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Elliptic Operators and Finite Groups

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0. Dirac operator

Definition 0.1. *The Clifford algebra C_n and the Lie group Spin(n) are defined by*

$$C_n = \sum_{k=0}^{n} \otimes^k \mathbb{R}^n / \{v \otimes v + |v|^2 \cdot 1\} \quad (v_1 \cdots v_m = [v_1 \otimes \cdots \otimes v_m] \in C_n),$$
$$C_n \supset Spin(n) = \{v_1 \cdots v_m \,;\, |v_i| = 1\,(\forall i) \ \text{and} \ m : even\},$$

and the double covering $\pi : Spin(n) \to SO(n)$ (universal covering if $n \geq 3$) is defined by $\pi(v_1 \cdots v_m)(w) = v_1 \cdots v_m \cdot w \cdot v_m \cdots v_1 \in \mathbb{R}^n \subset C_n\,(\forall w \in \mathbb{R}^n)$. The Lie group $Spin^c(n)$ and the homomorphisms $\pi^c : Spin^c(n) \to SO(n)$, $\rho : Spin^c(n) \to S^1$ are defined by $Spin^c(n) = (Spin(n) \times S^1)/\mathbb{Z}_2$ (where $\mathbb{Z}_2 : (h, a) \sim (-h, -a)$), $\pi^c([h, a]) = \pi(h)$, $\rho([h, a]) = a^2$.

Now, assume that $n = 2m$ and that M is the $2m$-dimensional closed smooth oriented manifold with a Riemannian metric.

Definition 0.2. *Let Δ denote the 2^m-dimensional $\mathbb{C}$-subspace of $C_{2m} \otimes \mathbb{C}$ generated by 2^m-elements $\{(1 \pm e_{2m}) \cdots (1 \pm e_4)(1 \pm e_2)(1 + c_{2m-1}) \cdots (1 + c_3)(1 + c_1)\}$ where $\{e_i\}$: standard basis of $\mathbb{R}^{2m}$, $c_{2k-1} = i^k e_1 e_2 \cdots e_{2k-1}$. Since $e_i \cdot \Delta \subset \Delta$ (for $\forall i$), $C_{2m} \otimes \mathbb{C} \cdot \Delta \subset \Delta$. Moreover, it is known that $C_{2m} \otimes \mathbb{C} = End_{\mathbb{C}}(\Delta)$, and hence $Spin(2m) \subset End_{\mathbb{C}}(\Delta)$. $Spin^c(2m)$ also acts on Δ via Clifford multiplication $[(h, a)] \cdot \delta = ah \cdot \delta$ for $\delta \in \Delta$. $\Delta \supset \Delta_\pm$ are defined to be the ± 1-eigenspaces of τ where $\tau = i^m e_1 e_2 \cdots e_{2m}$ $(\tau^2 = 1)$. $\Delta_\pm$ are irreducible $Spin^c(2m)$-representations, and $v \cdot \Delta_+ \subset \Delta_-$ for $\forall v \in \mathbb{R}^{2m}$.*

Definition 0.3. *Assume that $w_2(M) \in Image\{H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}_2)\}$. Then there exists a $Spin^c(2m)$-structure $P \to M$ which is a principal $Spin^c(2m)$-bundle such that $P \times_{Spin^c(2m)} \mathbb{R}^{2m} = TM$. Then the associated complex line bundle η is defined by $\eta = P \times_{Spin^c(2m), \rho} \mathbb{C}$.*

It is known that any $2m$-dimensional Spin or almost complex manifold has a Spin$^c(2m)$-structure and that any closed oriented n-dimensional manifold has a Spin$^c(n)$-structure if $n \leq 4$. On the other hand, it is known that the 5-dimensional homogeneous space SU(3)/SO(3) does not admit any Spin^c-structure.

Definition 0.4. *Since $(h(v)) \cdot (h \cdot \Delta) = h(v \cdot \Delta)$ for any $h \in Spin^c(2m)$, we can define the Clifford multiplication $cm : TM \otimes S_+ \simeq T^*M \otimes S_+ \to S_-$ where $S_\pm = P \times_{Spin^c(2m)} \Delta_\pm$ and $\simeq$ is given by the Riemannian metric. Assume that there exists a $Spin^c(2m)$-structure $P \to M$ with a connection. For any complex vector bundle E with a connection, the E-valued ($Spin^c$-)Dirac operator D is defined by*

$$D_E \,:\, \Gamma(S_+ \otimes E) \xrightarrow{\nabla} \Gamma(T^*M \otimes S_+ \otimes E) \xrightarrow{cm} \Gamma(S_- \otimes E)$$

where ∇ *is the tensor product connection. In terms of a local orthonomal basis* $\{e_k\}$ *of* TM, D_E *is expressed as* $D_E(\gamma) = \sum_k e_k \cdot \nabla_{e_k}\gamma$.

Definition 0.5. *Let* E *and* F *be complex vector bundles over* M *and* $\Gamma(E)$ $(\Gamma(F))$ *the set of all smooth sections of* E (F). *A linear operator* $D : \Gamma(E) \to \Gamma(F)$ *is called a differential operator of order* m *iff*

$$D\left(\sum_j u^j \epsilon_j\right)(x) = \sum_{i,j} \sum_{|\alpha| \le m} a_\alpha^{ij}(x)(D_1^{\alpha_1} \cdots D_n^{\alpha_n} u^j)(x) f_i(x)$$

where $\{\epsilon_j\}$, $\{f_i\}$ *are local basis of* E, F *on* $U \subset M$, $x = (x^1, \cdots, x^n)$ *is a local coordinate system on* U, $D_k = -i(\partial/\partial x^k)$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ *and* $a_\alpha^{ij}(x)$ *is a smooth function on* U *such that* $a_\alpha^{ij}(x) \neq 0$ *for some* i, j, α *with* $|\alpha| = m$. *For any differential operator* D, *any* $q \in M$ *and any* $\xi = \sum_k \xi_k (dx^k)_q \in T_q^*M$, *a linear map* $\sigma(D)_\xi : E_q \to F_q$ *is defined by*

$$\sigma(D)_\xi\left(\sum_j u^j(q)\epsilon_j(q)\right) = \sum_{i,j}\left(\sum_{|\alpha|=m} a_\alpha^{ij}(q)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}\right) u^j(q) f_i(q)\,.$$

It is shown that the definition above is independent of the choice of the local coordinate systems and local basis, and the homomorphism $\sigma(D)_\xi$ *is determined only by* D *and* $\xi \in T_q^*M$. $\sigma(D)_\xi$ *is called the principal symbol of* D *at* ξ.

Definition 0.6. *A differential operator* D *is called an elliptic operator iff* $\sigma(D)_\xi$ *gives an isomorphism* $E_q \to F_q$ *for any* $q \in M$ *and any* $T_q^*M \ni \xi \neq 0$.

