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Rankin-Selberg convolution for Cohen type Eisenstein series

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1 Introduction

In this report , we give a meromorphic continuation and a functional equation of a convolution product of two Cohen type Eisenstein series . Cohen type Eisenstein series is a real analytic automorphic form of half integral weight with respect to $\Gamma_0(4)$ with a parameter σ . When the parameter is fixed at a certain value , this convolution product appears as one of factors in an explicit formula of Koecher-Maass series associated with Siegel-Eisenstein series given by Ibukiyama and Katsurada[I-K].

On a proof , the convolution product is a convolution of the Mellin transformation of automorphic forms , we want to apply the Rankin-Selberg method . In this case , Cohen type Eisenstein series is not rapid decay as $y = Imz$ to infinity . So we use the Zagier's method which gives Rankin-Selberg method for the automorphic functions which are not of rapid decay [Z]. This method gives , for a $SL_2(\mathbf{Z})$ invariant function $G(z)$ on the upper half plane which satisfies an increasing condition such that $\tilde{G(z)} = G(z) - \sum_{i=1}^{l} \frac{c_i}{n_i!} y^{\alpha_i} log^{n_i}(y) (z = x + iy)$ is rapid decay as $y \to \infty$, define $R(s)$ by $R(s) = \int_0^\infty \int_0^1 \tilde{G(z)} y^{s-2} dxdy$. Then this integral converges absolutely for $Re(s)$ sufficiently large , and by applying the unfolding method for a integral over the truncated domain , $R(s)$ can be expressed by sum of rational functions of s and integrals using Eisenstein series $E(z,s) = \frac{1}{2} \sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$ over certain domain . Then analytic proerties of $E(z,s)$ gives a meromorphic continuation and a functional equation of $R(s)$. S.Gupta[G] generalized it for invariant functions respect to congruence subgroup which satisfies the same type increasing condition at each cusps .

In section 2 , we apply their method for an automorphic form $G(z)$ respect to congruence subgroup Γ ($G(\gamma z) = (cz+d)^k G(z)$ for $\gamma \in \Gamma$, k is not nesesaly zero) which satisfies the same type increasing condition at each cusps of Γ . And in section 3 , as an application , we give a meromorphic continuation and a functional equation of a convolution product of Cohen type Eisenstein series .To obtain the functional equation , we use the plus condition of Fourier coefficients of Cohen type Eisenstein series and Kohnen's method on modular forms belonging to plus space .

2 Rankin-Selberg method for the automorphic forms which are not of rapid decay

Let $H = \{z = x + iy \in \mathbf{C} : y > 0\}$ be the complex upper half plane and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$ acts on $z \in H$ by $gz = \frac{az+b}{cz+d}$. Let Γ be a congruence subgroup which satisfies that

$$-I \in \Gamma \ \ and \ \ \Gamma_\infty^+ = \Gamma \cap \{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}; a \in \mathbf{R}\} = < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} > .$$

Let $\{x_1, ..., x_h\}$ be the set of representative of non equivalent cusps of Γ . Take some $g \in SL_2(\mathbf{R})$ such that $g\ i\infty = x_i$ and put

$$\Gamma_i = \Gamma \cap g\{\pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}; a \in \mathbf{R}\}g^{-1} \ \ , \ \ \Gamma_i^+ = \Gamma \cap g\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}; a \in \mathbf{R}\}g^{-1}.$$

Γ_i is independent of a choice of such $g \in SL_2(\mathbf{R})$. By the assumption $-I \in \Gamma$, we can take and fix such $g_i \in SL_2(\mathbf{R}); g\ i\infty = x_i$ so that

$$\Gamma_i^+ = g_i < \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} > g_i^{-1}$$

for each i (especially we put $x_1 = i\infty, g_1 = I$) .

Fix two integers k and p . For $z \in H$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$, put $\nu(g,z) = \frac{(cz+d)^k}{|cz+d|^p}$ .Let χ be a character of Γ . We assume that χ satisfies $\chi(-I) = (-1)^k$ for above k and $\chi(\Gamma_i^+) = \{1\}$ for all i .

