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A SURVEY ON EISENSTEIN SERIES IN RAMANUJAN'S LOST NOTEBOOK

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1. Introduction

In contemporary notation, the Eisenstein series $G_{2j}(\tau)$ and $E_{2j}(\tau)$ of weight $2j$ on the full modular group $\Gamma(1)$, where j is a positive integer exceeding one, are defined for $\operatorname{Im}\tau>0$ by

$$G_{2j}(\tau):=\sum_{\substack{m_1,m_2\in\mathbf{Z}\\ m_1,m_2\neq(0,0)}}(m_1\tau+m_2)^{-2j}$$

and

$$\begin{aligned} E_{2j}(\tau):=\frac{G_{2j}(\tau)}{2\zeta(2j)} &=1-\frac{4j}{B_{2j}}\sum_{k=1}^{\infty}\frac{k^{2j-1}e^{2\pi ik\tau}}{1-e^{2\pi ik\tau}}\\ &=1-\frac{4j}{B_{2j}}\sum_{r=1}^{\infty}\sigma_{2j-1}(r)q^{r}, \qquad q=e^{2\pi i\tau}, \end{aligned} \tag{1.1}$$

where $B_n, n\geq 0$, denotes the nth Bernoulli number, and where $\sigma_\nu(n)=\sum_{d|n}d^{\nu}$. The latter two representations in (1.1) can be established by using the Lipschitz summation formula or Fourier analysis. For these and other basic properties of Eisenstein series, see, for example, R. A. Rankin's text [38, Chap. 6].

In Ramanujan's notation, the three most relevant Eisenstein series are defined for $|q|<1$ by

$$P(q):=1-24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \tag{1.2}$$

$$Q(q):=1+240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}, \tag{1.3}$$

and

$$R(q):=1-504\sum_{k=1}^{\infty}\frac{k^5q^k}{1-q^k}. \tag{1.4}$$

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Thus, for $q = \exp(2\pi i\tau)$, $E_4(\tau) = Q(q)$ and $E_6(\tau) = R(q)$, which have weights 4 and 6, respectively. The function $P(q)$ is not a modular form. However, if $q = \exp(2\pi i\tau)$, with $\operatorname{Im} \tau = y$, then $P(q) - 3/(\pi y)$ satisfies the functional equation required of a modular form of weight 2 [38, p. 195]. In his notebooks [36], Ramanujan used the notations L, M, and N in place of P, Q, and R, respectively.

Ramanujan made many contributions to the theory and applications of Eisenstein series in his paper [32], [35, pp. 136–162]. Applications were made to the theory of the divisor functions σ_ν; the partition function $p(n)$, the number of ways the positive integer n can be represented as a sum of positive integers with the order of the summands irrelevant; and $r_k(n)$, the number of representations of the positive integer n as a sum of k squares. Since Ramanujan did not consider any further applications of Eisenstein series to $r_k(n)$ in his lost notebook, we shall not further discuss his work on sums of squares.

Among the most useful results on Eisenstein series established by Ramanujan in [32] are his differential equations [32, eqs. (30)], [35, p. 142]

$$q\frac{dP}{dq} = \frac{P^2(q) - Q(q)}{12}, \quad q\frac{dQ}{dq} = \frac{P(q)Q(q) - R(q)}{3}, \quad \text{and} \quad q\frac{dR}{dq} = \frac{P(q)R(q) - Q^2(q)}{2}. \tag{1.5}$$

Several results in the present paper depend upon the differential equations (1.5).

Ramanujan recorded numerous results on Eisenstein series in his notebooks as well. Chapter 15 in his second notebook contains some of the results from [32] and other results not recorded in [32]; see [2, pp. 326–333]. Chapter 21 in Ramanujan's second notebook [36], [3, pp. 454–488] is entirely devoted to Eisenstein series. Eisenstein series arise in Ramanujan's "cubic" theory of elliptic functions; see [5, pp. 105–108]. Our brief summary has been by no means exhaustive; for further results of Ramanujan on Eisenstein series, consult [2]–[5].