Example 0.7. *Let* $D_E : \Gamma(S_+ \otimes E) \to \Gamma(S_- \otimes E)$ *be the Dirac operator. Let* q *be any point in* M *and* $(x^1, \cdots, x^n)$ *a local coordinate system around* q *such that* $\{X_k = \left(\frac{\partial}{\partial x^k}\right)_q\}_{1 \le k \le n}$ *is orthonormal. Then*

$$D_E(\gamma)(q) = \sum_k X_k \cdot (\nabla_{X_k}\gamma)(q) = \sum_k \left(\frac{\partial}{\partial x^k}\right)_q \cdot \left(\frac{\partial \gamma}{\partial x^k}\right)(q) + \textit{higher order terms},$$

and hence, for any $T_q^*M \ni \xi = \sum_k \xi_k (dx^k)_q \simeq \sum_k \xi_k (\partial/\partial x^k)_q \in T_qM$, *we have*

$$\sigma(D_E)_\xi(\gamma(q)) = \sum_k \left(\frac{\partial}{\partial x^k}\right)_q \cdot i\xi_k\gamma(q) = i\sum_k \xi_k \left(\frac{\partial}{\partial x^k}\right)_q \cdot \gamma(q) = i\xi \cdot \gamma(q)\,.$$

Thus $\sigma(D_E)_\xi$ *is invertible for any* $\xi \neq 0$ *and* D_E *is an elliptic operator of order* 1.

Elliptic operators are "almost" invertible operators and it is known that the kernel and the cokernel of elliptic operators are finite dimensional.

1. MAIN THEOREM

Let M be a $2m$-dimensional closed oriented Riemannian manifold with a Spin^c-structure P and η the associated complex line bundle over M. Let G be a finite group. In this paper, we define an action of G as an orientation-preserving isometric faithful action of G on M which lifts to an action on the Spin^c-structure P. Assume that there exists an action of G on M. Then for any complex G-vector bundle E over M we can define the G-equivariant Spin^c-Dirac operator

$$D_E : \Gamma(S_+ \otimes E) \longrightarrow \Gamma(S_- \otimes E)$$

by using G-invariant metric connections of the tangent bundle TM and E, where $S_\pm$ are the half spinor bundles. Note that the operator D_E is equal to the non-twisted Spin^c-Dirac operator

$$D : \Gamma(S_+) \longrightarrow \Gamma(S_-)$$

if E is the trivial complex line bundle with the trivial G-action. Then the determinant of D_E evaluated at $g \in G$ is defined by

$$\det(D_E, g) = \det(g|\ker D_E)/\det(g|\operatorname{coker} D_E) \in S^1 \subset \mathbb{C}^* . \tag{1}$$

If $g^p = 1$ $(p \geq 2)$, as was proved in Appendix of [9], we have

$$\det(D_E, g) = \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}} \{\operatorname{Ind}(D_E) - \operatorname{Ind}(D_E, g^k)\} \tag{2}$$

where ξ_p is the primitive p-th root of unity,

$$\operatorname{Ind}(D_E, g^k) = \operatorname{Tr}(g^k|\ker D_E) - \operatorname{Tr}(g^k|\operatorname{coker} D_E) \in \mathbb{C}$$

is the equivariant index of D_E evaluated at $g^k \in G$ and

$$\operatorname{Ind}(D_E) = \operatorname{Ind}(D_E, 1) = \dim\ker D_E - \dim\operatorname{coker} D_E \in \mathbb{Z}$$

is the numerical index of D_E (cf. [1]).

Now since the real part of $(1-\xi_p^{-k})^{-1}$ is $1/2$ for any $p \geq 2$ and any $1 \leq k \leq p-1$, it follows from (2) that the equality

$$\frac{1}{2\pi i}\log\det(D_E, g) \equiv \frac{p-1}{2p}\operatorname{Ind}(D_E) - \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}}\operatorname{Ind}(D_E, g^k) . \quad (\text{mod } \mathbb{Z})$$

holds if $g^p = 1$ $(p \geq 2)$. Hence we can define $I(g) \in \mathbb{C}/\mathbb{Z}$ as follows.

Definition 1.1. *Assume that $g \in G$ satisfies $g^p = 1$. Then $I(g) \in \mathbb{C}/\mathbb{Z}$ is defined by*

$$I(g) = \frac{p-1}{2p} Ind(D_E) - \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\xi_p^{-k}} Ind(D_E, g^k) \quad (\text{mod } \mathbb{Z}) \tag{3}$$

if $g \neq 1$. If $g = 1$, we define $I(g) = 0$.

Then we have

$$I(g) \equiv \frac{1}{2\pi i}\log\det(D_E, g) \quad (\text{mod } \mathbb{Z}) \tag{4}$$

and hence $I(g)$ is independent of the choice of $p \geq 2$ such that $g^p = 1$.

Now since the equalities

$$\det(D_E, gh) = \det(D_E, g)\det(D_E, h)$$
$$\frac{1}{2\pi i}\log\det(D_E, g)^N \equiv N\frac{1}{2\pi i}\log\det(D_E, g) \ (\text{mod } \mathbb{Z})$$

hold, the next theorem follows from (4).

Thorem 1.2. *Assume that there exists an action of G on M. Then we have*

(a) $I(g) + I(h) - I(gh) = 0$ for any g, $h \in G$,

(b) $N\, I(g) = 0$ for any natural number N and any $g \in G$ such that $\det(D_E, g)^N = 1$.

We can calculate Ind(D_E) and Ind(D_E, h) and hence $I(h)$ by using the equivariant index theorem. For example, the next proposition is proved by the same argument as in [6].

Proposition 1.3. *Assume that the fixed point set of* h *consists of points* $q_1, q_2, \cdots, q_n$ *and the* $\mathbb{Z}_p$*-action on* M *lifts to an action on a complex line bundle* L *over* M. *Suppose that the eigenvalues of* $h|T_{q_j}M$ *are* $(\xi_p^{\tau_{j1}}, \xi_p^{-\tau_{j1}}, \cdots, \xi_p^{\tau_{jm}}, \xi_p^{-\tau_{jm}})$ *with respect to an oriented orthonormal basis of* $T_{q_j}M$. *Then we have*

$$Ind(D_L) = e^{c_1(L)+c_1(\eta)}\widehat{A}(TM)[M] \ , \quad Ind(D_L, h) = \sum_{j=1}^{n} \frac{\xi_p^{\lambda_j}}{\prod_{i=1}^{m}(1-\xi_p^{-\tau_{ji}})}$$

where $c_1(L)$, $c_1(\eta) \in H^2(M;\mathbb{Z})$ *are the first Chern classes of* L *and* η *respectively,* $\widehat{A}$ *is the* $\widehat{A}$*-class and* λ_j *is an integer.*

When M has an almost complex structure, the next proposition follows from the Riemann-Roch theorem (4.3) and the holomorphic Lefschetz theorem (4.6) in [1] (see also Theorem 3.5.10 in [3]).