Then we define Eisenstein series associated to the cusp x_i by

$$E_i(z,s,\chi)=\sum_{\gamma\in\Gamma_i\backslash\Gamma}\overline{\chi(\gamma)}\nu(g_i^{-1}\gamma,z)Im(g_i^{-1}\gamma z)^s.$$

This series converges absolutly for $Re(s)+\frac{p-k}{2}>1$.
Further for another cusp x_j , we put

$$E_{i,j}(z,s,\chi)=\nu(g_j,z)E_i(g_jz,s,\chi).$$

Then this has a period 1 and has a Fourier expansion . Especially its constant term can be written as

$$a_{0,i,j}(y,s,\chi)=\delta_{i,j}y^s+\varphi_{i,j}(s-\frac{k-p}{2},\chi)y^{1-s+k-p},$$

where $\delta_{i,j}$ is the Kronecker delta , and $\varphi_{i,j}(s,\chi)$ is a certain function . Then we put

$$\Phi(s,\chi)=(\varphi_{i,j}(s,\chi))_{1\leq i,j\leq l}.$$

Theorem 1 *We use assumptions and notations as above .*
Let $\xi(z)$ be a continuous function on H which satisfies the following two conditions (a) and (b) .
(a)

$$\xi(\gamma z)=\overline{\chi(\gamma)}\nu(\gamma,z)\xi(z)=\overline{\chi(\gamma)}\frac{(cz+d)^k}{|cz+d|^p}\xi(z),$$

for all $z\in H,\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$.
(b)For each i $(i=1,..,h)$, put $\xi_i(z)=\nu(g_i,z)^{-1}\xi(g_iz)$ and assume that $\xi_i(z)=\psi_i(y)+O(y^{-N})$ for any N as $y\to\infty$, where $\psi_i(y)$ is a function of the form

$$\psi_i(y)=\sum_{m_i:finite\ sum}\frac{c_{i,m_i}}{n_{i,m_i}!}y^{\alpha_{i,m_i}}log^{n_{i,m_i}}(y)\quad,\quad c_{i,m_i},\alpha_{i,m_i}\in\mathbf{C},n_{i,m_i}\in\mathbf{N}\geq 0.$$

For such function $\xi(z)$, define the Rankin-Serberg transformation of $\xi(z)$ at the cusp x_i by

$$R_i(s,\xi)=\int_0^\infty\int_0^1(\xi_i(z)-\psi_i(y))y^{s-2}dxdy.$$

Then we claim ,

(i)$R_i(s,\xi)$ converges absolutely for $Re(s)$ sufficiently large .

When we take (it is possible) the fundamental domain D for the action of Γ in the form

$$D = D_0 \cup (\cup_{j=1}^{h} g_j D^Y),$$

where D_0 is a relative compact open subset in H and

$$D^Y = \{z = x + iy; 0 < x < 1, y > Y\}$$

for sufficiently large Y , and put

$$D_Y = \{x + iy; 0 < x < 1, y \leq Y\}, d\mu = \frac{dxdy}{y^2},$$

then we have a following expression .

$$\begin{aligned} R_i(s,\xi) &= \sum_{j=1}^{h}\{\int\int_{D^Y} \xi_j(z)(E_{i,j}(z,s,\chi) - a_{0,i,j}(y,s,\chi))d\mu \\ &\quad + \int\int_{D^Y}(\xi_j(z) - \psi_j(y))a_{0,i,j}(y,s,\chi)d\mu \\ &\quad + \int\int_{D^Y} \psi_j(y)\varphi_{ij}(s - \frac{k-p}{2},\chi)y^{1-s+k-p}d\mu \quad \} \\ &+ \int\int_{D_0} \xi(z)E_i(z,s,\chi)d\mu \\ &- \int\int_{D_Y} \psi_i(y)y^s d\mu. \end{aligned}$$

(ii)If we assume further that each Eisenstein series $E_i(z,s,\chi)(i = 1,..,h)$ can be meromorphically continued to all s-plane,then so is $R_i(s,\xi)$ and we have a following functional equation .

$$\begin{aligned} \mathcal{R}(s,\xi) &= \begin{pmatrix} R_1(s,\xi) \\ \vdots \\ R_h(s,\xi) \end{pmatrix} \\ &= \Phi(s-a,\chi)\mathcal{R}(1-s+2a,\xi). \end{aligned}$$

Proof is same as in Gupta[G] .