The present survey of results on Eisenstein series found mostly in Ramanujan's lost notebook [37] comprises entries of different natures. In Section 2, we discuss Ramanujan's formulas for the coefficients of quotients of Eisenstein series. These were communicated to G. H. Hardy from a nursing home in 1918 and greatly extend the content of their last joint paper [21], [35, pp. 310–321]. In Section 3, we examine the role of Eisenstein series in proving congruences for the partition function $p(n)$. This material is found in a manuscript of Ramanujan on the partition and tau functions first published in handwritten form in [37] and then in [10] with commentary. The results in these two sections bring us to the natural investigation of possible congruences for the coefficients of quotients of Eisenstein series upon which we briefly focus in Section 4. At various places in his lost notebook, Ramanujan gives formulas for Eisenstein series in terms of quotients of Dedekind eta-functions. Two examples are given in Section 5. In [32], Ramanujan expresses families of Eisenstein series and related series as polynomials in P, Q, and R. A page in the lost notebook [37, p. 188] is devoted to another family of series, which in this case is related to the pentagonal number theorem; see Section 6 for some of these series. On another page in the lost notebook [37, p. 211], Ramanujan cryptically relates some formulas for Eisenstein series, which yield approximations to π in a spirit not unlike that for approximations to π given in his famous paper on modular equations and approximations to π [31]. We also indicate in Section 7 how Ramanujan's ideas lead to new series representations for $1/\pi$. In the final section, we discuss integrals of Eisenstein series associated with Dirichlet characters.

Lastly, we introduce further notation that we use in the sequel. As usual, set

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

Define, after Ramanujan,

$$f(-q) := (q;q)_\infty =: e^{-2\pi i\tau/24}\eta(\tau), \qquad q=e^{2\pi i\tau}, \qquad \operatorname{Im}\tau>0, \tag{1.6}$$

where η denotes the Dedekind eta-function.

2. Formulas for the Coefficients of Quotients of Eisenstein Series

In their famous paper [20], [35, pp. 276–309], Hardy and Ramanujan found an asymptotic formula for the partition function $p(n)$, which arises from the power series coefficients of the reciprocal of the Dedekind eta-function, a modular form of weight $\frac{1}{2}$. As they indicated near the end of their paper, their methods also apply to several analogues of the partition function generated by modular forms of negative weight that are analytic in the upper half-plane. In their last published paper [21], [35, pp. 310–321], they considered a similar problem for the coefficients of modular forms of negative weight having one simple pole in a fundamental region, and, in particular, they applied their theorem to find interesting series representations for the coefficients of the reciprocal of the Eisenstein series $E_6(\tau)$. The ideas in [21] have been greatly extended by H. Poincaré [29], H. Petersson [26], [27], [28], and J. Lehner [23], [24].

While confined to nursing homes and sanitariums during his last two years in England, Ramanujan wrote several letters to Hardy about the coefficients in the power series expansions of certain quotients of Eisenstein series. These letters are photocopied in [37, pp. 97–126] and set to print with commentary in the book by Berndt and Rankin [11, pp. 175–191]. The letters contain many formulas for the coefficients of quotients of Eisenstein series not examined by Hardy and Ramanujan in [21]. Many of Ramanujan's claims do not fall under the purview of the main theorem in [21]. Ramanujan obviously wanted another example to be included in their paper [21], for in his letter of 28 June 1918 [11, pp. 182–183], he wrote, "I am sending you the analogous results in case of g_2. Please mention them in the paper without proof. After all we have got only two neat examples to offer, viz. g_2 and g_3. So please don't omit the results." This letter was evidently written after galley proofs for [21] were printed, because Ramanujan's request went unheeded. The functions g_2 and g_3 are the familiar invariants in the theory of elliptic functions and are constant multiples of the Eisenstein series $E_4(\tau)$ and $E_6(\tau)$, respectively.

Because the example from Hardy and Ramanujan's paper [21] is necessary for us in our examination of the deepest result from Ramanujan's letters, we state it below.