Proposition 1.4. *Let* M *be an almost complex manifold with the natural Spin*^{c}*-structure and the action of a finite group* G, L *a complex* G*-line bundle over* M *and* h *an element of* G. *Assume that the* G*-action preserves the almost complex structure and that the fixed point set of* h *consists only of points* $q_1, q_2, \cdots, q_n$. *Suppose that* h *acts on the tangent space* $T_{q_j}M$ *via multiplication by a diagonal matrix with diagonal entries* $(\xi_p^{\tau_{j1}}, \cdots, \xi_p^{\tau_{jm}})$ *and acts on the fiber* $L|q_j$ *via multiplication by* $\xi_p^{\mu_j}$. *Then we have*

$$Ind(D_\ell) = Ch(L^\ell)\,Td(TM)[M] \ , \quad Ind(D_\ell, h) = \sum_{j=1}^{n} \frac{\xi_p^{\mu_j \ell}}{\prod_{i=1}^{m}(1-\xi_p^{-\tau_{ji}})}$$

where D_ℓ *denotes* D_{L^ℓ}, *Ch is the Chern character, Td is the Todd class and* $[M]$ *is the fundamental cycle of* M.

2. Finite subgroup of the mapping class group

Let M be a compact Riemann surface of genus $\sigma \geq 2$. In this section, we define an action of a finite group G on M as a biholomorphic action of G with respect to some complex structure of M. Then it is known that G is not a subgroup of the mapping class group Γ_σ if M does not admit an action of G (see [7]).

Assume that M admits an action of the cyclic group $\mathbb{Z}_p$ of order p generated by g and suppose that the quotient map $\pi : M \longrightarrow M/\mathbb{Z}_p$ is a branched covering with b branch points $y_1, \cdots, y_b \in M/\mathbb{Z}_p$ of order $(n_1, \cdots, n_b)$. For $1 \leq i \leq b$, set $r_i = p/n_i$. Then the Riemann-Hurwitz equation

$$(5) \qquad 2\sigma - 2 = p(2\overline{\sigma} - 2) + \sum_{i=1}^{b}(p - r_i)$$

holds where $\overline{\sigma}$ is the genus of $M/\mathbb{Z}_p$. In this section, applying Theorem 1.2 and the Riemann-Hurwitz equation, we examine whether M admits actions of cyclic groups and dihedral groups.

Let L be the tangent bundle of M and D_ℓ the L^ℓ-valued Spin^c-Dirac operator on M. Under the notation above, we have the next theorem.

Thorem 2.1. *Assume that M admits an action of $G = \mathbb{Z}_p = \langle g \rangle$. Then for $1 \le i \le b$ there exists a natural number $1 \le t_i \le n_i - 1$ which is prime to n_i such that*

$$\varphi_{\ell,z}(t_1, \cdots, t_b) \in \mathbb{Z}\,, \quad N\psi_{\ell,z}(t_1, \cdots, t_b) \in \mathbb{Z}$$

for any ℓ and for any z $(1 \le z < p)$ which is prime to p where

$$\begin{aligned} &\varphi_{\ell,z}(t_1, \cdots, t_b) \\ &= (1-z)\frac{p-1}{2p}(1-\sigma)(2\ell+1) - \sum_{i=1}^{b} \frac{1}{n_i} \sum_{j=1}^{n_i-1} \frac{1}{1-\xi_{n_i}^{-j}} \left(\frac{\xi_{n_i}^{jzt_i\ell}}{1-\xi_{n_i}^{-jzt_i}} - z\frac{\xi_{n_i}^{jt_i\ell}}{1-\xi_{n_i}^{-jt_i}} \right), \\ &\psi_{\ell,z}(t_1, \cdots, t_b) = \frac{p-1}{2p}(1-\sigma)(2\ell+1) - \sum_{i=1}^{b} \frac{1}{n_i} \sum_{j=1}^{n_i-1} \frac{\xi_{n_i}^{jzt_i\ell}}{(1-\xi_{n_i}^{-j})(1-\xi_{n_i}^{-jzt_i})} \end{aligned}$$

and N is a natural number such that $\det(D_\ell, g)^N = 1$.

Proof. We have

$$\mathrm{Ch}(L^\ell) = 1 + \ell x\,, \quad \mathrm{Td}(TM) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x$$

where x is the first Chern class $c_1(TM)$ of the tangent bundle TM. Moreover since $x[M] = c_1(TM)[M] = 2 - 2\sigma$, it follows from Proposition 1.4 that

$$\mathrm{Ind}(D_\ell) = \left(\ell + \frac{1}{2}\right) x[M] = (1-\sigma)(2\ell+1)\,.$$

Now let $\Omega(k)$ be the fixed point set of g^k $(1 \le k \le p-1)$ and q_i any point in $\pi^{-1}(y_i)$. Then we can see that $\pi^{-1}(y_i)$ consists of r_i points q_i, $g \cdot q_i$, $\cdots$, $g^{r_i - 1} \cdot q_i$, which are fixed points of g^{r_i} and therefore it follows that

$$\pi^{-1}(y_i) \subset \Omega(k) \iff \pi^{-1}(y_i) \cap \Omega(k) \ne \phi \iff k = r_i j \quad (j = 1, 2, \cdots, n_i - 1)\,.$$

Since g acts transitively on $\pi^{-1}(y_i)$, g^{r_i} acts on the tangent space of each point in $\pi^{-1}(y_i)$ via the same rotation and therefore we can suppose that g^{r_i} acts on the tangent space of each point in $\pi^{-1}(y_i)$ via multiplication by $\xi_p^{r_i t_i}$ where $1 \le t_i \le n_i - 1$ and t_i is prime to n_i. Let z be any integer with $1 \le z < p$ such that $gcd(z,p) = 1$ and ξ_{n_i} the primitive n_i-th root of unity. Then since the order of g^z is p, $M/\langle g^z \rangle$ coincides with $M/\langle g \rangle$ and $(g^z)^{r_i}$ acts on the tangent space of each point in $\pi^{-1}(y_i)$ via multiplication by $\xi_p^{zr_i t_i}$, it follows from Propotion 1.4 that

$$\begin{aligned} I(g^z) &\equiv \frac{p-1}{2p}(1-\sigma)(2\ell+1) - \frac{1}{p}\sum_{i=1}^{b} r_i \sum_{j=1}^{n_i-1} \frac{\xi_p^{r_i jzt_i\ell}}{(1-\xi_p^{-r_i j})(1-\xi_p^{-r_i jzt_i})} \\ &= \frac{p-1}{2p}(1-\sigma)(2\ell+1) - \sum_{i=1}^{b} \frac{1}{n_i} \sum_{j=1}^{n_i-1} \frac{\xi_{n_i}^{jzt_i\ell}}{(1-\xi_{n_i}^{-j})(1-\xi_{n_i}^{-jzt_i})} \pmod{\mathbb{Z}}\,. \end{aligned}$$