3 An application for Cohen type Eisenstein series

We apply above result in case

$$\Gamma = \Gamma_0(4) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) : c \equiv 0(mod\ 4)\},$$

and $\xi(z) = F_1(z)\overline{F_2(z)}$ where $F_j(z)(j = 1, 2)$ are Cohen type Eisenstein series defined as follows .

For each $\sigma \in \mathbf{C}$, an odd integer k , define Eisenstein series $E(k, \sigma, z)$ as

$$E(k, \sigma, z) = y^{\sigma/2} \sum_{d=1, odd}^{\infty} \sum_{c=-\infty}^{\infty} (\frac{4c}{d})\epsilon_d^{-k}(4cz + d)^{k/2}|4cz + d|^{-\sigma}$$

where for $\gamma = \begin{pmatrix} a & b \\ 4c & d \end{pmatrix} \in \Gamma_0(4)$, we put

$$j(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)} = \epsilon_d^{-1}(\frac{4c}{d})(4cz + d)^{1/2} \quad , \quad \theta(z) = \sum_{n \in \mathbf{Z}} e^{2\pi i n^2 z}$$

is well known automorphic factor of $\Gamma_0(4)$. This $E(k, \sigma, z)$ converges absolutely and uniformly for $-k+2\sigma-4 > 0$. And put $E^*(k, \sigma, z) = E(-\frac{1}{4z})(-2iz)^{k/2}$. Then Cohen type Eisenstein series is defined by

$$F(k, \sigma, z) = E(k, \sigma, z) + 2^{k/2-\sigma}(e(k/8) + e(-k/8))E^*(k, \sigma, z).$$

This satisfies

$$F(k, \sigma, \gamma z) = j(\gamma, z)^{-k} F(k, \sigma, z) \quad \gamma \in \Gamma_0(4)$$

and has a Fourier expansion as

$$F(k, \sigma, z) = y^{\sigma/2} + \sum_{d=-\infty}^{\infty} c(d, \sigma, k) y^{\sigma/2} e(dx) \tau_d(y, \frac{\sigma - k}{2}, \frac{\sigma}{2}),$$

where $e(x) = e^{2\pi i x}$ and $\tau_d(y, \alpha, \beta)$ is same as in [I-S] . $\tau_d(y, \alpha, \beta)$ relate to the Whittaker function $W_{\alpha,\beta}(y) = y^{\alpha} e^{-\frac{y}{2}} \omega(y, \frac{1}{2} + \alpha + \beta, \frac{1}{2} - \alpha + \beta)$, where $\omega(y, \alpha, \beta) = \frac{y^{\beta}}{\Gamma(\beta)} \int_0^{\infty} (1 + u)^{\alpha-1} u^{\beta-1} e^{-yu} du$ as

$$\tau_d(y,\alpha,\beta) = i^{\beta-\alpha}$$

$$\times \begin{cases} (2\pi)^{\alpha}\Gamma(\alpha)^{-1}d^{\alpha-1}e^{-2\pi dy}(2y)^{-\beta}(4\pi dy)^{\frac{\beta-\alpha}{2}}e^{2\pi dy}W_{\frac{\alpha-\beta}{2},\frac{\alpha+\beta-1}{2}}(4\pi dy), \\ (2\pi)^{\beta}\Gamma(\beta)^{-1}|d|^{\beta-1}e^{-2\pi|d|y}(2y)^{-\alpha}(4\pi|d|y)^{\frac{\alpha-\beta}{2}}e^{2\pi|d|y}W_{\frac{\beta-\alpha}{2},\frac{\beta+\alpha-1}{2}}(4\pi|d|y), \\ (2\pi)^{\alpha+\beta}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta}, \end{cases}$$

It is known that , $W_{\alpha,\beta}(y)$ is continued holomorphically to whole $(\alpha,\beta) \in \mathbf{C}^2$, and

$$lim_{y\to\infty} sup_{(\alpha,\beta)\in K}|y^{-\alpha}e^{\frac{y}{2}}W_{\alpha,\beta}(y) - 1| = 0$$

for any compact subset K in $\mathbf{C}^2$.

