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Cocycle Knot Invariants, Quandle Extensions, and Alexander Matrices

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Abstract

The theory of quandle (co)homology and cocycle knot invariants is rapidly being developed. We begin with a summary of these recent advances. One such advance is the notion of a dynamical cocycle. We show how dynamical cocycles can be used to color knotted surfaces that are obtained from classical knots by twist-spinning. We also demonstrate relations between cocycle invariants and Alexander matrices.

1 Introduction

The first half of this paper is a survey of the rapidly growing area of knot invariants and knotted surface invariants that are defined via quandles and their cocycles. Several key examples are closely examined from the viewpoint of these recent developments. In particular, dynamical cocycles are used to color knotted surfaces that are obtained from classical knots by twist-spinning, and relations between cocycle invariants and Alexander matrices are demonstrated. A quandle is a set with a self-distributive binary operation (defined below) whose definition was partially motivated from knot theory. A (co)homology theory was defined in [7] for quandles, which is a modification of rack (co)homology defined in [14]. The cohomology theory has found applications to the classification

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of Nichols algebras [1]. State-sum invariants, called the quandle cocycle invariants, using quandle cocycles as weights are defined [7] and computed for important families of classical knots and knotted surfaces [8]. Other survey articles on this subject are available [10, 21].

In this paper, first we give a short overview of the subject in Sections 2, 3, 4, and 5. New extensions called extensions by dynamical cocycles defined in [1] are studied in relation to colorings of twist-spun knots in Section 6. Then, we define a generalized cocycle invariant in the form of a family of vectors, which encorporates the refined version given in [28], with detailed computations for a few examples in Section 7. Relations of these cocycle invariants and Alexander matrices are given in Section 8.

2 Quandles and Quandle Colorings

In this section we define quandles and quandle colorings.

A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ such that

(I) For any $a \in X$, $a * a = a$.

(II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.

(III) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation that satisfies (II) and (III).

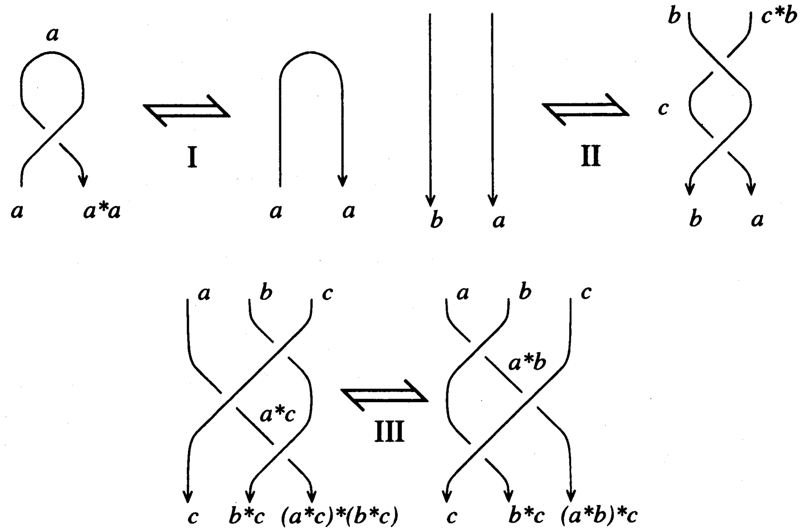

Figure 1: Reidemeister moves and quandle conditions

Racks and quandles have been studied in, for example, [2, 13, 20, 22, 29]. The axioms for a quandle correspond respectively to the Reidemeister moves of type I, II, and III (see Fig. 1 and [13, 22], for example). Quandle structures have been found in areas other than knot theory, see [1] and [2] for example. Recently, it was pointed out in [40] that simple curves on a surface have a quandle structure via an action by Dehn twists.

A function $f : X \to Y$ between quandles or racks is a *homomorphism* if $f(a * b) = f(a) * f(b)$ for any $a, b \in X$. The following are typical examples of quandles.

1. A group $X = G$ with n-fold conjugation as the quandle operation: $a * b = b^{-n}ab^n$.

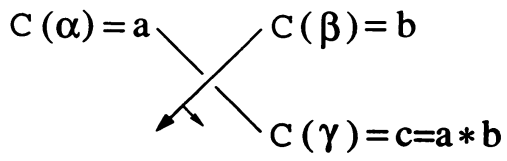

Figure 2: Quandle relation at a crossing

2. Any subset of G that is closed under conjugation.

3. Let n be a positive integer. For elements $i, j \in \{0, 1, \ldots, n-1\}$, define $i * j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n. This set can be identified with the set of reflections of a regular n-gon with conjugation as the quandle operation.

4. Any $\Lambda(= \mathbb{Z}[T, T^{-1}])$-module M is a quandle with $a * b = Ta + (1 - T)b$, $a, b \in M$, called an *Alexander quandle*. Furthermore for a positive integer n, a *mod-n Alexander quandle* $\mathbb{Z}_n[T, T^{-1}]/(h(T))$ is a quandle for a Laurent polynomial $h(T)$. It is finite if the coefficients of the highest and lowest degree terms of h are units in $\mathbb{Z}_n$. The dihedral quandle R_n can be identified with $\mathbb{Z}_n[T, T^{-1}]/(T + 1)$.

Let X be a fixed quandle. Let K be a given oriented classical knot or link diagram, and let $\mathcal{R}$ be the set of (over-)arcs. The normals are given in such a way that (tangent, normal) matches the orientation of the plane, see Fig. 2. A (quandle) *coloring* $\mathcal{C}$ is a map $\mathcal{C} : \mathcal{R} \to X$ such that at every crossing, the relation depicted in Fig. 2 holds. More specifically, let β be the over-arc at a crossing, and α, γ be under-arcs such that the normal of the over-arc points from α to γ. Then it is required that $\mathcal{C}(\gamma) = \mathcal{C}(\alpha) * \mathcal{C}(\beta)$. The color $\mathcal{C}(\gamma)$ depends only on the choice of orientation of the over-arc; therefore this rule defines the coloring at both positive and negative crossings.

For example, Fox's n-coloring [15] is a quandle coloring by the dihedral quandle R_n. The classical result that a knot is non-trivially Fox n-colorable (for n prime) if $n|\Delta(-1)$ (where $\Delta(T)$ denotes the Alexander polynomial) has been generalized by Inoue [19] to the following:

Let $\Delta_K^{(i)}(T)$ denote the greatest common divisor of all $(n - i - 1)$ minor determinants of the presentation matrix for the knot module obtained via the Fox calculus.

Theorem 2.1 [19] *Let p be a prime number, J an ideal of the ring $\Lambda_p = \mathbb{Z}_p[T, T^{-1}]$. For each $i \geq 0$, put $e_i(T) = \Delta_K^{(i)}(T)/\Delta_K^{(i+1)}(T)$. Then the number of colorings by the Alexander quandle Λ_p/J is equal to the cardinality of the module $\Lambda_p/J \oplus \oplus_{i=0}^{n-2}\{\Lambda_p/(e_i(T), J)\}$.*

Alternatively, a coloring can be described as a quandle homomorphism as follows. Classical knots have fundamental quandles that are defined via generators and relations. The theory of quandle presentations is given a complete treatment in [13]. The quandle relation $a * b = c$ holds where a is the generator that corresponds to the under-arc away from which the normal to the over-arc points, b is the generator that corresponds to the over-arc and c corresponds to the under-arc towards which the transversal's normal points, see Fig. 2. A coloring of a classical knot diagram by a quandle X gives rise to a quandle homomorphism from the fundamental quandle to the quandle X.

Using Waldhausen's theorem Joyce shows:

Theorem 2.2 **[20, 29]** *If two knots in $\mathbb{R}^3$ have isomorphic fundamental quandles, then the knots are equivalent up to orientations of $\mathbb{R}^3$ and the knots.*

This fundamental fact was generalized:

Theorem 2.3 **[13]** *The fundamental augmented rack is a complete invariant for irreducible semi-framed links in closed connected 3-manifolds.*

See [13] for a full account of the notation and terminology. Using an interpretation of cocycle knot invariants in terms of the *canonical* class $c(L)$ of a link L, the above theorem was further generalized to:

Theorem 2.4 **[34]** *If L, M are two links in S^3 such that there is an isomorphism ϕ of fundamental racks with $\phi_*(c(L)) = c(M)$, then L and M are isotopic.*

3 Quandle Homology and Cohomology Theories

In this section, we present the ordinary quandle homology theory. Originally, rack homology and homotopy theory were defined and studied in [14], and a modification to quandle homology theory was given in [7] to define a knot invariant in a state-sum form. Then they were generalized to a twisted theory in [4]. The most general form of the quandle homology known to date is given in [1]. Computations are found in [8, 9] and also in [12, 27, 30] by other authors.

Let $C_n^{\mathrm{R}}(X)$ be the free abelian group generated by n-tuples $(x_1, \dots, x_n)$ of elements of a quandle X. Define a homomorphism $\partial_n : C_n^{\mathrm{R}}(X) \to C_{n-1}^{\mathrm{R}}(X)$ by

$$\begin{aligned}
&\partial_n(x_1, x_2, \dots, x_n) \\
&\quad = \sum_{i=2}^{n} (-1)^i \left[(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \right. \\
&\quad \left. - (x_1 * x_i, x_2 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n) \right] \qquad (1)
\end{aligned}$$

for $n \geq 2$ and $\partial_n = 0$ for $n \leq 1$. Then $C_*^{\mathrm{R}}(X) = \{C_n^{\mathrm{R}}(X), \partial_n\}$ is a chain complex.