Therefore it follows from Theorem 1.2 (a) that

$$0 = I(g^z) - zI(g) \equiv \varphi_{\ell,z}(t_1, \cdots, t_b) \pmod{\mathbb{Z}}$$

and it follows from Theorem 1.2 (b) that

$$0 = NI(g^z) \equiv N\psi_{\ell,z}(t_1, \cdots, t_b) \pmod{\mathbb{Z}}\,.$$

Example 2.2. *Let M be a compact Riemann surface of genus σ. Then the necessary and sufficient condition for M to admit a $\mathbb{Z}_p$-action is given in Theorem 4 in* [5] *(see also Proposition 2.2 in* [4]*). In this example, we consider one hundred cases that $2 \leq \sigma, p \leq 11$. Then if*

$$(\sigma, p) = (2,7),\ (2,11),\ (3,11),\ (4,11),\ (5,7),\ (7,11),\ (8,11),\ (9,11)\,, \tag{6}$$

the Riemann-Hurwitz equation is not satisfied for any $\overline{\sigma}$, b, r_i and hence M does not admit a $\mathbb{Z}_p$-action. Moreover using Thoerem 4 in [5]*, we can see that M does not admit an action of $\mathbb{Z}_p$ if and only if (σ, p) is contained in (6) or*

$$(\sigma, p) = (2,9),\ (3,5),\ (3,10),\ (4,7),\ (5,9)\,, (6,11),\ (11,7)\,. \tag{7}$$

In this example, using the Riemann-Hurwitz equation and Theorem 2.1, we prove that M does not admit a $\mathbb{Z}_p$-action for (σ, p) in (7).

Now using the Riemann-Hurwitz equation, we can see that

$$\begin{aligned}
&(\sigma, p) = (2,9) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{3,3,9\})\\
&(\sigma, p) = (3,5) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (1, \{5\})\\
&(\sigma, p) = (3,10) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{5,5,5\}),\ (4, \{2,2,2,10\})\\
&(\sigma, p) = (4,7) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (1, \{7\})\\
&(\sigma, p) = (5,9) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (4, \{3,3,3,9\}),\ (1, \{9\})\\
&(\sigma, p) = (6,11) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (1, \{11\})\\
&(\sigma, p) = (11,7) \Longrightarrow (b, \{n_1, \cdots, n_b\}) = (1, \{7\})\,.
\end{aligned}$$

When $(\sigma, p) = (2,9)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{3,3,9\})$, direct computation using a computer shows that

$$\begin{aligned}
&1 < \varphi_{1,2}(1,1,1) < 2\,,\ 1 < \varphi_{1,2}(2,1,1) = \varphi_{1,2}(1,2,1) < 2\,,\ 0 < \varphi_{1,2}(2,2,1) < 1\,,\\
&2 < \varphi_{1,2}(1,1,2) < 3\,,\ 1 < \varphi_{1,2}(2,1,2) = \varphi_{1,2}(1,2,2) < 2\,,\ 0 < \varphi_{1,2}(2,2,2) < 1\,,\\
&2 < \varphi_{1,2}(1,1,4) < 3\,,\ 1 < \varphi_{1,2}(2,1,4) = \varphi_{1,2}(1,2,4) < 2\,,\ 1 < \varphi_{1,2}(2,2,4) < 2\,,\\
&1 < \varphi_{1,2}(1,1,5) < 2\,,\ 1 < \varphi_{1,2}(2,1,5) = \varphi_{1,2}(1,2,5) < 2\,,\ 0 < \varphi_{1,2}(2,2,5) < 1\,,\\
&2 < \varphi_{1,2}(1,1,7) < 3\,,\ 1 < \varphi_{1,2}(2,1,7) = \varphi_{1,2}(1,2,7) < 2\,,\ 0 < \varphi_{1,2}(2,2,7) < 1\,,\\
&2 < \varphi_{1,2}(1,1,8) < 3\,,\ 1 < \varphi_{1,2}(2,1,8) = \varphi_{1,2}(1,2,8) < 2\,,\ 1 < \varphi_{1,2}(2,2,8) < 2\,,
\end{aligned}$$

and therefore none of $\varphi_{1,2}(t_1, t_2, t_3)$ is an integer. Hence it follows from Theorem 2.1 that the Riemann surface of genus 2 does not admit an action of $\mathbb{Z}_9$.

When $(\sigma, p) = (3,5)$, $(b, \{n_1, \cdots, n_b\}) = (1, \{5\})$, direct computation shows that

$$2 < \varphi_{1,2}(1)\,,\ \varphi_{1,2}(2)\,,\ \varphi_{1,2}(3)\,,\ \varphi_{1,2}(4) < 3\,.$$

Hence the Riemann surface of genus 3 does not admit an action of $\mathbb{Z}_5$. Hence it is clear that the Riemann surface of genus 3 does not admit an action of $\mathbb{Z}_{10}$.

When $(\sigma, p) = (4,7)$, $(b, \{n_1, \cdots, n_b\}) = (1, \{7\})$, direct computation shows that

$$3 < \varphi_{1,2}(1)\,,\ \varphi_{1,2}(4)\,,\ \varphi_{1,2}(5) < 4 < \varphi_{1,2}(2)\,,\ \varphi_{1,2}(3)\,,\ \varphi_{1,2}(6) < 5\,.$$

Hence the Riemann surface of genus 4 does not admit an action of $\mathbb{Z}_7$.

When $(\sigma, p) = (5,9)$, $(b, \{n_1, \cdots, n_b\}) = (4, \{3,3,3,9\})$, direct computation shows that none of $\varphi_{1,2}(t_1, t_2, t_3, t_4)$ is an integer for $1 \leq t_1 \leq t_2 \leq t_3 \leq 2, 1 \leq t_4 \leq 8, t_4 \neq 3, 6$. Moreover if $(\sigma, p) = (5,9)$, $(b, \{n_1, \cdots, n_b\}) = (1, \{9\})$, direct computation also shows *that none of* $\varphi_{1,2}(t_1)$ *is an integer for* $1 \leq t_1 \leq 8, t_1 \neq 3, 6$. *Hence the Riemann surface of genus* 5 *does not admit an action of* $\mathbb{Z}_9$.

When $(\sigma, p) = (6, 11)$, $(b, \{n_1, \cdots, n_b\}) = (1, \{11\})$, *direct computation shows that none of* $\varphi_{1,2}(t_1)$ *is an integer for* $1 \leq t_1 \leq 10$. *Hence the Riemann surface of genus* 6 *does not admit an action of* $\mathbb{Z}_{11}$.

