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Linear independence of the values of q-hypergeometric series

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In the present note we are interested in linear independence of the values of a certain class of q-hypergeometric series and its generalizations. We give a brief history on this topic in the first section, then state our results in the second and the third sections. Our results here are in [1], a joint work with K. Väänänen.

1. A brief history

Let us call here q-hypergeometric series the series of the form

$$f(z)=1+\sum_{n=1}^{\infty}\frac{q^{-s\binom{n}{2}}}{\prod_{k=0}^{n-1}P(q^{-k})}z^{n}, \tag{1.1}$$

where q is a complex number with absolute value greater than one, s is a positive integer, and $P(x)$ is a polynomial with complex coefficients satisfying $P(0)\neq 0$ and $P(q^{-n})\neq 0$ $(n=0,1,2,\ldots)$. Note that $f(z)$ represents an entire function. By defining $R(x)=x^{s}P(1/x)$, the series (1.1) can be expressed as

$$f(z)=1+\sum_{n=1}^{\infty}\frac{z^{n}}{\prod_{k=0}^{n-1}R(q^{k})}.$$

Then, under the assumption that $\deg P\leq s$ (or equivalently, $R(x)$ is a polynomial), $f(z)$ satisfies the q-difference equation

$$\{R(D/q)-z\}f(z)=R(1/q),\qquad Df(z):=f(qz). \tag{1.2}$$

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The cases $R(x) = qx$ and $R(x) = qx - 1$ correspond to the Tschakaloff function $T_q(z)$ and the q-exponential function $E_q(z)$, respectively.

The study of the arithmetical nature of the values of the function $T_q(z)$ goes back to Tschakaloff [10] in 1921. He proved the linear independence over the rational number field $\mathbf{Q}$ of the numbers $1,\ T_q(\alpha_j)\ (j = 1,..,m)$ under a certain condition on $q \in \mathbf{Q}$, where α_j are nonzero rational numbers satisfying $\alpha_i/\alpha_j \neq q^n\ (n \in \mathbf{Z})$ for any $i \neq j$, while Skolem [8] proved a similar result involving the derivatives of the function. The former result was refined in a quantitative form by Bundschuh and Shiokawa [4], and the later result by Katsurada [5]. Note that both results are valid for $q \in \mathbf{K}$ and numbers $\alpha_j \in \mathbf{K}$ with certain conditions, here and in what follows $\mathbf{K}$ denotes $\mathbf{Q}$ or an imaginary quadratic number field. Then Stihl [9] generalized the result of Bundschuh and Shiokawa to $f(z)$ having $P(x) \in \mathbf{K}[x]$ with $\deg P < s$, and proved the linear independence over $\mathbf{K}$ of the numbers

$$1,\ f(q^k\alpha_j) \quad (j = 1,..,m; k = 0,1,...,s-1)$$

in quantitative form under a certain condition on $q \in \mathbf{K}$, where α_j are nonzero elements of $\mathbf{K}$ satisfying the same conditions as above. Since the functional equation (1.2) for $f(z)$ with $\deg P \leq s$ has the order s with respect to the q-difference operator D, this result is best possible in qualitative nature. Further, Katsurada [6] put the derivatives of the function in Stihl's result to get the linear independence over $\mathbf{K}$ of the numbers

$$(1.3) \qquad 1,\ f^{(i)}(q^k\alpha_j) \quad (i = 0,1,...,\ell; j = 1,..,m; k = 0,1,...,s-1)$$

in quantitative form under the same conditions as Stihl's on q and α_j's, where ℓ is a nonnegative integer.

We now come to the general case in which the degree of $P(x)$ is not necessarily less than s. In this direction Lototsky [7] in 1943 proved an irrationality result on $E_q(\alpha)$ with $q \in \mathbf{Z}$ at a rational point α different from $q^n\ (n \in \mathbf{N})$. A quantitative refinement of this result with $q \in \mathbf{K}$ was obtained by Bundschuh [3]. After the work of Stihl [9], on noting that $\{R(q^k)\}$ is a linear recurrent sequence, Bézivin [2] introduced a class of entire series as follows. Let $\{A(n)\}$ be a linear recurrent sequence of the form

$$(1.4) \qquad A(n) = \lambda_1\theta_1^n + \cdots + \lambda_h\theta_h^n \qquad (n = 0,1,2,...),$$

where θ_i are nonzero algebraic integers and λ_i are nonzero algebraic numbers. Assume that $A(n)$ belong to $\mathbf{K}^\times$, and that

$$(1.5) \qquad |\theta_1| > |\theta_2| \geq \cdots \geq |\theta_h| \geq 1 \quad \text{and} \quad 1 = \theta_h < |\theta_{h-1}| \text{ if } |\theta_h| = 1.$$

Then we define an entire function $\Phi(z)$ by

$$(1.6) \qquad \Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{\prod_{k=0}^{n} A(k)}.$$