Let $C_n^{\mathrm{D}}(X)$ be the subset of $C_n^{\mathrm{R}}(X)$ generated by n-tuples $(x_1, \dots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \dots, n-1\}$ if $n \geq 2$; otherwise let $C_n^{\mathrm{D}}(X) = 0$. If X is a quandle, then $\partial_n(C_n^{\mathrm{D}}(X)) \subset C_{n-1}^{\mathrm{D}}(X)$ and $C_*^{\mathrm{D}}(X) = \{C_n^{\mathrm{D}}(X), \partial_n\}$ is a sub-complex of $C_*^{\mathrm{R}}(X)$. Put $C_n^{\mathrm{Q}}(X) = C_n^{\mathrm{R}}(X)/C_n^{\mathrm{D}}(X)$ and $C_*^{\mathrm{Q}}(X) = \{C_n^{\mathrm{Q}}(X), \partial'_n\}$, where ∂'_n is the induced homomorphism. Henceforth, all boundary maps will be denoted by ∂_n.

For an abelian group G, define the chain and cochain complexes

$$C_*^{\mathrm{W}}(X; G) = C_*^{\mathrm{W}}(X) \otimes G, \qquad \partial = \partial \otimes \mathrm{id}; \qquad (2)$$

$$C_{\mathrm{W}}^*(X; G) = \mathrm{Hom}(C_*^{\mathrm{W}}(X), G), \qquad \delta = \mathrm{Hom}(\partial, \mathrm{id}) \qquad (3)$$

in the usual way, where W = D, R, Q.

The groups of cycles and boundaries are denoted respectively by $\ker(\partial) = Z_n^{\mathrm{W}}(X; G) \subset C_n^{\mathrm{W}}(X; G)$ and $\mathrm{Im}(\partial) = B_n^{\mathrm{W}}(X; G) \subset C_n^{\mathrm{W}}(X; G)$ while the cocycles and coboundaries are denoted respectively by $\ker(\delta) = Z_{\mathrm{W}}^n(X; G) \subset C_{\mathrm{W}}^n(X; G)$ and $\mathrm{Im}(\partial) = B_{\mathrm{W}}^n(X; G) \subset C_{\mathrm{W}}^n(X; G)$. In particular, a quandle 2-cocycle is an element $\phi \in Z^2_{\mathrm{Q}}(X;G)$, and the equalities

$$\begin{aligned} \phi(x,z) + \phi(x*z, y*z) &= \phi(x*y,z) + \phi(x,y) \\ \text{and} \quad \phi(x,x) &= 0 \end{aligned}$$

are satisfied for all $x, y, z \in X$.

The nth *quandle homology group* and the nth *quandle cohomology group* [7] of a quandle X with coefficient group G are

$$\begin{aligned} H^{\mathrm{Q}}_n(X;G) &= H_n(C^{\mathrm{Q}}_*(X;G)) = Z^{\mathrm{Q}}_n(X;G)/B^{\mathrm{Q}}_n(X;G), \\ H^n_{\mathrm{Q}}(X;G) &= H^n(C^*_{\mathrm{Q}}(X;G)) = Z^n_{\mathrm{Q}}(X;G)/B^n_{\mathrm{Q}}(X;G). \end{aligned} \tag{4}$$

The following developments have been made recently.

- The conjecture made in [9] on the long exact homology sequence was proved by Litherland and Nelson in [27]: the long exact sequence of quandle homology

  $$\cdots \to H^{\mathrm{D}}_n(X;A) \to H^{\mathrm{R}}_n(X;A) \to H^{\mathrm{Q}}_n(X;A) \to H^{\mathrm{D}}_{n-1}(X;A) \to \cdots$$

  splits into short exact sequences

  $$0 \to H^{\mathrm{D}}_n(X;A) \to H^{\mathrm{R}}_n(X;A) \to H^{\mathrm{Q}}_n(X;A) \to 0.$$

- Mochizuki has computed several key cohomology groups. We highlight some of his results. First,

  $$H^3_{\mathrm{Q}}(R_p;\mathbb{Z}_p) \cong \mathbb{Z}_p,$$

  and he gives an explicit expression for a generating cocycle. He gives explicit 2-cocycles for Alexander quandles over a field K thereby computing $H^2_{\mathrm{Q}}(K[T,T^{-1}]/(T-\omega);K)$ for $\omega \neq 0, 1$. He shows that dihedral quandles of odd order have vanishing rational cohomology in all dimensions. This was shown independently in [27].

- Etingof and Graña [12] have computed the rational cohomology of any finite quandle in all dimensions. In particular, they show that the bounds on the rank of the betti numbers given in [9] are equalities. Furthermore, they relate the 2-dimensional quandle homology to group cohomology. Let X be a quandle and G_X be its enveloping group: $G_X = \langle x \in X : y^{-1}xy = x*y \rangle$. If A is a trivial G_X-module (as is the case with ordinary quandle homology), then

  $$H^2_{\mathrm{Q}}(X;A) \cong H^1(G_X;\mathrm{Fun}(X,A))$$

  where $\mathrm{Fun}(X,A)$ denotes the set of functions.

- Andruskiewitsch and Graña [1] have developed the theory of quandle and rack cohomology further. They have developed a cohomology theory that encompasses those in [4] and Ohtsuki's theory [32]. Furthermore, their primary interest is in the classification of certain pointed Hopf algebras called Nichols algebras. The quandle cohomology plays a central role here.

- Using dynamical cocycles, Graña [17] classified indecomposable racks of order p^2 for any prime p. Another classification theorem was proved by Nelson [31] who described isomorphism classes of Alexander quandles by the submodules $\mathrm{Im}(1-T)$.

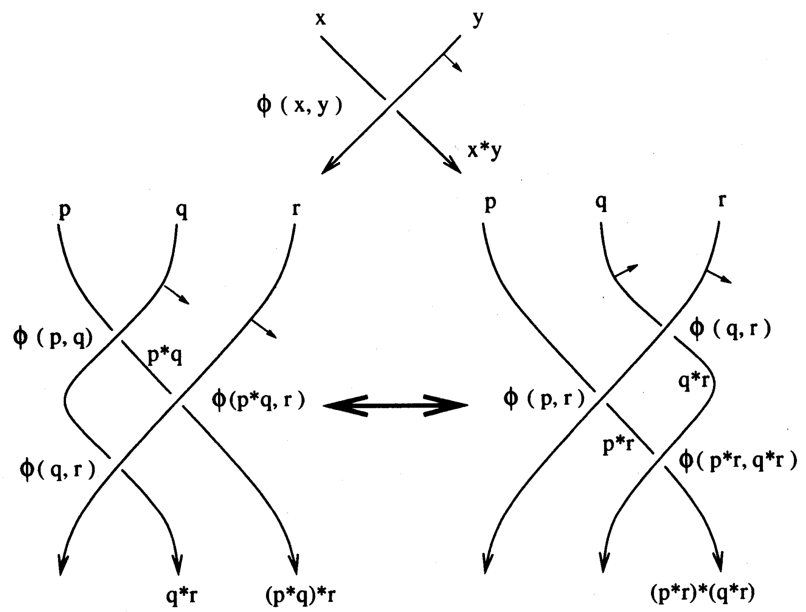

Figure 3: The untwisted 2-cocycle condition and type III move

4 Cocycle Knot Invariants

Cocycle knot invariants of classical and virtual knots

Let K be a classical knot or link diagram. Let a finite quandle X, and an (untwisted) quandle 2-cocycle $\phi \in Z^2_{\mathrm{Q}}(X; A)$ be given. A *(Boltzmann) weight*, $B(\tau, \mathcal{C})$ (that depends on ϕ), at a crossing τ is defined as follows. Let $\mathcal{C}$ denote a coloring $\mathcal{C} : \mathcal{R} \to X$. Let β be the over-arc at τ, and α, γ be under-arcs such that the normal to β points from α to γ, see Fig. 2. Let $x = \mathcal{C}(\alpha)$ and $y = \mathcal{C}(\beta)$. Then define $B(\tau, \mathcal{C}) = \phi(x, y)^{\epsilon(\tau)}$, where $\epsilon(\tau) = 1$ or -1, if (the sign of) the crossing τ is positive or negative, respectively. By convention, the crossing in Fig. 2 is positive if the orientation of the under-arc points downward.

The *(quandle) cocycle knot invariant* is defined by the state-sum expression

$$\Phi(K) = \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C}).$$

The product is taken over all crossings of the given diagram K, and the sum is taken over all possible colorings. The values of the partition function are taken to be in the group ring $\mathbb{Z}[A]$ where A is the coefficient group written multiplicatively. The state-sum depends on the choice of 2-cocycle ϕ. This is proved in [7] to be a knot invariant. Figure 3 shows the invariance of the state-sum under the Reidemeister type III move. The sums of cocycles, equated before and after the move, is the 2-cocycle condition.

Relations to braid group representations and quantum invariants are studied in [16], see also [10] for a viewpoint from the bracket state-sum form and Dijkgraaf-Witten invariants.

Cocycle Invariants for Knotted Surfaces

The state-sum invariant is defined in an analogous way for oriented knotted surfaces in 4-space using their projections and diagrams in 3-space. Specifically, the above steps can be repeated as follows, for a fixed finite quandle X and a knotted surface diagram K.

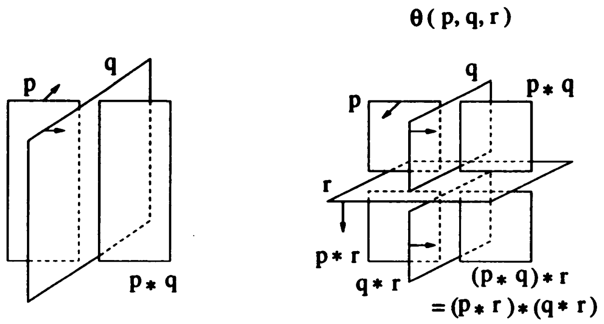

Figure 4: Colors at double curves and 3-cocycle at a triple point

- The diagrams consist of double curves and isolated branch and triple points [11]. Along the double curves, the coloring rule is defined using normals in the same way as classical case, as depicted in the left of Fig. 4.
- The sign $\epsilon(\tau)$ of a triple point τ is defined [11] in such a way that it is positive if and only if the normals to top, middle, bottom sheets, in this order, match the orientation of 3-space.
- For a coloring $\mathcal{C}$, the Boltzman weight at a triple point τ is defined by $B(\tau, \mathcal{C}) = \theta(x, y, z)^{\epsilon(\tau)}$, where θ is a 3-cocycle, $\theta \in Z^3_{\mathrm{Q}}(X; A)$. In the right of Fig. 4, the triple point τ is positive, so that $B(\tau, \mathcal{C}) = \theta(p, q, r)$.
- The state-sum is defined by $\Phi(K) = \sum_{\mathcal{C}} \prod_{\tau} B(\tau, \mathcal{C})$.