When $(\sigma, p) = (11, 7)$, $(b, \{n_1, \cdots, n_b\}) = (1, \{7\})$, *direct computation shows that none of* $\varphi_{1,2}(t_1)$ *is an integer for* $1 \leq t_1 \leq 6$. *Hence the Riemann surface of genus* 11 *does not admit an action of* $\mathbb{Z}_7$.

Example 2.3. *Let* M *be a compact Riemann surface of genus* σ $(2 \leq \sigma \leq 11)$ *which admits an action of* $\mathbb{Z}_p$ $(3 \leq p \leq 11)$. *Note that* M *always admits an action of* $\mathbb{Z}_2$ *because we can embed* M *symmetrically into* $\mathbb{R}^3$ *with respect to the* π*-rotation around* x*-axis. In this example, applying Theorem 2.1, we examine whether* M *admits an action of the dihedral group* $D(2p)$ *generated by* g, h *with the relation*

$$g^p = h^2 = 1\,, \quad h^{-1}gh = g^{-1}\,. \tag{8}$$

Note that M *clearly admits an action of the dihedral group* $D(2p)$ *if* $\sigma \equiv 0, 1 \pmod p$ *because we can embed* M *symmetrically into* $\mathbb{R}^3$ *with respect to the* $2\pi/p$*-rotation around* z*-axis.*

If M *admits an action of* $D(2p)$, *the relation (8) implies that*

$$\det(D_\ell, g) = \det(D_\ell, h^{-1}gh) = \det(D_\ell, g^{-1}) = \det(D_\ell, g)^{-1} \iff \det(D_\ell, g)^2 = 1\,.$$

Since we have $\det(D_\ell, g)^p = \det(D_\ell, g^p) = 1$, *it follows from Theorem 2.1 that*

$$\det(D_\ell, g)^2 = 1 \Longrightarrow 2\psi_{\ell,z}(t_1, \cdots, t_b) \in \mathbb{Z} \quad \textit{when } p \textit{ is even},$$
$$\det(D_\ell, g) = 1 \Longrightarrow \psi_{\ell,z}(t_1, \cdots, t_b) \in \mathbb{Z} \quad \textit{when } p \textit{ is odd}$$

for any ℓ *and any* z $(1 \leq z < p)$ *which is prime to* p.

Now it follows from the Riemann-Hurwitz equation and Thoerem 4 in [5] *that*

$$\begin{aligned}
(\sigma, p) = (2, 5) &\Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{5, 5, 5\}) \\
(\sigma, p) = (7, 5) &\Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{5, 5, 5\}) \\
(\sigma, p) = (3, 9) &\Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{3, 9, 9\}) \\
(\sigma, p) = (4, 9) &\Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{9, 9, 9\}) \\
(\sigma, p) = (11, 9) &\Longrightarrow (b, \{n_1, \cdots, n_b\}) = (5, \{3, 9, 9, 9, 9\}) \\
(\sigma, p) = (2, 10) &\Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{2, 5, 10\}) \\
(\sigma, p) = (7, 10) &\Longrightarrow (b, \{n_1, \cdots, n_b\}) = (4, \{2, 10, 10, 10\}),\ (5, \{2, 2, 2, 5, 10\}) \\
(\sigma, p) = (5, 11) &\Longrightarrow (b, \{n_1, \cdots, n_b\}) = (3, \{11, 11, 11\})\,.
\end{aligned}$$

When $(\sigma, p) = (2, 5)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{5, 5, 5\})$, *direct computation shows that*

$$-2 < \psi_{1,1}(t_1, t_2, t_3) < -1$$

for any $1 \leq t_1 \leq t_2 \leq t_3 \leq 4$ *and therefore none of* $\psi_{1,1}(t_1, t_2, t_3)$ *is an integer. Hence the Riemann surface of genus* 2 *does not admit an action of* $D(10)$.

When $(\sigma, p) = (7, 5)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{5, 5, 5\})$, *direct computation shows that*

$$-8 < \psi_{1,1}(t_1, t_2, t_3) < -7$$

for any $1 \leq t_1 \leq t_2 \leq t_3 \leq 4$ *and therefore none of* $\psi_{1,1}(t_1, t_2, t_3)$ *is an integer. Hence the Riemann surface of genus* 7 *does not admit an action of* $D(10)$.

When $(\sigma, p) = (3, 9)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{3, 9, 9\})$, *direct computation shows that*

$$\begin{aligned}(t_1, t_2, t_3) = {} & (1,1,1),\ (1,1,5),\ (1,1,7),\ (1,5,5),\ (1,5,7),\ (1,7,7),\\ & (2,1,1),\ (2,1,2),\ (2,1,4),\ (2,1,5),\ (2,1,7),\ (2,1,8),\\ & (2,2,5),\ (2,2,7),\ (2,4,5),\ (2,4,7),\ (2,5,5),\ (2,5,7),\\ & (2,5,8),\ (2,7,7),\ (2,7,8)\\ \Longrightarrow {} & -3 < \psi_{1,1}(t_1, t_2, t_3) < -2\end{aligned}$$

and that $-4 < \psi_{1,1}(t_1, t_2, t_3) < -3$ *for other* $1 \leq t_1 \leq 2$, $1 \leq t_2 \leq t_3 \leq 8$, $t_2, t_3 \neq 3, 6$. *Hence the Riemann surface of genus* 3 *does not admit an action of* $D(18)$.

When $(\sigma, p) = (4, 9)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{9, 9, 9\})$, *direct computation shows that none of* $\psi_{1,1}(t_1, t_2, t_3)$ *is an integer for* $1 \leq t_1 \leq t_2 \leq t_3 \leq 8$, $t_1, t_2, t_3 \neq 3, 6$. *Hence the Riemann surface of genus* 4 *does not admit an action of* $D(18)$.

When $(\sigma, p) = (11, 9)$, $(b, \{n_1, \cdots, n_b\}) = (5, \{3, 9, 9, 9, 9\})$, *direct computation shows that none of* $\psi_{1,1}(t_1, t_2, t_3, t_4, t_5)$ *is an integer for* $1 \leq t_1 \leq 2$, $1 \leq t_2 \leq t_3 \leq t_4 \leq t_5 \leq 8$, $t_2, t_3, t_4, t_5 \neq 3, 6$. *Hence the Riemann surface of genus* 11 *does not admit an action of* $D(18)$.

When $(\sigma, p) = (2, 10)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{2, 5, 10\})$, *direct computation shows that none of* $2\psi_{1,1}(t_1, t_2, t_3)$ *is an integer for* $t_1 = 1$, $1 \leq t_2 \leq 4$, $1 \leq t_3 \leq 9$, $t_3 \neq 2, 4, 5, 6, 8$. *Hence the Riemann surface of genus* 2 *does not admit an action of* $D(20)$.