Fourier coefficient $c(d,\sigma,k)$ is explicitly calculated in [I-S] and it is proven in [I-S]p10 that Fourier coefficients satisfy the plus condition ; if $(-1)^{\frac{k+1}{2}}d \equiv 2,3(mod\ 4)$ then $c(d,\sigma,k) = 0$.

For odd integer k_j and $\sigma_j \in \mathbf{C}$ $(j = 1,2)$, we put $F_j(z) = F(k_j,\sigma_j,z)$.

Theorem 2 *If* $k_1 \equiv k_2(mod\ 4)$,$-k_j + 2\sigma_j - 4 > 0(j = 1,2)$,

$$\Omega(s) = 2^{2s}\pi^{-s}\Gamma(s+\frac{k_1}{2})\zeta(2s+\frac{k_1}{2}+\frac{k_2}{2})R_{\infty}(s,F_1\overline{F_2})$$

($\Gamma(s)$ is gamma function and $\zeta(s)$ is Riemann zeta function) can be meromorphically continued to all s-plane and satisfies

$$\Omega(s) = \Omega(1-\frac{k_1}{2}-\frac{k_2}{2}-s).$$

(proof)We will check the conditions in Theorem 1 for $\Gamma = \Gamma_0(4)$, and $\xi(z) = F_1(z)\overline{F_2(z)}$.

$$F_1(\gamma z)\overline{F_2(\gamma z)} = \frac{(cz+d)^{\frac{k_2-k_1}{2}}}{|cz+d|^{k_2}}F_1(z)\overline{F_2(z)} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

is the condition (a) if we put $p = k_2, k = \frac{k_2-k_1}{2}$, and $\chi = id(mod\ 4)$. As the set of representative of non equivalent cusps , and the element $g \in SL_2(\mathbf{R})$ which transforms $i\infty$ to each cusps , we take $\{i\infty, 0, \frac{1}{2}\}$ and

$$g_{\infty} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_0 = \begin{pmatrix} 0 & -1/2 \\ 2 & 0 \end{pmatrix}, g_1 = \begin{pmatrix} 1 & -1/2 \\ 2 & 0 \end{pmatrix}.$$

Each Eisenstein series $E_i(z,s) = E_i(z,s,id)$ associated to cusp have a meromorphic continuation to whole s-plane by Fourier expansion .

Some components of the matrix $\Phi(s,id)$ which we need for a functional equation are given by

$$\phi_{\infty,\infty}(s) = \frac{2^{1-4s}}{1-2^{-2s}} \frac{\zeta(2(1-s))\Gamma((1-s)-\frac{k}{2})\pi^{-(1-s)}}{\zeta(2s)\Gamma(s-\frac{k}{2})\pi^{-s}} \quad , \quad k = \frac{k_2-k_1}{2}$$

$$\phi_{\infty,0}(s) = \phi_{\infty,1}(s) = 2^{-2s}\frac{1-2^{1-2s}}{1-2^{-2s}} \frac{\zeta(2(1-s))\Gamma((1-s)-\frac{k}{2})\pi^{-(1-s)}}{\zeta(2s)\Gamma(s-\frac{k}{2})\pi^{-s}}.$$

To check the condition (b) , and to obtain a Dirichlet series expansion of Rankin-Selberg transformations , we use a Fourier expansion of $(2z)^{\frac{k}{2}}F(k,\sigma,g_i z)$ (i $0,1$) which can be obtained by a method of Kohnen [K]. Next result is proven in [I-S] .

Proposition 1 (I-S,Lemma3.1) *For any function $f(z)$ on H , define*

$$(f|_k W_4)(z) = f(-\frac{1}{4z})(-2iz)^{k/2} \quad , \quad (f|U_4)(z) = \frac{1}{4}\sum_{\nu=0}^{3} f(\frac{z+\nu}{4})$$

Then for $F(z) = F(k,\sigma,z)$,

$$F|_k W_4 = 2^{\frac{k+1}{2}}(-1)^{\frac{k^2-1}{8}} F|U_4 \tag{1}$$

$$(F|U_4)(z) = \frac{1}{2}(F(\frac{z}{4}) + F(\frac{z+2}{4})) \tag{2}$$

Using these relations , we obtain Fourier expansion of $(2z)^{\frac{k}{2}}F(k,\sigma,g_i z)$.