Recall that a function $\theta : X \times X \times X \to A$ is a quandle 3-cocycle if

$$\begin{aligned} \theta(p,r,s) + \theta(p*r, q*r, s) + \theta(p,q,r) &= \theta(p*q, r, s) + \theta(p,q,s) + \theta(p*s, q*s, r*s) \\ \theta(p,p,q) &= 0 \\ \theta(p,q,q) &= 0 \end{aligned}$$

By checking the analogues of Reidemeister moves for knotted surface diagrams, called Roseman moves, it was shown in [7] that $\Phi(K)$ is an invariant, called the *(quandle) cocycle invariant* of knotted surfaces.

The value of the state-sum invariant depends only on the cohomology class represented by the defining cocycle. In particular, a coboundary will simply count the number of colorings of a knot or knotted surface by the quandle X.

Applications

Important topological applications have been obtained using the cocycle invariants for knotted surfaces.

- The 2-twist spun trefoil K and its orientation-reversed counterpart $-K$ have shown to have distinct cocycle invariants using a cocycle in $Z^3_{\mathrm{Q}}(R_3;\mathbb{Z}_3)$, providing a proof that K is non-invertible [7]. The higher genus surfaces obtained from K by adding arbitrary number of trivial 1-handles are also non-invertible, since such handle additions do not alter the cocycle invariant. This result in higher genus cases is not immediately obtained from [18, 35], although higher genus generalizations of the Farber-Levine pairing [25] can be used.

- Cocycle invariants for twist spun $(2;n)$-torus knots were computed using Maple [8] for some quandles. Computer-free calculations and general formulas were obtained later in [37] using explicit formulas of 3-cocycles provided in [30] for dihedral quandles. Mochizuki's formulas were also used for the following geometric application.

- The projection of the 2-twist spun trefoil was shown to have at least four triple points [38].

- The projection of the 3-twist spun trefoil was shown to have at least six triple points [39]. The cocycle employed here appeared in [8]. Satoh and Shima gave a set of linear equations among numbers of colored triple points to give algebraic lower bounds on the number of triple points. They have developed a computer program to compute these bounds.

  Colorings of knotted surfaces in relation to dynamical cocycles are discussed in Section 6, that would provide the first step towards extending their results.

5 Extension theory of quandles

Let X be a quandle, and for a given abelian coefficient group A, take a 2-cocycle $\phi \in Z^2_{\mathrm{Q}}(X;A)$. Let $E = A \times X$ and define a binary operation by $(a_1, x_1) * (a_2, x_2) = (a_1 + \phi(x_1, x_2), x_1 * x_2)$. It was shown in [4] that $(E, *)$ defines a quandle, called an *abelian* (or *central) extension*, and is denoted by $E = E(X, A, \phi)$. (This is in parallel to central extension of groups, see Chapter IV of [3].) The following examples were given in [4].

- For any positive integer q and m, the quandle $E = W_{m+1} = \mathbb{Z}_q[T,T^{-1}]/(1-T)^{m+1}$ is an abelian extension of $X = W_m = \mathbb{Z}_q[T,T^{-1}]/(1-T)^m$ over $\mathbb{Z}_q$: $E = E(X,\mathbb{Z}_q,\phi)$, for some $\phi \in Z^2_{\mathrm{Q}}(X;\mathbb{Z}_q)$.

- For any positive integers q and m, $E = U_{m+1} = \mathbb{Z}_{q^{m+1}}[T,T^{-1}]/(T-1+q)$ is an abelian extension $E = E(\mathbb{Z}_{q^m}[T,T^{-1}]/(T-1+q), \mathbb{Z}_q, \phi)$ of $X = U_m = \mathbb{Z}_{q^m}[T,T^{-1}]/(T-1+q)$ for some cocycle $\phi \in Z^2_{\mathrm{Q}}(X;\mathbb{Z}_q)$.

For these quandles, explicit formulas were obtained in [4] using extensions as follows.

- Represent elements of $E = W_{m+1}$ by $A = A_m(1-T)^m + \cdots + A_1(1-T) + A_0$, where $A_j \in \mathbb{Z}_q$, $j = 0, \ldots, m$. Define $f : E = W_{m+1} \to \mathbb{Z}_q \times X (= W_m)$ by

$$f(A) = (A_m \ (\mathrm{mod}\ q),\ \overline{A}\ (\mathrm{mod}\ (1-T)^{m-1})),$$

where $\overline{A} = \sum_{j=0}^{m-1} A_j(1-T)^j$. Then for $A, B \in E$, the quandle operation is computed by

$$\begin{aligned} A * B &= TA + (1-T)B \\ &= (A_m - A_{m-1} + B_{m-1})(1-T)^m + \sum_{j=0}^{m-1} (A_j - A_{j-1} + B_{j-1})(1-T)^j, \end{aligned}$$

where A_{-1}, B_{-1} are understood to be zeros in the last summation, and the coefficients are in $\mathbb{Z}_q$. In $\mathbb{Z}_q[T,T^{-1}]/(1-T)^m$, and we have $f(A * B) = (\phi(\overline{A},\overline{B}), \overline{A} * \overline{B})$ where $\phi(\overline{A},\overline{B}) = B_{m-1} - A_{m-1}$. Hence f yields an isomorphism.

- The cocycle ϕ has a description using a section. Let

$$s : \mathbb{Z}_q[T,T^{-1}]/(1-T)^m \to \mathbb{Z}_q[T,T^{-1}]/(1-T)^{m+1}$$

be a set-theoretic section defined by

$$s\left(\sum_{j=0}^{m-1} A_j(1-T)^j\right) = \sum_{j=0}^{m-1} A_j(1-T)^j \mod (1-T)^{m+1}.$$

Then we have $\overline{s(X)} * \overline{s(Y)} = \overline{s(X * Y)}$ for any $X, Y \in \mathbb{Z}_q[T,T^{-1}]/(1-T)^m$, so that $[s(X) * s(Y) - s(X * Y)]$ is divisible by $(1-T)^m$, and we have

$$\phi(\overline{A},\overline{B}) = [s(A) * s(B) - s(A * B)]/(1-T)^m \in \mathbb{Z}_q.$$

- For $E = U_{m+1}$, represent elements of $\mathbb{Z}_{q^{m+1}}$ by $\{0, 1, \dots, q^{m+1}-1\}$ and express them in their q^{m+1}-ary expansion:

$$A = A_m q^m + \cdots + A_1 q + A_0 \in \mathbb{Z}_{q^{m+1}},$$

where $0 \le A_j < q$, $j = 0, \dots, m$. Then $E = U_{m+1}$ and $X = U_m$ have a similar description as above.

An extension theory of quandles for "twisted" cohomology cocycles was developed in [5], and it provided more general extension theories. In the twisted case, the coefficient group is taken to be a Λ-module, thus has an Alexander quandle structure, and the extension $AE(X, A, \phi) = (A \times X, *)$ is defined by $(a_1, x_1) * (a_2, x_2) = (a_1 * a_2 + \phi(x_1, x_2), x_1 * x_2)$ for $\phi \in Z^2_{\mathrm{TQ}}(X; A)$, and is called an *Alexander extension* of X by (A, ϕ), where the subscript TQ represents the twisted theory. For example, R_{p^m} is an Alexander extension of $R_{p^{m-1}}$ by R_p: $R_{p^m} = AE(R_{p^{m-1}},\ R_p,\ \phi)$, for some $\phi \in Z^2_{\mathrm{TQ}}(R_{p^{m-1}}; R_p)$.

Ohtsuki [32] defined a new cohomology theory for quandles and an extension theory, together with a list of problems in the subject. Further generalizations of extensions by dynamical cocycles as defined in [1] will be discussed and used for coloring twist spun knots in the next section.

6 Extensions of quandles and colorings of twist-spun knots

Extensions by dynamical cocycles

This subsection is a brief summary of a quandle extension theory by Andruskiewitch and Graña [1]. The notation has been changed below from that given in [1] to match our conventions in this paper. Let X be a quandle and S be a non-empty set. Let $\alpha : X \times X \to \mathrm{Fun}(S \times S, S) = S^{S\times S}$ be a function, so that for $\sigma, \tau \in X$ and $a, b \in S$ we have $\alpha_{\sigma,\tau}(s,t) \in S$.

Then it is checked by computations that $S \times X$ is a quandle by the operation $(a,\sigma) * (b,\tau) = (\alpha_{\sigma,\tau}(a,b), \sigma * \tau)$, where $\sigma * \tau$ denotes the quandle operation in X, if and only if α satisfies the following conditions:

1. $\alpha_{\sigma,\sigma}(a,a) = a$ for all $\sigma \in X$ and $a \in S$;

2. $\alpha_{\sigma,\tau}(-,b) : S \to S$ is a bijection for all $\sigma, \tau \in X$ and for all $b \in S$;

3. $\alpha_{\sigma*\tau,\eta}(\alpha_{\sigma,\tau}(a,b),c) = \alpha_{\sigma*\eta,\tau*\eta}(\alpha_{\sigma,\eta}(a,c), \alpha_{\tau,\eta}(b,c))$ for all $\sigma, \tau, \eta \in X$ and $a, b, c \in S$.