When $(\sigma, p) = (7, 10)$, $(b, \{n_1, \cdots, n_b\}) = (4, \{2, 10, 10, 10\})$, *direct computation shows that none of* $2\psi_{1,1}(t_1, t_2, t_3, t_4)$ *is an integer for* $t_1 = 1$, $1 \leq t_2 \leq t_3 \leq t_4 \leq 9$, $t_2, t_3, t_4 \neq 2, 4, 5, 6, 8$. *When* $(\sigma, p) = (7, 10)$, $(b, \{n_1, \cdots, n_b\}) = (5, \{2, 2, 2, 5, 10\})$, *direct computation also shows that none of* $2\psi_{1,1}(t_1, t_2, t_3, t_4, t_5)$ *is an integer for* $t_1 = t_2 = t_3 = 1$, $1 \leq t_4 \leq 4$, $1 \leq t_5 \leq 9$, $t_5 \neq 2, 4, 5, 6, 8$. *Hence the Riemann surface of genus* 7 *does not admit an action of* $D(20)$.

When $(\sigma, p) = (5, 11)$, $(b, \{n_1, \cdots, n_b\}) = (3, \{11, 11, 11\})$, *direct computation shows that*

$$\{(t_1, t_2, t_3) \mid \psi_{1,1}(t_1, t_2, t_3) \in \mathbb{Z}\} \cap \{(t_1, t_2, t_3) \mid \psi_{2,1}(t_1, t_2, t_3) \in \mathbb{Z}\} = \phi\,.$$

Hence the Riemann surface of genus 5 *does not admit an action of* $D(22)$.

Theorem 2.1 is useful in determining the rotation angles around the fixed points of the action of an element of the mapping class group.

Example 2.4. *Assume that a Riemann surface* M *of genus* σ $(2 \leq \sigma \leq 11)$ *admits an action of* $\mathbb{Z}_3$ *generated by* g *and let* $q_1, \cdots, q_b \in M$ *be the fixed points of* g. *Note that* $b = 0$ *if* g *acts freely on* M. *In this example, we use Theorem 2.1 to determine the rotation angle* $\frac{2\pi t_i}{3}$ *of* $g|T_{q_i}M$, *where we can assume that* $2 \geq t_1 \geq t_2 \geq \cdots \geq t_b \geq 1$. *If* g *acts on* $T_{q_b}M$ *via rotation of* $\frac{4\pi}{3}$, *then* g^2 *acts on* $T_{q_b}M$ *via rotation of* $\frac{2\pi}{3}$. *Hence it suffices to determine* $t_1, t_2, \cdots, t_b$ *under the condition that* $t_b = 1$, *which we assume*

Now it follows from the Riemann-Hurwitz equation and Theorem 4 in [5] *that a Riemann surface* M *of genus* σ $(2 \le \sigma \le 11)$ *admits an action of* $\mathbb{Z}_3$ *if and only if*

$$(9)\qquad \begin{aligned}(\sigma,b) =&(2,4),\ (3,2),\ (3,5),\ (4,0),\ (4,3),\ (4,6),\ (5,4),\ (5,7),\ (6,2),\ (6,5),\ (6,8),\\ &(7,0),\ (7,3),\ (7,6),\ (7,9),\ (8,4),\ (8,7),\ (8,10),\ (9,2),\ (9,5),\ (9,8),\ (9,11),\\ &(10,0),\ (10,3),\ (10,6),\ (10,9),\ (10,12),\ (11,4),\ (11,7),\ (11,10),\ (11,13).\end{aligned}$$

If $(\sigma, b) = (4,3)$, *then the direct computation shows that*

$$\varphi_{\ell,z}(t_1,t_2,t_3) \in \mathbb{Z}\,, \quad 3\psi_{\ell,z}(t_1,t_2,t_3) \in \mathbb{Z}$$

for any $1 \le \ell, z \le 2$ *if and only if* $(t_1,t_2,t_3) = (1,1,1)$ *(or* $= (2,2,2)$*). Hence it follows from Theorem 2.1 that* g *acts on* $T_{q_1}M$, $T_{q_2}M$, $T_{q_3}M$ *via the rotation* $\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}$ *respectively. On the other hand, if* $(\sigma, b) = (4,6)$, *the direct computation shows that*

$$\varphi_{\ell,z}(t_1,t_2,t_3,t_4,t_5,t_6) \in \mathbb{Z}\,, \quad 3\psi_{\ell,z}(t_1,t_2,t_3,t_4,t_5,t_6) \in \mathbb{Z}$$

for any $1 \le \ell, z \le 2$ *if and only if* $(t_1,t_2,t_3,t_4,t_5,t_6) = (1,1,1,1,1,1)$ *or* $(2,2,2,1,1,1)$. *This result does not imply that there are two types of rotation angles because Theorem 2.1 gives only a necessary condition. But this result implies that there does not exist another type of rotation angles. Further computation leads to the next result.*

(10)

$$\begin{aligned}
&b=2 \Longrightarrow (t_1,t_2)=(2,1)\,,\quad b=3 \Longrightarrow (t_1,t_2,t_3)=(1,1,1)\,,\\
&b=4 \Longrightarrow (t_1,t_2,\cdots,t_4)=(2,2,1,1)\,,\quad b=5 \Longrightarrow (t_1,t_2,\cdots,t_5)=(2,1,1,1,1)\,,\\
&b=6 \Longrightarrow (t_1,t_2,\cdots,t_6)=(1,1,1,1,1,1) \text{ or } (2,2,2,1,1,1)\\
&b=7 \Longrightarrow (t_1,t_2,\cdots,t_7)=(2,2,1,1,1,1,1)\,,\\
&b=8 \Longrightarrow (t_1,t_2,\cdots,t_8)=(2,1,1,1,1,1,1,1) \text{ or } (2,2,2,2,1,1,1,1)\\
&b=9 \Longrightarrow (t_1,t_2,\cdots,t_9)=(1,1,1,1,1,1,1,1,1) \text{ or } (2,2,2,1,1,1,1,1,1)\\
&b=10 \Longrightarrow (t_1,t_2,\cdots,t_{10})=(2,2,1,1,1,1,1,1,1,1) \text{ or } (2,2,2,2,2,1,1,1,1,1)\\
&b=11 \Longrightarrow (t_1,t_2,\cdots,t_{11})=(2,1,1,1,1,1,1,1,1,1,1) \text{ or } (2,2,2,2,1,1,1,1,1,1,1)\\
&b=12 \Longrightarrow (t_1,t_2,\cdots,t_{12})=(1,1,1,1,1,1,1,1,1,1,1,1) \text{ or } (2,2,2,1,1,1,1,1,1,1,1,1)\\
&\qquad\qquad \text{ or } (2,2,2,2,2,2,1,1,1,1,1,1)\\
&b=13 \Longrightarrow (t_1,t_2,\cdots,t_{13})=(2,2,1,1,1,1,1,1,1,1,1,1,1)\\
&\qquad\qquad \text{ or } (2,2,2,2,2,1,1,1,1,1,1,1,1)\,.
\end{aligned}$$

3. Almost free action

In this section we call the action of a finite group G on M is almost free if the fixed point set of any $G \ni g \neq 1$ is empty or consists only of points. Note that M does not admit an almost free action of the cyclic group $\mathbb{Z}_{pq}$ if M does not admit an almost free action of $\mathbb{Z}_p$.