Theorem 2.1. *Define the coefficients p_n by*

$$\frac{1}{R(q^2)} =: \sum_{n=0}^{\infty} p_n q^{2n},$$

where, say, $|q| < q_0 < 1$. *Let*

$$\mu = 2^a \prod_{j=1}^{r} p_j^{a_j}, \tag{2.1}$$

where $a = 0$ *or* 1, p_j *is a prime of the form* $4m+1$, *and* a_j *is a nonnegative integer,* $1 \leq j \leq r$. *Then, for* $n \geq 0$,

$$p_n = \sum_{(\mu)} T_\mu(n),$$

where μ *runs over all integers of the form* (2.1), *and where*

$$T_1(n) = \frac{2}{Q^2(e^{-2\pi})} e^{2n\pi}, \tag{2.2}$$

$$T_2(n) = \frac{2}{Q^2(e^{-2\pi})} \frac{(-1)^n}{2^4} e^{n\pi}, \tag{2.3}$$

and, for $\mu > 2$,

$$T_\mu(n) = \frac{2}{Q^2(e^{-2\pi})} \frac{e^{2n\pi/\mu}}{\mu^4} \sum_{c,d} 2\cos\left((ac+bd)\frac{2\pi n}{\mu} + 8\tan^{-1}\frac{c}{d}\right), \tag{2.4}$$

where the sum is over all pairs (c, d), *where* (c, d) *is a distinct solution to* $\mu = c^2 + d^2$ *and* (a, b) *is any solution to* $ad - bc = 1$. *Also, distinct solutions* (c, d) *to* $\mu = c^2 + d^2$ *give rise to distinct terms in the sum in* (2.4).

All of Ramanujan's formulas for the coefficients of quotients of Eisenstein series were established in two papers by Berndt and Bialek [6] and by Berndt, Bialek, and Yee [7]. The results proved in the former paper require only a mild extension of Hardy and Ramanujan's principal theorem. However, those in the latter paper are more difficult to prove, and not only did the theorem of Hardy and Ramanujan [21] needed to be extended, but the theorems of Poincaré, Petersson, and Lehner needed to be extended to cover double poles.

As one can see from Theorem 2.1, the formulas for these coefficients have a completely different shape from those arising from modular forms analytic in the upper half-plane. Moreover, these series are very rapidly convergent, more so than those arising from modular forms analytic in the upper half-plane, so that truncating a series, even with a small number of terms, provides a remarkable approximation. This is really very surprising, for when one examines the formulas for these coefficients, one would never guess how such amazingly accurate approximations could be obtained from just two terms.

Now define

$$B(q) := 1 + 24 \sum_{k=1}^{\infty} \frac{(2k-1)q^{2k-1}}{1 - q^{2k-1}}, \qquad |q| < 1.$$

It is not difficult to see that $B(q)$ is the unique modular form of weight 2 with multiplier system identically equal to 1 on the modular group $\Gamma_0(2)$. It is also easy to show that $B(q)$ is related to $P(q)$ by the simple formula

$$B(q) = 2P(q^2) - P(q).$$

We now state one of our principal theorems from [7].

Theorem 2.2. *Define the coefficients* b'_n *by*

$$\frac{1}{B^2(q^2)} =: \sum_{n=0}^{\infty} b'_n q^{2n},$$

where, say, $|q| < q_0 < 1$. *Then,*

$$b'_n = 18 \sum_{(\mu_e)} \left(n + \frac{3\mu_e}{2\pi} \right) T_{\mu_e}(n), \tag{2.5}$$

where the sum is over all even integers μ_e *of the form* (2.1), *and where* $T_{\mu_e}(n)$ *is defined by* (2.2)-(2.4).