Denote by $\tilde{\mathcal{G}}$ the multiplicative group generated by $\theta_1, ..., \theta_h$, Bézivin [2] proved the linear independence over $\mathbf{K}$ of the numbers

$$(1.7) \qquad 1, \ \Phi^{(i)}(\alpha_j) \quad (i = 0, 1, ..., \ell; j = 1, ..., m),$$

where α_j are nonzero elements of $\mathbf{K}$ such that $\alpha_i/\alpha_j \notin \tilde{\mathcal{G}}$ for any $i \neq j$, and in addition that $\lambda_h \alpha_j \neq \tilde{\mathcal{G}}$ $(j = 1, ..., m)$ if $\theta_h = 1$. This result implies that, for $f(z)$ with $\deg P \leq s$ and an integer q in $\mathbf{K}$, the numbers (1.3) without powers of q are linearly independent over $\mathbf{K}$.

2. Generalizations of Bézivin's result

We can relax the condition (1.5) in Bézivin's result to get the following result.

Theorem 1. *Let $\theta_1, ..., \theta_h$ be nonzero algebraic integers such that*

$$|\theta_1| > 1, \quad |\theta_1| > |\theta_2| \geq \cdots \geq |\theta_h|,$$

and that $|\theta_h| < |\theta_{h-1}|$ if $|\theta_h| < 1$ and $\theta_h = 1 < |\theta_{h-1}|$ if $|\theta_h| = 1$. Let $\{A(n)\}$ be the recurrent sequence (1.4) with nonzero algebraic numbers $\lambda_1, ..., \lambda_h$, and assume that $A(n)$ belong to $\mathbf{K}^\times$ for all n. Let $\alpha_1, ..., \alpha_m$ be elements of $\mathbf{K}^\times$ satisfying $\alpha_i/\alpha_j \notin \tilde{\mathcal{G}}$ for any $i \neq j$. If $\theta_h = 1$, assume in addition that $\lambda_h \alpha_j^{-1} \notin \tilde{\mathcal{G}}$ $(j = 1, ..., m)$. Then the numbers (1.7) are linearly independent over $\mathbf{K}$.

We give an example of this theorem. Let $\{F_n\}$ be the Fibonacci sequence defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ $(n = 0, 1, 2, ...)$, which is expressed as

$$F_n = \lambda_1 \alpha^n + \lambda_2 \beta^n \quad (n = 0, 1, 2, ...),$$

where $\alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2, \lambda_1 = \alpha/\sqrt{5}, \lambda_2 = -\beta/\sqrt{5}$. Since $\beta = -\alpha^{-1}$, the multiplicative group generated by α^ν and β^ν with a positive integer ν is $\langle -1 \rangle \times \langle \alpha^\nu \rangle$ or $\langle \alpha^\nu \rangle$ according as ν is odd or even. Hence the numbers

$$1, \sum_{n=i}^{\infty} \frac{n(n-1)\cdots(n-i+1)\alpha_j^{n-i}}{F_0 F_\nu \cdots F_{n\nu}} \quad (i = 0, 1, \ldots, \ell;\ j = 1, \ldots, m)$$

are linearly independent over $\mathbf{Q}$, if ν is odd and α_j are nonzero rational numbers having distinct absolute values, or if ν is even and α_j are nonzero distinct rational numbers.

For the next result let $\theta_i, \lambda_i \in \mathbf{K}$ in the above, and assume that $\tilde{\mathcal{G}}$ is a free abelian group. We take a free abelian group $\hat{\mathcal{G}}$ of finite rank satisfying $\tilde{\mathcal{G}} \subseteq \hat{\mathcal{G}} \subset \bar{\mathbf{Q}}^\times$. Let r be the rank of $\hat{\mathcal{G}}$, and $\Theta_1, \ldots, \Theta_r$ be a set of generators of $\hat{\mathcal{G}}$. By using these generators we can express θ_i as

$$\theta_i = \Theta_1^{e(i,1)} \cdots \Theta_r^{e(i,r)} \quad (i = 1, \ldots, h).$$

Define

$$\hat{\mathcal{S}} = \{\Theta_1^{\nu_1} \cdots \Theta_r^{\nu_r} \mid 0 \le \nu_j < s_j, j = 1, \ldots, r\},$$

where

$$s_j = \max(0, e(1,j), \ldots, e(h,j)) - \min(0, e(1,j), \ldots, e(h,j)) \quad (j = 1, \ldots, r).$$

Note that $s_j \ge 1$ for all j. Then we have the following result.

Theorem 2. *Let the notations and the assumptions be as above. Let $\alpha_1, \ldots, \alpha_m$ be nonzero elements of* $\mathbf{K}$ *satisfying $\alpha_i/\alpha_j \notin \hat{\mathcal{G}}$ for any $i \ne j$. If $\theta_h = 1$, assume in addition that $\lambda_h \alpha_j^{-1} \notin \hat{\mathcal{G}}$ $(j = 1, \ldots, m)$. Then the numbers*

$$1,\ \Phi^{(i)}(\lambda\alpha_j) \quad (i = 0, 1, \ldots, \ell; j = 1, \ldots, m; \lambda \in \hat{\mathcal{S}})$$

are linearly independent over $\mathbf{K}$.