Now let $\mathbb{Z}_p$ be the cyclic group of prime order p generated by g and L a complex $\mathbb{Z}_p$-line bundle over M. Then since the fixed point set of g^k is independent of k, the number n of the fixed points of g^k is independent of k and the action of $\mathbb{Z}_p$ is almost free if and only if the fixed point set of g is empty or consists only of points. In this section, applying Theorem 1.2, we examine whether $\mathbb{Z}_p$ can act almost freely on M.

First we have the next theorem for $p = 2$.

Thorem 3.1. *Assume that M admits an almost free action of $\mathbb{Z}_2$. Then we have the following results.*
(1) If the almost free action of $\mathbb{Z}_2$ lifts to an action on a complex line bundle L over M and Ind(D_L) is an odd number, then we have $n \geq 2^m$.
(2) If M has an almost complex structure and the almost free action of $\mathbb{Z}_2$ preserves the almost complex structure, then we have $n = 0$ or $n \geq 2^m$.

Proof. (1) It follows from Proposition 1.3 that

$$2I(g) \equiv \frac{1}{2}\left(\mathrm{Ind}(D_L) - \frac{1}{2^m}\sum_{j=1}^{n}(-1)^{\lambda_j}\right) \quad (\mathrm{mod}\ \mathbb{Z}).$$

The right-hand side of the equality above is not an integer if $n < 2^m$ because Ind(D_L) is an odd number. Hence it follows from Theorem 1.2 (b) that $n \geq 2^m$.
(2) It follows from Proposition 1.4 that

$$2I(g) \equiv \frac{2-1}{2}\mathrm{Ind}(D) - \frac{1}{1-(-1)}\sum_{j=1}^{n}\frac{1}{(1-(-1))^m} = \frac{1}{2}\left(\mathrm{Ind}(D) - \frac{n}{2^m}\right) \quad (\mathrm{mod}\ \mathbb{Z}).$$

The right-hand side of the equality above is not an integer if $0 < n < 2^m$. Hence it follows from Theorem 1.2 (b) that $n = 0$ or $n \geq 2^m$. □

Remark 3.2. *Let L be the trivial complex line bundle over M. Then any action of $\mathbb{Z}_p$ lifts to the trivial action on L.*

Remark 3.3. *Professor Akio Hattori has pointed out to the author that (2) of the theorem above is also deduced from the equivariant index theorem by using the fact that the equivariant index of any involution is an integer.*

Example 3.4. *Let $M = \mathbb{CP}^m$ be the m-dimensional complex projective space with the Spin^c-structure determined by the condition that $c_1(\eta) = (m+1+2s)x$ where s is an integer and x is the positive generator of $H^2(M;\mathbb{Z}) \cong \mathbb{Z}$. Assume that M admits an almost free action of a finite group G and let g be any element of G. Then $g^*x = \pm x$, $(m+1+2s)g^*x = (m+1+2s)x$ and $(g^*x)^m = x^m$ imply that $g^*x = x$. Hence it follows from the Lefschetz fixed point theorem that g has $m+1$ fixed points. For example, if $m < p$, the fixed point set of the action of $\mathbb{Z}_p = \langle g\rangle$ on M defined by*

$$g \cdot [z_0 : z_1 : z_2 : \cdots : z_m] \longrightarrow [z_0 : \xi_p z_1 : \xi_p^2 z_2 : \cdots : \xi_p^m z_m]$$

consists of $m+1$ points and hence the action is almost free. Moreover it follows from

Proposition 1.3 that

$$\begin{aligned} Ind(D) &= e^{\frac{(m+1+2s)x}{2}} \widehat{A}(M)[M] \\ &= x^m\text{-coefficient of } e^{sx}\left(\frac{x}{1-e^{-x}}\right)^{m+1} = \frac{1}{2\pi i}\oint_{C(z)} \frac{e^{(m+s)z}}{(e^z-1)^{m+1}} e^z\, dz \end{aligned}$$

(*where $C(z)$ is a sufficiently small counterclockwise loop around the origin*)

$$= \frac{1}{2\pi i}\oint_{C(u)} \frac{(u+1)^{m+s}}{u^{m+1}}\, du$$

(*via the substitution $u = e^z$, where $C(u)$ is a counterclockwise loop around the origin*)

$$= u^m\text{-coefficient of } (u+1)^{m+s} = \binom{m+s}{m}$$

Now we assume that $m \geq 2$, which implies that $m+1 < 2^m$. Then it follows from Theorem 3.1 (1) that M does not admit an almost free involution which preserves the Spin^c-structure of M if the number $\binom{m+s}{m}$ is odd.

Example 3.5. *Let $M = S^6$ be the 6-dimentional sphere with any almost complex structure. Note that any orientation-preserving free involution has two fixed points. Then since $2 < 2^m = 8$, it follows from Theorem 3.1 (2) that S^6 does not admit any almost free involution which preserves the almost complex structure. On the other hand, S^6 clearly admits an orientation-preserving almost free involution defined by*

$$\mathbb{R}^7 \supset S^6 \ni (x_1, \cdots, x_6, x_7) \longrightarrow (-x_1, \cdots, -x_6, x_7),$$

which preserves the unique Spin^c-structure of S^6. Note that the involution above has two fixed points and that $Ind(D)$ is equal to 0 because $Ind(D) = \widehat{A}(TM)[M]$ is a Pontrjagin number of S^6.

For $p = 3, 5$, we have the next theorem.