Such a function α is called a *dynamical quandle cocycle.* The quandle constructed above is denoted by $S \times_\alpha X$, and is called the *extension* of X by a dynamical cocycle α. The construction is general, as they show:

Lemma 6.1 [1] *Let $p : Y \to X$ be a surjective quandle homomorphism such that the cardinality of $p^{-1}(x)$ is a constant for all $x \in X$. Then Y is isomorphic to an extension $S \times_\alpha X$ of X by some dynamical cocycle on a set S.*

Quandle extensions in wreath products

Let

$$0 \to N \xrightarrow{i} G \xrightarrow{\pi} H \to 1$$

be a split short exact sequence of groups that expresses the finite group G as a semi-direct product $G = N \rtimes H$, so that we have a homomorphism $s : H \to G$ with $\pi \circ s = 1_H$. The elements of G can be written as pairs (x, σ) where $x \in N$ and $\sigma \in H$. The multiplication rule in G is given by $(x,\sigma) \cdot (y,\tau) = (x\sigma(y), \sigma\tau)$, where $\sigma(y)$ denotes the action of H on N that gives G the structure of a semi-direct product.

Let Q denote a subquandle of $\mathrm{Conj}(H)$ (the group H with the quandle structure given by conjugation). Thus Q is a subset of H that is closed under conjugation. Let $\widetilde{Q} = \{(x,\sigma) : \sigma \in Q\}$, then $\widetilde{Q}$ is a quandle by conjugation in G. The group homomorphism $\pi : G \to H$ induces the quandle homomorphism (denoted by the same letter) $\pi : \widetilde{Q} \to Q$. Lemma 6.1 implies

Lemma 6.2 *Suppose the cardinality $|\pi^{-1}(\sigma)|$ is independent of $\sigma \in Q$. Then $\widetilde{Q}$ is an extension Q by a dynamical cocycle $\alpha : Q \times Q \to S^{S\times S}$ where S is a set with cardinality $|\pi^{-1}(\sigma)|$.*

Now we specialize to the case that Q is a subquandle of $\mathrm{Conj}(\Sigma_n)$, where Σ_n denotes the symmetric group on n letters, and $N = (\mathbb{Z}_v)^n$ for some $v \in \{0, 1, \ldots\}$. (In case $v = 0$, then N is the direct product of the integers, and when $v = 1$ then N is trivial.) The action of Σ_n is given by permutation of the factors $\sigma(x_1, \ldots, x_n) = \sigma(\vec{x}) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, for $\sigma \in \Sigma_n$ and $\vec{x} = (x_j)_{j=1}^n \in (\mathbb{Z}_v)^n$. In this situation, $G = (\mathbb{Z}_v)^n \rtimes \Sigma_n$ is also called a wreath product and denoted by $G = (\mathbb{Z}_v) \wr \Sigma_n$. Hence the group operation in G is written by

$$(\vec{x}, \sigma) \cdot (\vec{y}, \tau) = (\vec{x}\sigma(\vec{y}), \sigma\tau)$$

where $\sigma(\vec{y}) = (y_{\sigma(1)}, \ldots, y_{\sigma(n)})$.

It is well known that elements of G can be represented by matrices with entries in $\{x^j\}$ as follows. First, we represent the cyclic group $(\mathbb{Z}_v)$ multiplicatively as $\langle x | x^v = 1 \rangle$, and represent $\vec{x} \in (\mathbb{Z}_v)^n$ by $(x^{i_1}, \ldots, x^{i_n})$. Represent $\sigma \in \Sigma_n$ by an $(n \times n)$-matrix $M(\sigma)$ acting on vectors of n letters from the left. It is a matrix with exactly one nonzero entry in each row and column, and each non-zero entry is 1. The pair $(\vec{x}, \sigma)$ is represented as a matrix $M(\vec{x}, \sigma)$ obtained from $M(\sigma)$ by replacing the non-zero entry in the jth row by x^{i_j}. The group composition in G is matrix multiplication of $M(\vec{x}, \sigma)$s where $0 \cdot x^i = 0$ and $x^i x^j = x^{i+j}$. For example, we write

$$((x^i, x^j, x^k), (12)) = \begin{pmatrix} 0 & x^i & 0 \\ x^j & 0 & 0 \\ 0 & 0 & x^k \end{pmatrix},$$

and

$$((x^\ell, x^m, x^p), (123)) = \begin{pmatrix} 0 & 0 & x^\ell \\ x^m & 0 & 0 \\ 0 & x^p & 0 \end{pmatrix}.$$

The matrix product evaluates as

$$\begin{pmatrix} 0 & x^i & 0 \\ x^j & 0 & 0 \\ 0 & 0 & x^k \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & x^\ell \\ x^m & 0 & 0 \\ 0 & x^p & 0 \end{pmatrix} = \begin{pmatrix} x^{i+m} & 0 & 0 \\ 0 & 0 & x^{\ell+j} \\ 0 & x^{k+p} & 0 \end{pmatrix}$$

which corresponds to $((x^{i+m}, x^{\ell+j}, x^{k+p}), (23))$. Meanwhile, the product

$$((x^\ell, x^m, x^p), (123)) \cdot ((x^i, x^j, x^k), (12)) = ((x^{\ell+k}, x^{m+i}, x^{p+j}), (13))$$

is represented by the matrix

$$\begin{pmatrix} 0 & 0 & x^{\ell+k} \\ 0 & x^{m+i} & 0 \\ x^{p+j} & 0 & 0 \end{pmatrix}.$$

Note that the cycles of Σ_n act on n-letters from the left in this convention.

We take our subquandle Q to be, for example, the dihedral quandle, R_n, or a subset of a given conjugacy class that is itself closed under conjugation, such as $QS_4 = \{(123), (142), (134), (243)\} \subset \Sigma_4$. We are interested in applications to knots herein, so we assume that n is odd. Then for both of $Q = R_n$ or QS_4, an element $\sigma \in Q$ has a fixed point in $\{1, \ldots, n\}$, and the matrix representation $M(\sigma, \vec{x})$ of $(\sigma, \vec{x})$ has exactly one element along the diagonal. It is easy to see that the exponent of the diagonal element is fixed under the conjugation action, so we restrict our attention to the subquandle $Q(v)$ of $\widetilde{Q}$ in which the non-zero diagonal element is 1. By Lemma 6.1, we see that $Q(v)$ is also an extension of Q by a dynamical cocycle.

In the case of the dihedral quandle $Q = R_n$ for n odd, we simplify the notation further. Consider a regular n-gon whose vertices labeled with $\{1, \ldots, n\}$ in this order, on which R_n acts as reflections. For $i = 1, \ldots, n$, let σ_i denote the reflection of a regular n-gon which fixes the vertex labeled i. Then $\sigma_i * \sigma_j = \sigma_{2j-i}$. Denote the element $(\vec{x}, \sigma_j) \in G$ where $\vec{x} = (x^{i_1}, \ldots, x^{i_{j-1}}, 1, x^{i_{j+1}}, \ldots, x^{i_n})$, by $\sigma_j(i_1, \ldots, \widehat{\imath_j}, \ldots, i_n)$ where $\widehat{\imath_j}$ indicates that the jth element in this list is missing. For example, set $a = \sigma_1$, $b = \sigma_2$, and $c = \sigma_3$ in R_3 and the elements of $R_3(v)$ are denoted by $a_{j,k}$, $b_{i,k}$, and $c_{i,j}$, where $i, j, k \in \mathbb{Z}/v$. For convenience we summarize the multiplication table for $R_3(v)$ as follows, where $r * c$ indicates that the table represents (row)$*$(column).

$r * c$	$a_{n,\ p}$	$b_{m,\ p}$	$c_{m,\ n}$
$a_{j,\ k}$	$a_{k+n-p,\ j-n+p}$	$c_{k-p,\ j+p}$	$b_{j-n,\ k+n}$
$b_{i,\ k}$	$c_{i+p,\ k-p}$	$b_{k+m-p,\ i-m+p}$	$a_{i-m,\ k+m}$
$c_{i,\ j}$	$b_{i+n,\ j-n}$	$a_{j+m,\ i-m}$	$c_{j+m-n,\ i-m+n}$

Coloring twist-spun knots by extended quandles

Now we use the above extensions of quandles to color twist-spun knots.

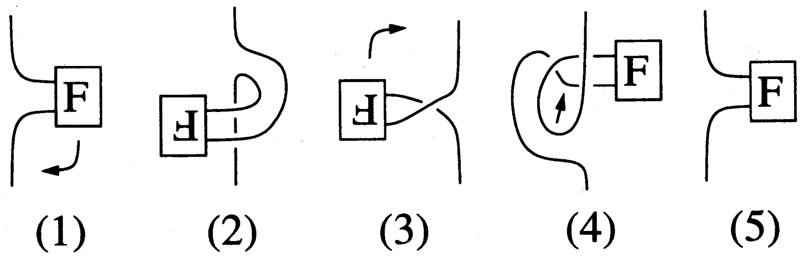

Figure 5: The general method of twist spinning

Example 6.3 *1. The $2v$-twist spun trefoil is non-trivially colorable by the quandle $R_3(v)$.*

2. The $3u$-twist spun trefoil is non-trivially colorable by the quandle $QS_4(u)$.

3. The $2v$-twist spun figure 8 knot is non-trivially colorable by $R_5(v)$.

Proof. The schematic diagram indicated in Fig. 5 illustrates Satoh's method [36] for obtaining the twist spun knot from a $(1-1)$-tangle F whose closure is a given classical knot K.

From left to right in the diagram, a movie of one full twist of a tangle is depicted. After repeating k full twists, the tangle is identified with the original one to form the k-twist spun knot of K. Reidemeister moves performed in the course of the isotopy correspond to critical or singular points on the projection, see [11], for example, for details. In particular, triple points on the projection correspond to type III moves, and they appear when the tangle F goes over and under the arc of axis, in steps between (1) and (2), (3) and (4) in the figure. The quandle colors assigned to F changes when F goes under the axis, between (3) and (4), and every quandle element b assigned to an arc in F changes to the quandle element $b * a$ if the arc of axis is colored by a.