Using *Mathematica*, we calculated $b'_n, 1 \le n \le 10$, and the first two terms in (2.5). The accuracy is remarkable.

n	b'_n	$18\left((n+\frac{3}{\pi})T_2(n) + (n+\frac{15}{\pi})T_{10}(n)\right)$
1	-48	-48.001187
2	$1,680$	$1,679.997897$
3	$-52,032$	$-52,031.997988$
4	$1,508,496$	$1,508,496.002778$
5	$-41,952,672$	$-41,952,671.998915$
6	$1,133,840,832$	$1,133,840,831.996875$
7	$-30,010,418,304$	$-30,010,418,304.008563$
8	$781,761,426,576$	$781,761,426,576.003783$
9	$-20,110,673,188,848$	$-20,110,673,188,847.986981$
10	$512,123,093,263,584$	$512,123,093,263,584.006307$

3. Eisenstein Series and Partitions

Ramanujan utilized Eisenstein series to establish congruences for the partition function $p(n)$. Among his most famous identities connected with partitions are the following four identities:

$$\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2} = q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}, \tag{3.1}$$

$$1 - 5\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^n}{1-q^n} = \frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}, \tag{3.2}$$

$$\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) q^n \frac{1+q^n}{(1-q^n)^3} = q(q;q)_\infty^3(q^7;q^7)_\infty^3 + 8q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}, \tag{3.3}$$

and

$$8 - 7\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{n^2q^n}{1-q^n} = 49q(q;q)_\infty^3(q^7;q^7)_\infty^3 + 8\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}, \tag{3.4}$$

where $|q| < 1$ and $\left(\frac{n}{p}\right)$ denotes the Legendre symbol. Identity (3.1) can be utilized to prove Ramanujan's celebrated congruence $p(5n+4) \equiv 0 \pmod 5$, while (3.3) can be employed to establish Ramanujan's equally famous congruence $p(7n+5) \equiv 0 \pmod 7$. The identities in (3.2) and (3.4) are Eisenstein series on the congruence subgroups $\Gamma_0(5)$ and $\Gamma_0(7)$, respectively. These four identities were first recorded in an unpublished manuscript of Ramanujan on the partition and tau functions, first published in handwritten form with Ramanujan's lost notebook [37, pp. 135–177]. Equivalent forms of (3.1) and (3.3) are offered by Ramanujan in his first paper on congruences for $p(n)$ [33], [35, pp. 210–213]. Hardy extracted some of the material from this unpublished manuscript for Ramanujan's posthumously published paper [34], [35, pp. 232–238]. A typed version of Ramanujan's unpublished manuscript, together with proofs and commentary, has been prepared by Berndt and K. Ono [10]. The latter paper and the new edition of Ramanujan's *Collected Papers* [35, pp. 372–375] contain several references to proofs of (3.1)–(3.4). H. H. Chan [18] has shown the equivalence of (3.1) and (3.2), as well as the equivalence of (3.3) and (3.4). For proofs of several congruences for $p(n)$ using Eisenstein series, see [34] and [10].

4. Congruences for the Coefficients of Quotients of Eisenstein Series

In calculating the coefficients of the quotients of the Eisenstein series which appear in [6] and [12], and which we briefly discussed in Section 2, we noticed that for some quotients the coefficients in certain arithmetic progressions are divisible by prime powers, usually a power of 3. In view of Ramanujan's famous congruences for $p(n)$, it seemed natural for us to systematically investigate congruences of this type for Eisenstein series. In some cases, it was very easy to establish our observations, but in other cases, the task was considerably more difficult. We summarize what we have accomplished [14] in the following table. For each quotient, set $F(q) = \sum_{n=0}^{\infty} a_n q^n$.

$F(q)$	$n \equiv 2 \pmod 3$	$n \equiv 4 \pmod 8$
$1/P(q)$	$a_n \equiv 0 \pmod{3^4}$	
$1/Q(q)$	$a_n \equiv 0 \pmod{3^2}$	
$1/R(q)$	$a_n \equiv 0 \pmod{3^3}$	$a_n \equiv 0 \pmod{7^2}$
$P(q)/Q(q)$	$a_n \equiv 0 \pmod{3^3}$	
$P(q)/R(q)$	$\alpha_n \equiv 0 \pmod{3^2}$	$a_n \equiv 0 \pmod 7$
$Q(q)/R(q)$	$\alpha_n \equiv 0 \pmod{3^3}$	
$P^2(q)/R(q)$	$a_n \equiv 0 \pmod{3^5}$	