Thorem 3.6. *Assume that M admits an almost free action of $\mathbb{Z}_p$ where p is an odd prime number and that the action lifts to an action on a complex line bundle L over M. Let d be the distance from $\frac{p-1}{2} Ind(D_L)$ to $p\mathbb{Z}$ defined by $d = \min_{s\in\mathbb{Z}} \left| sp - \frac{p-1}{2} Ind(D_L) \right|$. Then for any real number γ such that $0 \leq \gamma \leq d$, we have*

$$n \geq \frac{\gamma}{3(p-1)}\left(2\sin\frac{\pi}{p}\right)^{m+1}$$

Moreover if $\det(D_L, g) = 1$, then we have

$$n \geq \frac{\gamma}{p-1}\left(2\sin\frac{\pi}{p}\right)^{m+1}$$

Proof. Set

$$K_1 = \sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}} \left\{ \mathrm{Ind}(D_L, g^{2k}) - 2\mathrm{Ind}(D_L, g^k) \right\}, \quad K_2 = \sum_{k=1}^{p-1} \frac{1}{1-\xi_p^{-k}} \mathrm{Ind}(D_L, g^k).$$

Then since $|1-\xi_p^t| \geq |1-\xi_p|$ for any integer t which is not a multiple of p, it follows from Proposition 1.3 that

$$\begin{aligned} |K_1| &\leq \sum_{k=1}^{p-1}\sum_{j=1}^{n} \frac{1}{|1-\xi_p^{-k}|}\left\{\frac{1}{\prod_{i=1}^m |1-\xi_p^{-2k\tau_{ji}}|} + 2\frac{1}{\prod_{i=1}^m |1-\xi_p^{-k\tau_{ji}}|}\right\} \\ &\leq \frac{3n(p-1)}{|1-\xi_p|^{m+1}} = \frac{3n(p-1)}{\left(2\sin\frac{\pi}{p}\right)^{m+1}}. \end{aligned}$$

On the other hand, it follows from Theorem 1.2 (a) that

$$\begin{aligned} 2I(g) - I(g^2) &= \frac{p-1}{2p}\mathrm{Ind}(D_L) + \frac{1}{p}K_1 \equiv 0 \pmod{\mathbb{Z}} \\ \Longleftrightarrow \frac{p-1}{2}&\mathrm{Ind}(D_L) + K_1 \equiv 0 \pmod{p}. \end{aligned}$$

Hence we have $|K_1| \geq \gamma$ and therefore it follows that

$$\frac{3n(p-1)}{\left(2\sin\frac{\pi}{p}\right)^{m+1}} \geq \gamma \Longleftrightarrow n \geq \frac{\gamma}{3(p-1)}\left(2\sin\frac{\pi}{p}\right)^{m+1}$$

If $\det(D_L, g) = 1$, then we have

$$I(g) = \frac{p-1}{2p}\mathrm{Ind}(D_L) - \frac{1}{p}K_2 \equiv 0 \pmod{\mathbb{Z}} \Longleftrightarrow \frac{p-1}{2}\mathrm{Ind}(D_L) - K_2 \equiv 0 \pmod{p},$$

which implies that $|K_2| \geq \gamma$. Hence it follows from the same argument as above that

$$\gamma \leq |K_2| \leq \frac{n(p-1)}{\left(2\sin\frac{\pi}{p}\right)^{m+1}} \Longrightarrow n \geq \frac{\gamma}{p-1}\left(2\sin\frac{\pi}{p}\right)^{m+1}$$

□

Remark 3.7. *Note that if M admits a free action of $\mathbb{Z}_p$, then Ind(D_L) is a multiple of p and hence $\gamma = 0$.*

Example 3.8. *Let $M = \mathbb{CP}^m$ be the m-dimensional complex projective space with the Spin^c-structure determined by the condition that $c_1(\eta) = (m+1+2s)x$. As was seen in Example 3.4, we have Ind(D) $= \binom{m+s}{m}$ and hence we can set $\gamma = 1$ unless $\binom{m+s}{m}$ is a multiple of p. Therefore it follows from Theorem 3.6 that*

$$3(m+1)(p-1) \geq \left(2\sin\frac{\pi}{p}\right)^{m+1}$$

if p is an odd prime number and $\binom{m+s}{m}$ is not a multiple of p. This inequality implies that M does not admit any almost free actions of $\mathbb{Z}_3$, $\mathbb{Z}_5$ if $m \geq 6$, $m \geq 37$ respectively. Moreover if $p = 5$ and $\binom{m+s}{m} \equiv 1, 4 \pmod 5$, then we can set $\gamma = 2$ and hence it follows that M does not admit any almost free actions of $\mathbb{Z}_5$ if $m \geq 32$.

Example 3.9. *Let $M = \mathbb{CP}^m$ be the m-dimensional complex projective space with the Spin^c-structure determined by the condition that $c_1(\eta) = (m+1+2s)x$, p an odd prime number and $D(2p)$ the dihedral group generated by g, h with the relation in* (8).

Then there exists an action of $D(2p)$ on M defined by

$$g : [z_0 : z_1 : \cdots : z_m] \longrightarrow \left[z_0 : \xi_p z_1 : \cdots : \xi_p^{\frac{m}{2}} z_{\frac{m}{2}} : \xi_p^{p-\frac{m}{2}} z_{\frac{m}{2}+1} : \cdots : \xi_p^{p-1} z_m\right],$$

$$h : [z_0 : z_1 : \cdots : z_m] \longrightarrow \left[z_0 : z_m : \cdots : z_{\frac{m}{2}+1} : z_{\frac{m}{2}} : \cdots : z_1\right]$$

if m is even, and

$$g : [z_0 : z_1 : \cdots : z_m] \longrightarrow \left[\xi_p z_0 : \xi_p^2 z_1 : \cdots : \xi_p^{\frac{m+1}{2}} z_{\frac{m-1}{2}} : \xi_p^{p-\frac{m+1}{2}} z_{\frac{m-1}{2}+1} : \cdots : \xi_p^{p-1} z_m\right],$$

$$h : [z_0 : z_1 : \cdots : z_m] \longrightarrow \left[z_m : \cdots : z_{\frac{m-1}{2}+1} : z_{\frac{m-1}{2}} : \cdots : z_1 : z_0\right]$$

if m is odd. Note that the action of $\mathbb{Z}_p = \langle g \rangle$ defined above is almost free if $m < p$.

On the other hand, the same argument as in Example 2.3 shows that $\det(D_L, g) = 1$ *for any action of $D(2p)$ on M. Therefore as in the previous example, it follows from Theorem 3.6 that the inequality*

$$(m+1)(p-1) \geq \gamma \left(2\sin\frac{\pi}{p}\right)^{m+1}$$

holds if M admits an almost free action of $\mathbb{Z}_p = \langle g \rangle$. If $\begin{pmatrix} m+s \\ m \end{pmatrix}$ is not a multiple of p, then we can set $\gamma = 1$ and the inequality above implies that M does not admit any action of $D(2p) = \langle g, h \rangle$ such that the action of $\mathbb{Z}_p = \langle g \rangle$ is almost free if $p = 3$, $m \geq 3$ or $p = 5$, $m \geq 29$. Moreover if $p = 5$ and $\begin{pmatrix} m+s \\ m \end{pmatrix} \equiv 1, 4 \pmod 5$, then we can set $\gamma = 2$ and the inequality above implies that M does not admit any action of $D(10) = \langle g, h \rangle$ such that the action of $\mathbb{Z}_5 = \langle g \rangle$ is almost free if $m \geq 23$.

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