First we consider even twist spun trefoils. If we color the arcs of the trefoil as in the top left of Fig. 6 with color $a_{j,k}$ on the main arc of the axis of rotation, and color $b_{m,p}$ on the right, then such a coloring extends to the entire trefoil if $j + k = m + p$. After each pair of twists the indices a, b, and c return to the arcs of the diagram, but the subscripts $m, p, k - p$ and $j + p$ are incremented to $k + j + m$, $p - k - j$, $k + m$, and $p - k$ respectively. After $2v$ full twists, the colors on the b-arc and the c-arc become $b_{m+v(j+k),\ p-v(k+j)}$ and $c_{(k-p)+v(k+j),\ (p+j)-v(p+m)}$ as indicated in the figure.

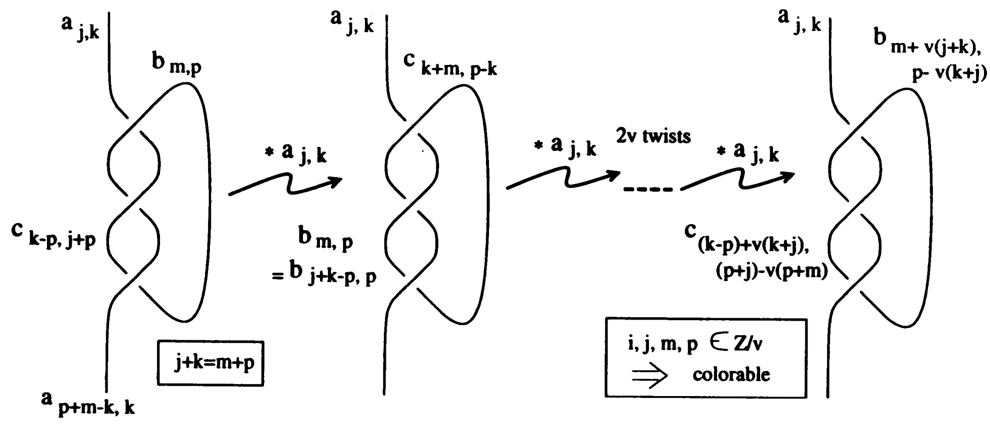

Figure 6: The $2v$-twist spun trefoil

(The subscripts in the figure are subjected to the identity $j + k = m + p$ to obtain this result.) Thus the extension colors the $2v$-twist spun trefoil if $v \equiv 0$.

Next we consider a coloring of the trefoil by QS_4. We label the elements of $(QS)_4(u)$ as follows:

$$[0,j,k,\ell] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & x^j & 0 \\ 0 & 0 & 0 & x^k \\ 0 & x^\ell & 0 & 0 \end{pmatrix}, \quad [i,0,k,\ell] = \begin{pmatrix} 0 & 0 & 0 & x^i \\ 0 & 1 & 0 & 0 \\ x^k & 0 & 0 & 0 \\ 0 & 0 & x^\ell & 0 \end{pmatrix}$$

$$[i,j,0,\ell] = \begin{pmatrix} 0 & x^i & 0 & 0 \\ 0 & 0 & 0 & x^j \\ 0 & 0 & 1 & 0 \\ x^\ell & 0 & 0 & 0 \end{pmatrix}, \quad [i,j,k,0] = \begin{pmatrix} 0 & 0 & x^i & 0 \\ x^j & 0 & 0 & 0 \\ 0 & x^k & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

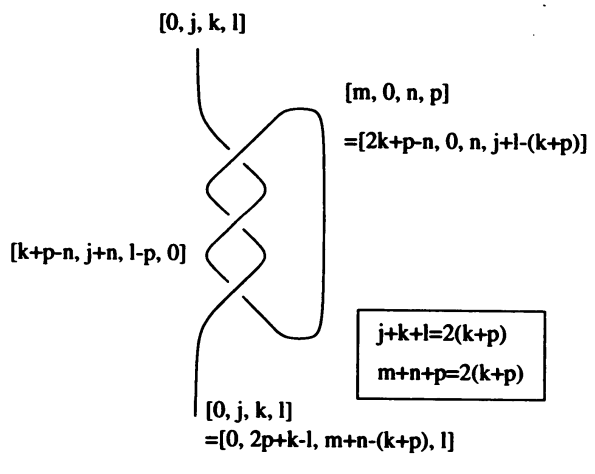

Figure 7: Coloring the trefoil with $\widetilde{(QS)}_4$

In the illustration of Fig. 7, an extension to a coloring by $(QS)_4(u)$ is indicated. The main arc is colored $[0, j, k, \ell]$ and the right hand arc is colored $[m, 0, n, p]$. These colors induce the color

$[k-n+p, j+n, \ell-p, 0]$ on the remaining arc of the diagram. A sufficient condition for this to be a coloring of trefoil is that $j+k+\ell = m+n+p = 2(k+p)$.

We follow the coloring of the right hand arc (colored $[m,0,n,p]$) for 6 twists.

$$\begin{aligned}
[m,\ 0,\ n,\ p] &\overset{*[0,j,k,\ell]}{\longrightarrow} [\ell+m,\ k-\ell+p,\ 0,\ n-k] \\
&\overset{*[0,j,k,\ell]}{\longrightarrow} [j+\ell+m,\ n-k-\ell,\ k+p-j,\ 0] \\
&\overset{*[0,j,k,\ell]}{\longrightarrow} [j+k+\ell+m,\ 0,\ -j-k-\ell+n,\ p] \\
&\overset{*[0,j,k,\ell]}{\longrightarrow} [j+k+2\ell+m,\ k-\ell+p,\ 0,\ n-j-2k-\ell] \\
&\overset{*[0,j,k,\ell]}{\longrightarrow} [2j+k+2\ell+m,\ -j-2k-2\ell+n,\ -j+k+p,\ 0] \\
&\overset{*[0,j,k,\ell]}{\longrightarrow} [2(j+k+l)+m,\ 0,\ n-2(j+k+\ell),\ p]
\end{aligned}$$

Observe that the 3-twist spun trefoil colors non-trivially with $QS_4(2)$, since $2(k+p) = 0 = (j+k+\ell)$ in this case. The result follows by induction.

A similar calculation applies to the figure 8 knot and R_5. We leave the details to the reader. ■

7 Variations of cocycle knot invariants

The following variations of cocycle knot invariants for classical knots have been considered.

- For a link $L = K_1 \cup \cdots \cup K_n$, let $\mathcal{T}_i$, $i = 1, \ldots, n$, be the set of crossings at which the under-arcs belong to the component K_i. Then it was observed [4] that $\vec{\Phi}(K) = (\sum_{\mathcal{C}} \prod_{\tau \in \mathcal{T}_i} B(\tau, \mathcal{C}))_{i=1}^{n}$ is a link invariant, strictly stronger than the single state-sum.

- Lopes [28] observed that the family $\{\prod_\tau B(\tau, \mathcal{C})\}_{\mathcal{C} \in \mathrm{Col}}$ is a knot invariant, without taking summation. Here, Col denotes the set of colorings. In particular, infinite quandles can be used for coloring in this case. He also defined for links $L = K_1 \cup \cdots \cup K_n$ the vector version $(\{\prod_{\tau \in \mathcal{T}_i} B(\tau, \mathcal{C})\}_{\mathcal{C} \in \mathrm{Col}})_{i=1}^{n}$.

Now we combine these variations to define the following generalized cocycle invariant.

Definition 7.1 Let X be a quandle, $\phi \in Z^2_{\rm Q}(X; A)$, where A is an abelian group, $\mathcal{C}$ a coloring of L by X, and $B(\tau, \mathcal{C})$ the Boltzmann weight at a crossing τ for a coloring $\mathcal{C}$. Let $L = K_1 \cup \cdots \cup K_r$ be a link and $\mathcal{T}_i$, $i = 1, \ldots, r$, be the set of crossings of L such that the under-arcs belong to K_i. Define

$$\vec{\Psi}(L) = \left\{ \left(\prod_{\tau \in \mathcal{T}_1} B(\tau, \mathcal{C}), \ldots, \prod_{\tau \in \mathcal{T}_r} B(\tau, \mathcal{C}) \right) \right\}_{\mathcal{C} \in \mathrm{Col}}.$$

This version of a family of vectors is potentially stronger than Lopes's version of a vector of families. For example, the two distinct families of vectors $\{(1,t),(t,1)\}$ and $\{(1,1),(t,t)\}$ give rise to the same vector of families $(\{1,t\},\{1,t\})$. As examples, we evaluate the invariants for Whitehead link and Borromean rings, using extension cocycles constructed in Section 5. We use the coefficient group $A = \mathbb{Z}_q = \{t^n | n = 0, 1, \ldots, q-1\}$ for a positive integer q.