Proposition 2 *If $F(z) = F(k,\sigma,z) = \sum_{d=-\infty}^{\infty} c(d,y)e(dx)$ then*

$$(2z)^{k/2}F(g_i z) = e(k/8)(-1)^{(k^2-1)/8}2^{(k+1)/2} \sum_{d\equiv i (mod\ 2)} c(d,y/4)e(dx/4)$$

(proof)For $i = 0,1$, put $F^i(z) = \sum_{d\equiv i(mod\ 2)} c(d,y/4)e(dx/4)$. Then (2) implies $F|U_4(z) = F^0(z)$....(2') . By definition of W_4 and (1), (2') , we obtain

$$(-2iz)^{\frac{k}{2}}F(g_0 z) = (-1)^{(k^2-1)/8}2^{(k+1)/2}F^0(z) \tag{3}$$

This proves the case $i=0$. Next by definition ,

$$F^0(z)+F^1(z)=F(\frac{z}{4}) \qquad (4)$$

$$F(z+\frac{1}{2})=F^0(4z)-F^1(4z) \qquad (5)$$

So

$$\begin{aligned}
&(-2iz)^{\frac{k}{2}}F(g_1z)\\
=\ &(-2iz)^{k/2}F(-\frac{1}{4z}+\frac{1}{2})\\
=\ &(-2iz)^{\frac{k}{2}}(F^0(-\frac{1}{z})-F^1(-\frac{1}{z})) \qquad (\ by(5)\ )\\
=\ &(-2iz)^{\frac{k}{2}}(2F^0(-\frac{1}{z})-F(-\frac{1}{4z})) \qquad (use\ (4)\ for\ F^1(-\frac{1}{z}))\\
=\ &(-1)^{(k^2-1)/8}2^{(k+1)/2}(F(\frac{z}{4})-F^0(z)) \qquad (use\ (3)\ for\ each\ terms)\\
=\ &(-1)^{(k^2-1)/8}2^{(k+1)/2}F^1(z) \qquad by(4)
\end{aligned}$$

this gives the case $i=1$.

If we take

$$\psi_\infty(y)=c_1(0,y)\overline{c_2(0,y)}\ ,\quad \psi_1(y)=0$$

$$\psi_0(y)=e(\frac{k_1-k_2}{8})\prod_{i=1}^{2}(-1)^{(k_i^2-1)/8}2^{(k_i+1)/2}c_1(0,y/4)\overline{c_2(0,y/4)},$$

where $c_j(0,y)$ is 0-th Fourier coefficient of $F_j(z)$, then by the facts on $\tau_d(y,\alpha,\beta)$ stated above and I-S Corollary 2.4;

$$c(0,y)=y^{\sigma/2}+2^{3k/2-3\sigma+7/2}(-1)^{(k^2-1)/8}\pi\frac{\Gamma(\sigma-k/2-1)\zeta(2\sigma-k-2)}{\Gamma((\sigma-k)/2)\Gamma(\sigma/2)\zeta(2\sigma-k-1)}y^{1-\sigma/2+k/2}$$

, the condition(b) in theorem is satisfied . Thus each Rankin-Selberg transformation at each cusps $R_j(s,F_1\overline{F_2})(j=\infty,0,1)$ is continued meromorphcally to all s-plane and these become

$$R_\infty(s,F_1\overline{F_2})=\sum_{d\neq 0}c(d,\sigma_1,k_1)\overline{c(d,\sigma_2,k_2)}I_d(s,\sigma_1,\sigma_2,k_1,k_2)$$

$$R_i(s,F_1\overline{F_2}) = e(\frac{k_1-k_2}{8})\prod_{i=1}^{2}(-1)^{(k_i^2-1)/8}2^{(k_i+1)/2}4^{s-1} \times \sum_{d\neq 0, d\equiv i(mod\ 2)} c(d,\sigma_1,k_1)\overline{c(d,\sigma_2,k_2)}I_d(s,\sigma_1,\sigma_2,k_1,k_2)$$

