i_a U^i_k U^j_\ell \theta^\ell$$
$$+ \sum_{a<b} \omega_a \wedge \omega_b (2U^i_k U^j_\ell w^k_a w^\ell_b)).$$

Then we can check that the matrix U^i_j satisfies the following:

$$\begin{aligned} I_{ij}U^i_kU^j_\ell &= a_{11}I_{k\ell} + a_{21}J_{k\ell} + a_{31}K_{k\ell} = I'_{k\ell}. \\ J_{ij}U^i_kU^j_\ell &= a_{12}I_{k\ell} + a_{22}J_{k\ell} + a_{32}K_{k\ell} = J'_{k\ell}. \\ K_{ij}U^i_kU^j_\ell &= a_{13}I_{k\ell} + a_{23}J_{k\ell} + a_{33}K_{k\ell} = K'_{k\ell}. \end{aligned} \tag{6.1}$$

Using this, there is the following general formula under the change of the element of $\mathrm{Aut}(M)$:

(6.2)
$$d\omega' + \omega' \wedge \omega' = (I'_{ij}\boldsymbol{i} + J'_{ij}\boldsymbol{j} + K'_{ij}\boldsymbol{k})(u^2\theta^i \wedge \theta^j + \sum_a \omega_a \wedge 2u w^i_a \theta^j$$
$$+ \sum_{a<b} \omega_a \wedge \omega_b (2w^i_a w^j_b)).$$

We consider the equation of the connection form corresponding to (1.4).

$$d\theta^i = \theta^j \wedge \varphi'^i_j + \sum_a \omega_a \wedge \tau'^i_a, \tag{6.3}$$

and define 1-forms ν_a^i by the following equations:

$$\text{(6.4)} \qquad \begin{pmatrix} \nu_1^i \\ \nu_2^i \\ \nu_3^i \end{pmatrix} = u^{-2} \cdot A^{-1} \begin{pmatrix} \tau'^i_1 \\ \tau'^i_2 \\ \tau'^i_3 \end{pmatrix}.$$

Then, we can define the fourth order tensor up to the terms $\omega_1, \omega_2, \omega_3$:

(6.5)

$$T'^i_{jk\ell}\theta^k \wedge \theta^\ell \equiv d\varphi'^i_j - \varphi'^\sigma_j \wedge \varphi'^i_\sigma - \sum_a u^2 \cdot I'^a_{jk}\theta^k \wedge \nu_a^i + \frac{1}{3}\sum_a u^2 \cdot I'^a_{jk}\nu_a^k \wedge \theta^i.$$

In order to determine this tensor uniquely, we assume **tracefree** condition of $T' = (T'^i_{jk\ell})$.

$$\text{(6.6)} \qquad T'_{j\ell} = T'^i_{ji\ell} = 0.$$

Using this, a calculation shows that

(6.7)

$$\begin{aligned} T'^i_{jk\ell} =& R^i_{jk\ell} - \{(\delta_{j\ell}\delta^i_k - \delta_{jk}\delta^i_\ell) + [I_{j\ell}I_{ik} - I_{jk}I_{i\ell} + 2I_{ij}I_{k\ell} \\ &+ J_{j\ell}J_{ik} - J_{jk}J_{i\ell} + 2J_{ij}J_{k\ell} + K_{j\ell}K_{ik} - K_{jk}K_{i\ell} + 2K_{ij}K_{k\ell}]\}. \end{aligned}$$

Hence, the fourth order curvature tensor T coincides under the change $\omega' = \bar{\lambda} \cdot \omega \cdot \lambda$ ($T = (T^i_{jk\ell}) = (T'^i_{jk\ell})$); T is an invariant tensor.
Moreover, the **tracefree** condition implies that $T = (T^i_{jk\ell})$ belongs to $\mathcal{R}_0(\mathrm{Sp}(n) \cdot \mathrm{Sp}(1))$ $(n > 1)$.

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