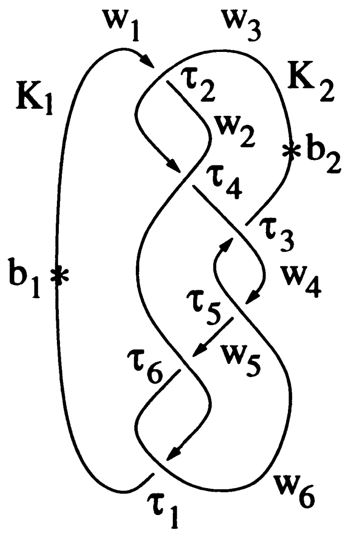

Figure 8: The Whitehead link

Example 7.2 Let $X = W_m = \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ or $X = \dot{U}_m = \mathbb{Z}_{q^m}[T, T^{-1}]/(T-1+q)$, and L the Whitehead link. Then the generalized cocycle invariant is

$$\vec{\Psi}(L) = \begin{cases} \{\underbrace{(1,1), \ldots, (1,1)}_{q^{2m} \text{ copies}}\} & \text{for } m = 1, 2, \\ \{\underbrace{(t^n, t^{-n}), \ldots, (t^n, t^{-n})}_{q^{m+2} \text{ copies}}\}_{n \in \{0,1,\ldots,q-1\}} & \text{for } m \geq 3. \end{cases}$$

Consequently,

$$\vec{\Phi}(L) = \begin{cases} (q^{2m}, q^{2m}) & \text{for } m = 1, 2, \\ (q^{m+2}(t^{q-1} + \cdots + t + 1), q^{m+2}(t^{q-1} + \cdots + t + 1)) & \text{for } m \geq 3. \end{cases}$$

Proof. Let $X = \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$. The case for $X = \mathbb{Z}_{q^m}[T, T^{-1}]/(T-1+q)$ is similar. Pick base points b_1 and b_2 on the components K_1 and K_2, respectively, of the Whitehead link $L = K_1 \cup K_2$ as depicted in Fig. 8, and trace each component in the given orientation of the link. The colors (elements of X) assigned to the arcs that appear in this order are w_1, w_2 for K_1, and $w_3, \ldots, w_6$ for K_2 as depicted. The crossing at the initial point of the arc colored by w_i is defined to be τ_i. First we determine the set of colorings: For $m \geq 3$, for two elements w_1, $w_3 \in X$ assigned to the top two arcs of the Whitehead link L as shown in Fig. 8, there is a coloring of L by X which restricts to the given w_1, w_3 if and only if

$$w_3 - w_1 \equiv 0 \pmod{(1-T)^{m-3}} \text{ for } w_1, w_3 \in \mathbb{Z}_q[T, T^{-1}]/(1-T)^m.$$

For $m = 1, 2$, there is such a coloring for any $w_1, w_3 \in X$. This can be computed as follows.

Represent the elements of $X = \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ by $a = a_{m-1}(1-T)^{m-1} + \cdots + a_1(1-T) + a_0$, where $a_j \in \mathbb{Z}_q$. Note that $(1-(1-T))(1+(1-T)+\cdots+(1-T)^{m-1}) = 1$ in X, so $T^{-1} = 1 + (1-T) + \cdots + (1-T)^{m-1}$. Note also that $a \mathbin{\bar{*}} b = T^{-1}a + (1-T^{-1})b$.

We have the following calculations for each arc:

$$\begin{aligned} w_2 &= w_1 * w_3 = w_1 + (1-T)(w_3 - w_1) \\ w_4 &= w_3 * w_2 = w_3 + (1-T)(w_2 - w_3) = (w_3 - w_1)(1-T)^2 - (w_3 - w_1)(1-T) + w_3 \\ w_6 &= w_3 \bar{*} w_4 = T^{-1} w_3 + (1 - T^{-1}) w_4 = (w_3 - w_1)(1-T)^2 + w_3 \\ w_5 &= w_4 * w_6 = w_4 + (1-T)(w_6 - w_4) = 2(w_3 - w_1)(1-T)^2 - (w_3 - w_1)(1-T) + w_3. \end{aligned}$$

These relations are obtained using the top four crossings (τ_2, τ_4, τ_3, and τ_5, respectively). The bottom two crossings (τ_6 and τ_1) of the link give rise to the next two relations. The first relation is $w_6 * w_2 = w_5$ for the second bottom crossing, giving $(w_1 - w_3)(1-T)^3 \equiv 0 \pmod{(1-T)^m}$. The second relation that corresponds to the bottom crossing is $w_1 * w_6 = w_2$ giving $(w_3 - w_1)(1-T)^3 \equiv 0$ $(\text{mod } (1-T)^m)$, as claimed above.

Now we determine the contribution to the invariant for each coloring. Recall that $\phi(w_1, w_3) = [s(w_1) * s(w_3) - s(w_1 * w_3)]/(1-T)^m$. Since $(w_3 - w_1)(1-T)^3 \equiv 0 \pmod{(1-T)^m}$, we see that the contribution is

$$\begin{aligned} &\phi(w_1, w_3) - \phi(w_1, w_6) \\ &\quad = [s(w_1) * s(w_3) - s(w_1 * w_3)]/(1-T)^m \\ &\qquad -[s(w_1) * s(w_3) + (w_3 - w_1)(1-T)^3 - s(w_1 * w_3)]/(1-T)^m \\ &\quad = -(w_3 - w_1)(1-T)^3/(1-T)^m \pmod{q}, \end{aligned}$$

for the first component, and for the second component, computations show that

$$\phi(w_3, w_2) - \phi(w_6, w_2) + \phi(w_6, w_4) + \phi(w_4, w_6) = (w_3 - w_1)(1-T)^3/(1-T)^m \pmod{q}.$$

For $m = 1, 2$ the contributions for the first and the second component are both 0, and we have q^m choices for both w_1 and w_3, therefore $\vec{\Psi}(L) = \underbrace{((1,1), \ldots, (1,1))}_{q^{2m} \text{ copies}}$ and $\vec{\Phi}(L) = (q^{2m}, q^{2m})$.

For $m \geq 3$, if w_1 and w_3 color L, then $(w_3 - w_1)(1-T)^3$ is 0 as an element of X, so that $w_3 - w_1$ is uniquely written as $w_3 - w_1 = k(1-T)^{m-3}$, where $k = k_0 + k_1(1-T) + k_2(1-T)^2$, and $k_0, k_1, k_2 \in \{0, 1, \ldots, q-1\}$. Then

$$(w_3 - w_1)(1-T)^3 = k(1-T)^m = (k_0 + k_1(1-T) + k_2(1-T)^2)(1-T)^m = k_0(1-T)^m \in E.$$

Thus the contribution to the invariant for the first and second components are t^{-k_0} and t^{k_0}, respectively.

To find the number of colorings conrtibuting to t^{-k_0} and t^{k_0}, fix k_0. We have q^m choices for w_1 and q^2 choices for k. Then w_3 is uniquely determined by $w_3 = w_1 + k(1-T)^{m-3}$. In total, the contribution is $q^m q^2 = q^{m+2}$ for each t^{-k_0} and t^{k_0}. Setting $n = -k_0$ we obtain the result. ■

Example 7.3 Let $X = \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ or $X = \mathbb{Z}_{q^m}[T, T^{-1}]/(T-1+q)$, and L the Borromean rings. Then the generalized cocycle invariant is

$$\vec{\Psi}(L) = \begin{cases} \{\underbrace{(1,1,1), \ldots, (1,1,1)}_{q^{3m} \text{ copies}}\} & \text{for } m = 1, \\ \{\underbrace{(t^{-k_0}, t^{-\ell_0}, t^{k_0+\ell_0}), \ldots, (t^{-k_0}, t^{-\ell_0}, t^{k_0+\ell_0})}_{q^{m+2} \text{ copies}}\}_{k_0, \ell_0 \in \{0,1,\ldots,q-1\}} & \text{for } m \geq 2. \end{cases}$$

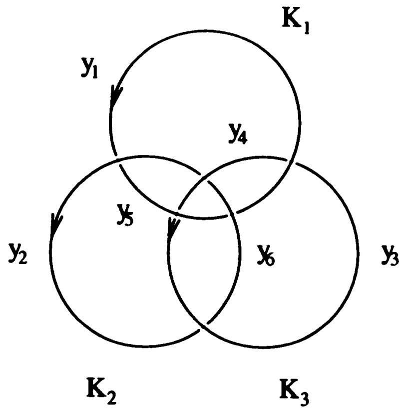

Figure 9: Borromean Rings

Consequently,

$$\vec{\Phi}(L) = \begin{cases} (q^{3m}, q^{3m}, q^{3m}) & \text{for } m = 1, \\ (q^{m+2}(t^{q-1} + \cdots + t + 1), q^{m+2}(t^{q-1} + \cdots + t + 1), q^{m+2}(t^{q-1} + \cdots + t + 1)) & \text{for } m \geq 2. \end{cases}$$

Proof. Let $X = \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ and let L be the Borromean rings as depicted in Fig. 9. The case for $X = \mathbb{Z}_{q^m}[T, T^{-1}]/(T - 1 + q)$ is similar. Calculations are similar to the preceding example and we give a sketch. First we determine the set of colorings: For three elements $y_1, y_2, y_3 \in X$ assigned to each outer arc in the diagram of L, there is a coloring of L by X which restricts to the given y_1, y_2, y_3 if and only if

$$(y_2 - y_3)(1-T)^2 \equiv 0, \quad (y_1 - y_2)(1-T)^2 \equiv 0 \quad \text{and} \quad (y_3 - y_1)(1-T)^2 \equiv 0.$$

The outer three crossings are used to describe y_4, y_5, y_6 in terms of the rest, and the inner three crossings give the above relations.

Contributions to the invariant are computed as follows. The contribution for the first component of L colored by y_1 is $\phi(y_1, y_2) - \phi(y_1, y_2 * y_3) = -(y_3 - y_2)(1-T)^2/(1-T)^m \pmod q$. For $m = 1$, the contribution is trivial, and the total number of colorings is q^{3m}. For $m \geq 2$, $(y_3 - y_2)(1-T)^2$ is divisible by $(1-T)^m$, so $y_3 - y_2$ is uniquely written as $y_3 - y_2 = k(1-T)^{m-2}$, where $k = k_0 + k_1(1-T)$ and $k_0, k_1 \in \{0, 1, \ldots, q-1\}$. So

$$(y_3 - y_2)(1-T)^2 = k(1-T)^m = (k_0 + k_1 q)(1-T)^m = k_0(1-T)^m,$$

and the first component contributes t^{-k_0} to the invariant. For the second component of L colored by y_2, similar calculations as above give the contribution $\phi(y_2, y_3) - \phi(y_2, y_3 * y_1) = -(y_1 - y_3)(1-T)^2$, which is divisible by $(1-T)^m$ so $y_1 - y_3 = (\ell_0 + \ell_1(1-T))(1-T)^{m-2}$ and therefore $-(y_1 - y_3)(1-T)^2 = -\ell_0(1-T)^m$. The we obtain $y_2 - y_1 = -[(k_0 + \ell_0) + (k_1 + \ell_1)(1-T)](1-T)^{m-2}$, so that the third component contributes $t^{k_0 + \ell_0}$. Finally, the contribution to the invariant is the vector $(t^{-k_0}, t^{-\ell_0}, t^{k_0 + \ell_0})$, where the entries correspond to the components K_1, K_2, K_3, respectively. The result follows. ■

In the above examples, we see that the cocycle invariant is non-trivial when the given link is colored by $X = \mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ but not by $E = \mathbb{Z}_q[T, T^{-1}]/(1-T)^{m+1}$, and the descrepancy in extending the coloring contributes to the invariant. This is the case in general, as proved in [4] for the knot case. We rephrase the theorem in our situation and include a similar proof for reader's convenience.