(i=0,1) for $Re(s)$ sufficiently large , where

$$I_d(s,\sigma_1,\sigma_2,k_1,k_2) = \int_0^\infty \tau_d(y,\frac{\sigma_1-k_1}{2},\frac{\sigma_1}{2})\overline{\tau_d(y,\frac{\sigma_2-k_2}{2},\frac{\sigma_2}{2})}y^{\frac{\sigma_1+\overline{\sigma_2}}{2}+s-2}dy$$
$$= |d|^{\frac{\sigma_1+\overline{\sigma_2}}{2}-\frac{k_1+k_2}{2}-1-s}e(\frac{k_1-k_2}{8})(2\pi)^{\sigma_1+\overline{\sigma_2}-\frac{k_1+k_2}{2}}(4\pi)^{-\frac{\sigma_1+\overline{\sigma_2}}{2}-s+1}$$
$$\times \int_0^\infty y^{\frac{k_1+k_2}{4}-2+s}W_{\frac{-sgn(d)k_1}{4},\frac{\sigma_1}{2}-\frac{k_1}{4}-\frac{1}{2}}(y)W_{\frac{-sgn(d)k_2}{4},\frac{\overline{\sigma_2}}{2}-\frac{k_2}{4}-\frac{1}{2}}(y)dy$$

$$\times\begin{cases}\Gamma(\frac{\sigma_1-k_1}{2})^{-1}\overline{\Gamma(\frac{\sigma_2-k_2}{2})}^{-1}, & \text{for } d>0\\ \Gamma(\frac{\sigma_1}{2})^{-1}\overline{\Gamma(\frac{\sigma_2}{2})}^{-1}, & \text{for } d<0\end{cases}$$

So by Theorem 1 , we obtain

$$R_\infty(s,F_1\overline{F_2})$$
$$= \{e(\frac{k_1-k_2}{8})\prod_{i=1}^{2}(-1)^{(k_i^2-1)/8}2^{(k_i+1)/2}4^{-\frac{k_1+k_2}{2}-s}\phi_{\infty,0}(s+\frac{k_1+k_2}{4})$$
$$+\phi_{\infty,\infty}(s+\frac{k_1+k_2}{4})\}R_\infty(1-s-\frac{k_1+k_2}{2},F_1\overline{F_2})$$
$$= \frac{2^{2(1-s-\frac{k_1+k_2}{2})}}{2^{2s}}\frac{\pi^s}{\pi^{1-s-\frac{k_1+k_2}{2}}}\frac{\Gamma(1-s-\frac{k_2}{2})}{\Gamma(s+\frac{k_1}{2})}\frac{\zeta(2-2s-\frac{k_1}{2}-\frac{k_2}{2})}{\zeta(2s+\frac{k_1}{2}+\frac{k_2}{2})}$$
$$\times R_\infty(1-s-\frac{k_1}{2}-\frac{k_2}{2},F_1\overline{F_2}),$$

in the second equation , we use $k_1\equiv k_2(mod\ 4)$.

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We set $k_j=\sigma_j=2\kappa_j$ where $\kappa_j\in\mathbf{Z}+\frac{1}{2},>2$ is a half integer for $j=0,1$,and observe by the formula of Fourier coefficients in [I-S] that if $\sigma\in\mathbf{R}$ then

$e(\frac{k}{8})c(d,\sigma,k) \in \mathbf{R}$ for $d \neq 0$, then Dirichlet series $D^*_{2\kappa_1-1}(s)\otimes D^*_{2\kappa_2-1}(s)$ which appers in an explicit formula of Koecher-Maass series for Siegel-Eisenstein series ([I-K]) is equal to

$$(-1)^{2\kappa_1-1}\pi^{1-2\kappa_1-2\kappa_2}\prod_{i=1}^{2}\Gamma(\kappa_i)\Gamma(\frac{2\kappa_i-1}{2})\zeta(2\kappa_i-1)$$
$$\times\frac{(4\pi)^s}{\Gamma(s)}\zeta(2s-\kappa_1-\kappa_2+2)R_\infty(s-\kappa_1-\kappa_2+1,F_1\overline{F_2}).$$

By the explicit form in [I-K] and Theorem 2 , we can check the functional equation of Koecher-Maass series of Siegel-Eisenstein series .

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