5. Representations of Eisenstein Series as Quotients of Dedekind Eta-functions

In Section 3, we offered some identities for Eisenstein series that are useful for the study of $p(n)$. The identities we present in this section are examples of those found primarily on pages 44, 50, 51, and 53 in Ramanujan's lost notebook [37] and were first proved by S. Raghavan and S. S. Rangachari [30]. Since they used the theory of modular forms, it seemed desirable to construct proofs in the spirit of Ramanujan to gain a better insight into how Ramanujan originally discovered them and to also seek a better understanding of the identities themselves. To that end, Berndt, Chan, J. Sohn, and S. H. Son [9] found proofs which depend *only* on material found in Ramanujan's notebooks [36]. All these identities for Eisenstein series are connected with modular equations of either degree 5 or degree 7. Z.-G. Liu [25] has recently constructed proofs of Ramanujan's septic identities and some new septic identities as well using the theory of elliptic functions, but employing methods from complex analysis. The methods in [9] do not use complex analysis.

We record only two examples of Ramanujan's identities involving Eisenstein series. See the paper by Berndt, Chan, Sohn, and Son [9] for further results and proofs.

Theorem 5.1 (p. 50). *For $Q(q)$ and $f(-q)$ defined by* (1.3) *and* (1.6), *respectively,*

$$Q(q)=\frac{f^{10}(-q)}{f^{2}(-q^5)}+250qf^4(-q)f^4(-q^5)+3125q^2\frac{f^{10}(-q^5)}{f^2(-q)}$$

and

$$Q(q^5)=\frac{f^{10}(-q)}{f^{2}(-q^5)}+10qf^4(-q)f^4(-q^5)+5q^2\frac{f^{10}(-q^5)}{f^2(-q)}.$$

6. Eisenstein Series and a Series Related to the Pentagonal Number Theorem

Page 188 of Ramanujan's lost notebook, in the pagination of [37], is devoted to the series

$$T_{2k}:=T_{2k}(q):=1+\sum_{n=1}^{\infty}(-1)^n\left\{(6n-1)^{2k}q^{n(3n-1)/2}+(6n+1)^{2k}q^{n(3n+1)/2}\right\},\quad |q|<1.$$

Note that the exponents $n(3n\pm 1)/2$ are the generalized pentagonal numbers. Ramanujan recorded formulas for T_{2k}, $k=1,2,\dots,6$, in terms of the Eisenstein series, P,Q, and R. The first three are given by

$$\text{(i)}\qquad \frac{T_2(q)}{(q;q)_\infty}=P,$$

$$\text{(ii)}\qquad \frac{T_4(q)}{(q;q)_\infty}=3P^2-2Q,$$

$$\text{(iii)}\qquad \frac{T_6(q)}{(q;q)_\infty}=15P^3-30PQ+16R.$$

Ramanujan's work on this page can be considered as a continuation of his study of representing certain kinds of series as polynomials in Eisenstein series in [32], [35, pp. 136–162].

The first formula, (i), has an interesting arithmetical interpretation. For $n\geq 1$, let $\sigma(n)=\sum_{d|n}d$, and define $\sigma(0)=-\frac{1}{24}$. Let n denote a nonnegative integer. Then

$$-24\sum_{j+k(3k\pm1)/2=n}(-1)^k\sigma(j)=\begin{cases}(-1)^r(6r-1)^2, & \text{if } n=r(3r-1)/2,\\ (-1)^r(6r+1)^2, & \text{if } n=r(3r+1)/2,\\ 0, & \text{otherwise,}\end{cases}\tag{6.1}$$

where the sum is over all nonnegative pairs of integers (j,k) such that $j+k(3k\pm1)/2=n$.

There are many identities involving the divisor sums $\sigma_k(n):=\sum_{d|n}d^k$ in the literature, but we have not previously seen (6.1). Besides (6.1), other identities of Ramanujan can be reformulated in terms of divisor sums. In particular, see [2, pp. 326–329] and the references cited there. The most thorough study of identities of this sort has been undertaken by J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams [22], where many references to the literature can also be found.