Let K be a classical or virtual knot or link. Let $\mathcal{C}$ be a coloring of K by X. Let E be a quandle with a surjective homomorphism $p : E \to X$. If there is a coloring $\mathcal{C}'$ of K by E such that for every arc a of K, it holds that $p(\mathcal{C}'(a)) = \mathcal{C}(a)$, then $\mathcal{C}'$ is called an *extension* of $\mathcal{C}$.

Theorem 7.4 *Let*

$$\vec{\Psi}(L) = \left\{ \left(\prod_{\tau \in \mathcal{T}_1} B(\tau, \mathcal{C}), \ldots, \prod_{\tau \in \mathcal{T}_r} B(\tau, \mathcal{C}) \right) \right\}_{\mathcal{C} \in \mathrm{Col}}$$

be the generalized cocycle invariant of a link $L = K_1 \cup \cdots \cup K_n$ with a quandle X and a cocycle $\phi \in Z^2_{\mathrm{Q}}(X; A)$ for an abelian group A. Then $\left(\prod_{\tau \in \mathcal{T}_1} B(\tau, \mathcal{C}), \ldots, \prod_{\tau \in \mathcal{T}_r} B(\tau, \mathcal{C})\right)$ is a vector with every entry 1 for a coloring $\mathcal{C}$ if and only if the coloring $\mathcal{C}$ extends to a coloring of L by $E(X, A, \phi)$.

Proof. Let $\mathcal{C}$ be a coloring whose contribution to $\vec{\Psi}(L)$ is $(1, \ldots, 1)$. Fix this coloring in what follows. Pick a base point b_0 on a component K_i of L. Let $x \in X$ be the color on the arc α_0 containing b_0. Let α_i, $i = 1, \ldots, n$, be the set of arcs that appear in this order when the diagram K is traced in the given orientation of K_i, starting from b_0. Pick an element $a \in A$ and give a color (a, x) on α_0, so that we define a coloring $\mathcal{C}'$ by E on α_0 by $\mathcal{C}'(\alpha_0) = (a, x) \in E$. We try to extend it to the entire diagram by traveling the diagram from b_0 along the arcs α_i, $i = 1, \ldots, n$, in this order, by induction.

Suppose $\mathcal{C}'(\alpha_i)$ is defined for $0 \leq i < k$. Define $\mathcal{C}'(\alpha_{k+1})$ as follows. Suppose that the crossing τ_k separating α_k and α_{k+1} is positive, and the over-arc at τ_k is α_j. Let $\mathcal{C}'(\alpha_k) = (a, x)$ and $\mathcal{C}(\alpha_j) = y \in X$. Then we have $\mathcal{C}(\alpha_{k+1}) = x * y \in X$. Define $\mathcal{C}'(\alpha_{k+1}) = (a\phi(x, y), x * y)$ in this case.

Suppose that the crossing τ_k is negative. Let $\mathcal{C}'(\alpha_k) = (a, x)$ and $\mathcal{C}(\alpha_j) = y \in X$. Then if $\mathcal{C}(\alpha_{k+1}) = z$, then we have $z * y = x$. Define $\mathcal{C}'(\alpha_{k+1}) = (a\phi(z, y)^{-1}, z)$ in this case.

Define $\mathcal{C}'(\alpha_i)$ inductively for all $i = 0, \ldots, n$. Regard α_0 as α_{n+1}, and repeat the above construction at the last crossing τ_n to come back to α_0. By the construction we have $\mathcal{C}'(\alpha_{n+1}) = (a \prod_\tau B(\tau, \mathcal{C}), \mathcal{C}(\alpha_0))$, where $\prod_\tau B(\tau, \mathcal{C})$ is the state-sum contribution (the product of Boltzmann weights over all crossings) of $\mathcal{C}$. This contribution is equal to 1 by the assumption that $\prod_\tau B(\tau, \mathcal{C}) = 1$, and we have a well-defined coloring $\mathcal{C}'$. Hence this color extends to $E(X, A, \phi)$.

Conversely, if a coloring $\mathcal{C}$ by X extends to a coloring by $E(X, A, \phi)$, then from the above argument, we have that $(a, x) = (a \prod_\tau B(\tau, \mathcal{C}), x)$, if (a, x) is the color on the base point b_0. Hence $\prod_\tau B(\tau, \mathcal{C}) = 1$. ■

8 Cocycle invariants and Alexander matrices

In this section we point out relations of the cocycle invariants to Alexander matrices. We examine closely Example 7.2 given in Section 7 from this new point of view.

Let $B_{D_L} = \sum_{i=1}^n B_i$ be an $(n \times n)$-matrix where B_i is the $(n \times n)$-matrix corresponding to each crossing point τ_i such that (k_i, i) entry is T^{ϵ_i}, (ℓ_i, i) entry is $1 - T^{\epsilon_i}$ and otherwise is 0. Here ϵ_i means the sign of the crossing point τ_i. Set $A_{D_L} = B_{D_L} - E_n$, where E_n denotes the n-dimensional identity matrix, then from the definitions it follows that A_{D_L} is an Alexander matrix. Recall that

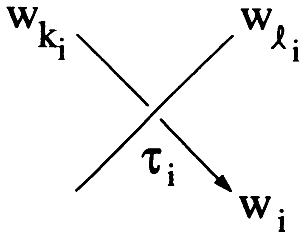

Figure 10: Labeling a crossing

a coloring is a function $C : R \to X$, where R is the set of over-arcs in the diagram and X is a fixed Alexander quandle Λ/J for an ideal J. A coloring which assigns y_i to an arc a_i ($C(a_i) = y_i$) is represented by the vector $\vec{y} = (y_1, \cdots, y_n)$ satisfying $\vec{y}A_{D_L}^{(X)} = \vec{0}$. These descriptions are given in [19] to prove Theorem 2.1.

Proposition 8.1 *Let $L = K_1 \cup \cdots \cup K_r$ be a link and $X = \Lambda_q/J$ be an Alexander quandle. Suppose $E = \Lambda_{q'}/J'$ is an abelian extension of X, where q, q' are positive integers. Let $A_{D_L}^{(X)}$ (respectively $A_{D_L}^{(E)}$) be the matrix A_{D_L} regarded as a matrix over X (respectively over E). Then a coloring $\vec{y}$ of L by X contributes a non-trivial value to the invariant $\vec{\Psi}(L)$ if and only if $\vec{y}A_{D_L}^{(X)} = \vec{0}$ and $s(\vec{y})A_{D_L}^{(E)} = \vec{x} \neq \vec{0}$, where $s : X \to E$ is the natural section.*

Proof. Let $\psi : (\Lambda_q/J)^n \to (\Lambda_q/J)^n$ be the map which takes a row vector $\vec{y}$ to $\vec{y}A_{D_L}$. By Inoue's description given above, the set of all quandle colorings is equal to $\ker A_{D_L}^{(X)}$. If $\vec{y}A_{D_L}^{(X)} = \vec{0}$ and $s(\vec{y})A_{D_L}^{(E)} = \vec{x} \neq 0$, then by Theorem 7.4 we obtain that $\vec{\Psi}(L)$ is non-trivial. ∎

Next, we compute the non-trivial contributions using Alexander matrices, for extensions discussed in Section 5. Let $X = W_m = \Lambda_q/(1-T)^m$ or $X = U_m = \Lambda_{q^m}/(T-1+q)$, and $E = W_{m+1}$ or $E = U_{m+1}$ be their abelian extensions, respectively. For this purpose, we fix the following convention in numbering crossings and arcs of a given diagram.

Let $L = K_1 \cup \cdots \cup K_r$ be a link with n crossings. Pick a base point b_i on K_i, for $i = 1, \ldots, r$. Let $a_1, \cdots, a_{i_1}$ be the arcs of K_1 such that a_1 contains b_1 and they appear in this order when one traces K_1 in the given orientation of K_1 starting from b_1. Then let a_{i_1+1} be the arc of K_2 containing b_2 and $a_{i_1+2}, \cdots, a_{i_2}$ be the arcs of K_2 similarly defined from the given orientation. Repeat this process for the remaining components to obtain the arcs $a_1, \cdots, a_{i_1}, a_{i_1+1}, \cdots, a_{i_2}, a_{i_2+1}, \cdots, a_{i_{r-1}+1}, \cdots, a_{i_r} = a_n$. Let $C : R \to X$ be a coloring of L by X. Let $w_i = C(a_i)$ and τ_i be the crossing such that the outcoming under-arc is a_i for $i = 1, \ldots, n$ (see Fig. 10). This convention is used in Fig. 8.

Let $s : X \to E$ be the section defined in Section 5 respectively by

$$s\left(\sum_{j=0}^{m-1} A_j(1-T)^j \mod (1-T)^m\right) = \sum_{j=0}^{m-1} A_j(1-T)^j \mod (1-T)^{m+1} \quad \text{for } W_m, \text{ and}$$

$$s\left(\sum_{j=0}^{m-1} X_j q^j\right) = 0 \cdot q^m + \sum_{j=0}^{m-1} X_j q^j \quad \text{for } U_m.$$

For the following theorem, let $\vec{\Psi}(L)$ be the generalized cocycle invariant defined with the cocycle $\phi \in Z^2_Q(X;\mathbb{Z}_q)$ corresponding to the extension $p: E \to X$ specified above.