3. q-hypergeometric series

We can apply Theorem 2 for considering the values of a series generalizing the series (1.1). Let $q_1, \ldots, q_r$ be r nonzero multiplicatively independent integers in $\mathbf{K}$

with $|q_i| > 1$ for all i, and $\mathcal{G}$ be the multiplicative group generated by them. Let $P(x_1, ..., x_r)$ be an element of $\mathbf{K}[x_1, ..., x_r]$ satisfying

$$(3.1) \qquad P(0, ..., 0) \neq 0, \quad P(q_1^{-n}, ..., q_r^{-n}) \neq 0 \quad (n = 0, 1, 2, ...).$$

Then, for positive integers $t_1, ..., t_r$, we define

$$(3.2) \qquad \phi(z) = 1 + \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{r} q_i^{-t_i \binom{n}{2}}}{\prod_{k=0}^{n-1} P(q_1^{-k}, ..., q_r^{-k})} z^n.$$

This series is a particular case of the series (1.6), and reduces to the series (1.1) when $r = 1$. We first restrict ourselves to the case $\deg_{x_i} P \leq t_i$ $(i = 1, ..., r)$.

Theorem 3. *Let q_i be as above, and $\phi(z)$ be the series (3.2) with $\deg_{x_i} P \leq t_i$ $(i = 1, ..., r)$. Let $\alpha_1, ..., \alpha_m$ be nonzero elements of* $\mathbf{K}$ *such that $\alpha_i/\alpha_j \notin \mathcal{G}$ for any $i \neq j$, and assume in addition that $p_{t_1,...,t_r}\alpha_i^{-1} \notin \mathcal{G}$ $(i = 1, ..., m)$ if $p_{t_1,...,t_r} \neq 0$, where $p_{t_1,...,t_r}$ is the coefficient of $x_1^{t_1} \cdots x_r^{t_r}$ in $P(x_1, ..., x_r)$. Then the numbers*

$$(3.3) \qquad 1, \ \phi^{(i)}(\lambda\alpha_j) \qquad (i = 0, 1, ..., \ell; j = 1, ..., m; \lambda \in \mathcal{S}_1)$$

are linearly independent over $\mathbf{K}$, *where*

$$\mathcal{S}_1 = \{q_1^{k_1} \cdots q_r^{k_r} \mid 0 \leq k_i < t_i \ (i = 1, ..., r)\}$$

To give a result without the condition $\deg_{x_i} P \leq t_i$ $(i = 1, ..., r)$ we assume that $P(x_1, ..., x_r)$ is a product of polynomials $P_i(x_i) \in \mathbf{K}[x_i]$.

Theorem 4. *Let $\phi(z)$ be the series (3.2) with $P(x_1, ..., x_r) = P_1(x_1) \cdots P_r(x_r)$, where $P_i(x_i) \in \mathbf{K}[x_i]$ and the condition (3.1) is satisfied. Let $\alpha_1, ..., \alpha_m$ be nonzero elements of* $\mathbf{K}$ *such that $\alpha_i/\alpha_j \neq \mathcal{G}$ for any $i \neq j$, and assume in addition that $p_{1,t_1} \cdots p_{r,t_r}\alpha_j^{-1} \neq \mathcal{G}$ $(i = 1, ..., m)$ if $p_{1,t_1} \cdots p_{r,t_r} \neq 0$, where p_{i,t_i} is the coefficient of $x_i^{t_i}$ in $P_i(x_i)$. Then the numbers (3.3) with $\mathcal{S}_2$ instead of $\mathcal{S}_1$ are linearly independent over* $\mathbf{K}$, *where*

$$\mathcal{S}_2 = \{q_1^{k_1} \cdots q_r^{k_r} \mid 0 \leq k_i < s_i \ (i = 1, ..., r)\}, \qquad s_i = \max(t_i, \deg P_i).$$

The following is a direct consequence of Theorem 4, which generalizes Katsurada's result [6] in qualitative form.

Corollary. *Let q be an integer in* $\mathbf{K}$ *with* $|q| > 1$. *Let* $f(z)$ *be the series (1.1) with* $P(z) \in K[z]$ *satisfying* $P(0) \neq 0, P(q^{-n}) \neq 0$ $(n = 0, 1, 2, \ldots)$. *Let* $\alpha_1, \ldots, \alpha_m$ *be nonzero elements of* K *such that* $\alpha_i/\alpha_j \neq q^n$ $(n \in \mathbf{Z})$ *for any* $i \neq j$. *Assume in addition that* $p_s \alpha_j^{-1} \neq q^n$ $(n \in \mathbf{Z}, j = 1, \ldots, m)$ *if* $p_s \neq 0$, *where* p_s *is the coefficient of* x^s *in* $P(x)$. *Then the numbers (1.3) are linearly independent over* $\mathbf{K}$.

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