In general, $T_{2k}(q)$ can be represented as a polynomial in terms of the form $P^{2a}Q^{4b}R^{6c}$, where $2a+4b+6c=2k$. It seems to be extremely difficult to find a general formula for this polynomial, but on page 188, Ramanujan does give the "first" five terms. A proof of this result and all claims on this page of the lost notebook can be found in a paper by Berndt and Yee [12].

7. Eisenstein Series and Approximations to π

On page 211 in his lost notebook, in the pagination of [37], Ramanujan listed eight integers, 11, 19, 27, 43, 67, 163, 35, and 51 at the left margin. To the right of each integer, Ramanujan recorded a linear equation in Q^3 and R^2. The arguments q in these identities were not revealed by Ramanujan, but $q=-\exp(-\pi\sqrt{n})$, where n is the integer at the left margin. To the right of each equation in Q^3 and R^2, Ramanujan entered an equality involving π and square roots. (For the integer 51, the linear equation and the equality involving π, in fact, are not recorded by Ramanujan.) The equalities in the third column lead to approximations of π that are remindful of approximations given by Ramanujan in his famous paper on modular equations and approximations to π [31], [35, p. 33].

We offer a general theorem from which all the equalities in the third column follow. Considerable notation is first needed, however.

Let

$$P_n := P(-e^{-\pi\sqrt{n}}), \quad Q_n := Q(-e^{-\pi\sqrt{n}}), \quad \text{and} \quad R_n := R(-e^{-\pi\sqrt{n}}). \tag{7.1}$$

Recall that the modular j-invariant $j(\tau)$ is defined by

$$j(\tau)=1728\frac{Q^3(q)}{Q^3(q)-R^2(q)}, \qquad q=e^{2\pi i\tau}, \qquad \operatorname{Im}\tau>0.$$

In particular, if n is a positive integer,

$$j_n := j\left(\frac{3+\sqrt{-n}}{2}\right)=1728\frac{Q_n^3}{Q_n^3-R_n^2},$$

where Q_n and R_n are defined by (7.1). Furthermore, set

$$J_n=-\frac{1}{32}\sqrt[3]{j_n}. \tag{7.2}$$

Next, define

$$b_n=\{n(1728-j_n)\}^{1/2} \tag{7.3}$$

and

$$a_n=\frac{1}{6}b_n\left\{1-\frac{Q_n}{R_n}\left(P_n-\frac{6}{\pi\sqrt{n}}\right)\right\}. \tag{7.4}$$

The numbers a_n and b_n arise in series representations for $1/\pi$ proved by D. V. and G. V. Chudnovsky [19] and J. M. and P. B. Borwein [16].

We now have sufficient notation to state our first theorem.

Theorem 7.1. *If P_n, b_n, a_n, and J_n are defined by* (7.1), (7.3), (7.4), *and* (7.2), *respectively, then*

$$\frac{1}{\sqrt{Q_n}}\left(\sqrt{n}P_n - \frac{6}{\pi}\right) = \sqrt{n}\left(1 - 6\frac{a_n}{b_n}\right)\left(\frac{\left(\frac{8}{3}J_n\right)^3 + 1}{\left(\frac{8}{3}J_n\right)^3}\right)^{1/2}. \tag{7.5}$$

To illustrate Theorem 7.1, we offer three of Ramanujan's eight examples.

Corollary 7.2. *We have*

$$\frac{1}{\sqrt{Q_{11}}}\left(\sqrt{11}P_{11} - \frac{6}{\pi}\right) = \sqrt{2},$$
$$\frac{1}{\sqrt{Q_{19}}}\left(\sqrt{19}P_{19} - \frac{6}{\pi}\right) = \sqrt{6},$$
$$\frac{1}{\sqrt{Q_{27}}}\left(\sqrt{27}P_{27} - \frac{6}{\pi}\right) = 3\sqrt{\frac{6}{5}}.$$

Theorem 7.1 leads to the following theorem giving approximations of π.