Proposition 8.2 *Let A_{D_L} be the Alexander matrix obtained from D_L with the above choice of order of w_i and τ_i.*

A given coloring represented by a vector $\vec{w}$ contributes a non-trivial vector to the invariant $\vec{\Psi}(L)$ if and only if $\vec{w}A^{(X)}_{D_L} = \vec{0}$ and $s(\vec{w})A^{(E)}_{D_L} = \vec{z} \neq \vec{0}$. This contribution is

$$(t^{\sum_{j=1}^{i_1} \eta(\tau_j)z_j/(1-T)^m}, \ldots, t^{\sum_{j=i_{r-1}+1}^{i_r} \eta(\tau_j)z_j/(1-T)^m}) \quad \text{for } X = W_m, \text{ and}$$

$$(t^{\sum_{j=1}^{i_1} \eta(\tau_j)z_j/q^m}, \ldots, t^{\sum_{j=i_{r-1}+1}^{i_r} \eta(\tau_j)z_j/q^m}) \quad \text{for } X = U_m, \text{ respectively}$$

where $\eta(\tau) = 1$ for a positive crossing τ and $\eta(\tau) = T$ for a negative crossing τ.

Proof. We consider the case $X = W_m$, as the other case is similar. Let $\psi : (\Lambda_q/(1-T)^m)^n \to (\Lambda_q/(1-T)^m)^n$ be the map which takes a row vector $\vec{w}$ to $\vec{w}A^{(X)}_{D_L}$. Assume that $\vec{w}A^{(X)}_{D_L} = \vec{0}$ and $s(\vec{w})A^{(E)}_{D_L} = \vec{z} \neq \vec{0}$. The contribution to the invariant at a positive crossing τ_i is given by

$$\begin{aligned} \phi(w_{k_i}, w_{\ell_i}) &= [s(w_{k_i}) * s(w_{\ell_i}) - s(w_{k_i} * w_{\ell_i})]/(1-T)^m \\ &= [s(w_{k_i}) * s(w_{\ell_i}) - s(w_i)]/(1-T)^m, \end{aligned}$$

where w_{ℓ_i} is the color on the over-arc at the crossing τ_i, and w_{k_i} is the color on the incoming under-arc at τ_i if τ_i is positive (see Fig. 10). Since $\vec{w}$ is in the kernel, $w_{k_i} * w_{\ell_i} - w_i = Tw_{k_i} + (1-T)w_{\ell_i} - w_i = 0 \bmod (1-T)^m$ and we have $[s(w_{k_i}) * s(w_{\ell_i}) - s(w_i)]/(1-T)^m = z_i/(1-T)^m$.

Suppose τ_i is negative. Then the contribution is

$$\begin{aligned} -\phi(w_i, w_{\ell_i}) &= -[s(w_i) * s(w_{\ell_i}) - s(w_i * w_{\ell_i})]/(1-T)^m \\ &= -[s(w_i) * s(w_{\ell_i}) - s(w_{k_i})]/(1-T)^m \\ &= -[Tw_i + (1-T)w_{\ell_i} - w_{k_i}]/(1-T)^m. \end{aligned}$$

On the other hand,

$$z_i = T^{-1}w_{k_i} + (1-T^{-1})w_{\ell_i} - w_i = -T^{-1}[Tw_i + (1-T)w_{\ell_i} - w_{k_i}]$$

so that the contribution is Tz_i in this case. Hence the total contribution of the invariant for the component K_r is

$$t^{\sum_{j=i_{r-1}+1}^{i_r} \eta(\tau_j)z_j/(1-T)^m},$$

where $\{z_1, \ldots, z_{i_r}\} \in K_r$. ∎

Example 8.3 We consider the Whitehead link $L = K_1 \cup K_2$ depicted in Fig. 8. Let $X = W_m$ and $E = W_{m+1}$. Use the letters w_i, $(i = 1, \ldots, 6)$ as depicted in the figure as colors assigned to the arcs, as well as generators for the Alexander matrix. Then the Alexander matrix $A_{D_L} = B_{D_L} - E_n$ with respect to the columns corresponding to $(\tau_1, \ldots, \tau_6)$ and rows corresponding to $(w_1, \ldots, w_6)$ is given by

$$A_{D_L} = \begin{pmatrix} -1 & T & 0 & 0 & 0 & 0 \\ T^{-1} & -1 & 0 & 1-T & 0 & 1-T^{-1} \\ 0 & 1-T & -1 & T & 0 & 0 \\ 0 & 0 & 1-T & -1 & T & 0 \\ 0 & 0 & 0 & 0 & -1 & T^{-1} \\ 1-T^{-1} & 0 & T & 0 & 1-T & -1 \end{pmatrix}.$$

After some row and column permutations we obtain

$$A_0 = \begin{pmatrix} -1 & 1-T & 0 & 0 & 1-T^{-1} & T^{-1} \\ 0 & -1 & 1-T & T & 0 & 0 \\ 0 & 0 & T & 1-T & -1 & 1-T^{-1} \\ 0 & 0 & 0 & -1 & T^{-1} & 0 \\ T & 0 & 0 & 0 & 0 & -1 \\ 1-T & T & -1 & 0 & 0 & 0 \end{pmatrix},$$

with respect to the columns corresponding to $(\tau_2, \tau_4, \tau_3, \tau_5, \tau_6, \tau_1)$ and rows corresponding to $(w_2, w_4, w_6, w_5, w_1, w_3)$. This permutation is performed so that we can diagonalize the first four rows and columns by column reductions to obtain

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -T & -T+T^2 & (1-T)^2 & 1-3T+2T^2 & -T^{-1}(1-T)^3 & T^{-1}(1-T)^3 \\ -1+T & -1+T-T^2 & -1-(1-T)^2 & -2+3T-2T^2 & T^{-1}(1-T)^3 & -T^{-1}(1-T)^3 \end{pmatrix}$$

The solution set

$$(w_2, w_4, w_6, w_5, w_1, w_3) A_1^{(X)} = (0,0,0,0,0,0),$$

is written by

$$\begin{aligned} w_2 &= Tw_1 + (1-T)w_3 \\ w_4 &= T(1-T)w_1 + (T + (1-T)^2)w_3 \\ w_6 &= -(1-T)^2 w_1 + (1 + (1-T)^2)w_3 \\ w_5 &= (T(1-T) - (1-T)^2)w_1 + (T + 2(1-T)^2)w_3 \\ 0 &= (w_3 - w_1)T^{-1}(1-T)^3 \end{aligned}$$

where $A_1^{(X)}$ denotes the matrix A_1 regarded as a matrix over X. The set of colorings is represented by vectors in the kernel of $A_1^{(X)}$. Specifically, the kernel is the set of vectors $\vec{w}$ with w_1 and w_3 satisfying $(1-T)^3(w_3 - w_1) = 0$ in X and w_2, w_4, w_6, w_5 determined accordingly as above. This matches the computations in Example 7.2. The contribution to the invariant is obtained by computing

$$\begin{aligned} \vec{z} &= s(\vec{w}) A_{D_L}^{(E)} \\ &= (-T^{-1}(1-T)^3(w_3 - w_1),\ 0,\ 0,\ 0,\ 0,\ T^{-1}(1-T)^3(w_3 - w_1)). \end{aligned}$$

By Proposition 8.2 the non-trivial contribution to $\vec{\Psi}(L)$ is $(t^{\sum_{j=1}^{2}\eta(\tau_j)z_j/(1-T)^3}, t^{\sum_{j=3}^{6}\eta(\tau_j)z_j/(1-T)^3}) = (t^{-s}, t^{s})$ for some s $(0 \leq s \leq q-1)$ depending on the value of $w_3 - w_1$, and this matches Example 7.2.

Finally we observe a relation to the Conway polynomial. Let $\Delta_L(T) \in \mathbb{Z}[T^{-\frac{1}{2}}, T^{\frac{1}{2}}]$ be the *Conway-normalized Alexander polynomial* [26]. In our case, let A'_{D_L} be the matrix obtained from A_{D_L} by deleting the jth column and jth row for some j, $j = 1, \ldots, n$, let $f(T) = \det(A'_{D_L}) \in \mathbb{Z}[T^1, T^{-1}]$ and μ and ν be the maximal and minimal degree of f respectively. Then $\Delta_L(T) = T^{-\frac{\mu+\nu}{2}} f(T)$. The *Conway polynomial* $\nabla_L(z) \in \mathbb{Z}[z]$ is defined by $\nabla_L(T^{-\frac{1}{2}} - T^{\frac{1}{2}}) = \Delta_L(T)$ where $z = T^{-\frac{1}{2}} - T^{\frac{1}{2}}$.

Proposition 8.4 *Let the minimal degree of* $\nabla_L(z)$ *be denoted by* min-deg$\nabla_L(z)$*, then it satisfies* min-deg$\nabla_L(z) \geq m$*, where* m *is the smallest integer such that the cocycle invariant defined from the extension of* $\mathbb{Z}_q[T, T^{-1}]/(1-T)^m$ *to* $\mathbb{Z}_q[T, T^{-1}]/(1-T)^{m+1}$ *is non-trivial.*

Proof. Assume that $\vec{y}A^{(X)}_{D_L} = \vec{0}$ and $s(\vec{y})A^{(E)}_{D_L} = \vec{x} \neq \vec{0}$. Then $\vec{y}$ contributes a non-trivial value to the invariant $\vec{\Psi}(L)$ as in Proposition 8.1. Since $\vec{x} \neq \vec{0}$ there exists i, $1 \leq i \leq n$, such that $x_i \neq 0$. Let j be an integer, $1 \leq j \leq n$, with $j \neq i$. Let $\vec{x'}$ be the vector $\vec{x}$ with the x_j entry deleted. Then there exists $\vec{y'} \neq \vec{0}$, where $\vec{y'}$ is the vector $\vec{y}$ with the jth entry deleted, such that $\vec{y'}A'^{(X)}_{D_L} = \vec{0}$. This implies that $\det A'^{(X)}_{D_L} = 0$. Hence $\det A'_{D_L} \equiv 0 \pmod{(1-T)^m}$, and we have min-deg$\nabla_L(z) \geq m$. ■

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