Theorem 7.3. *We have*

$$\pi \approx \frac{6}{\sqrt{n} - r_n},$$

with the error approximately equal to

$$144\frac{\sqrt{n} + 5r_n}{(\sqrt{n} - r_n)^2}e^{-\pi\sqrt{n}},$$

where r_n is the algebraic expression on the right side in (7.5), *i.e.,*

$$r_n = \sqrt{n}\left(\frac{\left(\frac{8}{3}J_n\right)^3 + 1}{\left(\frac{8}{3}J_n\right)^3}\right)^{1/2}.$$

To see how exact these approximations are, consult either [8] or [13].

The results on this page of the lost notebook led Berndt and Chan [8] to a new general series representation for $1/\pi$. This in turn led them to a new "World Record" series yielding 73 or 74 digits of $1/\pi$ per term.

8. Integrals of Eisenstein Series

At first glance, the next theorem does not appear to have any connection with Eisenstein series. Recall that the functions $f(-q)$ and $\eta(\tau)$ are defined in (1.6).

Theorem 8.1. *Suppose that* $0 < q < 1$. *Then*

$$q^{1/9}\prod_{n=1}^{\infty}(1 - q^n)^{\chi(n)n} = \exp\left(-C_3 - \frac{1}{9}\int_q^1 \frac{f^9(-t)}{f^3(-t^3)}\frac{dt}{t}\right), \tag{8.1}$$

where

$$C_3 = \frac{3\sqrt{3}}{4\pi}L(2,\chi) = L'(-1,\chi). \tag{8.2}$$

Here, $L(s,\chi)$ denotes the Dirichlet L-function defined for Re $s > 0$ by $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$ Furthermore, in (8.1) and (8.2), $\chi(n) = \left(\frac{n}{3}\right)$, the Legendre symbol.

This result, found on page 207 of Ramanujan's lost notebook, was first proved by Son [39], except that he did not establish Ramanujan's formula (8.2) for C_3. Berndt and Zaharescu [15] found another proof of Theorem 8.1, which included a proof of (8.2).

We now explain the connection of Theorem 8.1 with Eisenstein series. A key to the proof of (8.1) is the identity

$$\frac{f^9(-q)}{f^3(-q^3)} = 1 - 9\sum_{n=1}^{\infty}\sum_{d|n}\left(\frac{d}{3}\right) d^2 q^n, \tag{8.3}$$

which was first proved by L. Carlitz [17] and then by Son [39]. The series on the right side of (8.3) is an example of an Eisenstein series $E_{k,\chi}(\tau)$ of weight k, defined on the congruence subgroup $\Gamma_0(N)$ by

$$E_{k,\chi}(\tau) := 1 - \frac{2k}{B_{k,\chi}}\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^{k-1}q^n, \qquad q = e^{2\pi i\tau}, \tag{8.4}$$

where χ is a Dirichlet character of modulus N, and $B_{k,\chi}$ denotes the kth generalized Bernoulli number associated with χ.

Ramanujan's identity (3.4), when written in the form,

$$E_{3,\chi}(z) = 1 - \frac{7}{8}\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^2q^n = \frac{\eta^7(\tau)}{\eta(7\tau)} + \frac{49}{8}\eta^3(\tau)\eta^3(7\tau), \tag{8.5}$$

provides another example, where now $\chi(n)$ denotes the Legendre symbol modulo 7. Then (8.5) leads to the identity

$$q^{8/7}\prod_{n=1}^{\infty}(1-q^n)^{\chi(n)n} = \exp\left(-C_7 - \frac{8}{7}\int_q^1 \left\{\frac{f^7(-t)}{f(-t^7)} + \frac{49}{8}tf^3(-t)f^3(-t^7)\right\}\frac{dt}{t}\right), \tag{8.6}$$

where

$$C_7 = L'(-1,\chi). \tag{8.7}$$

Clearly (8.6)and (8.7) are analogues of (8.1) and (8.2), respectively.

S. Ahlgren, Berndt, Yee, and Zaharescu [1] have proved a general theorem from which (8.1) and (8.6) follow as special cases. The integral appearing in these authors' general theorem contains the Eisenstein series (8.4). It is only in certain instances that we have identities of the type (8.3) and (8.5), which enable us to reformulate the identities in terms of integrals of eta-functions, as we have in (8.1) and (8.6).

A more complete version of this paper can be found in [13].

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