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Toward a mathematical theory of renormalization

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1 Introduction

Renormalization transformations were developed by theoretical physicists in order to investigate first problems arising in quantum field theory and later in statistical mechanics, specifically phase transitions and critical phenomena appearing in systems of a large number of interacting components. In their latter version they provide a scheme of systematic reduction of complexity built up by the degrees of freedom, whose relevant number goes to infinity as the critical state of the system is approached. Renormalization schemes may be applied on various levels: in position space, in momentum space, and in various other ways, being presently a technique reaching far beyond its original purpose.

In this review I will look only at position space renormalization applied to interacting spin systems on a lattice. In physicists' practice spins are grouped into some collections according to a set rule, and after rescaling and summing over them in each block (i.e., performing a specific block-spin transformation) one computes the renormalized ("effective") potential for the "super-spin" originating from the replacing of the spins in a block with a block-spin. By iteration of this procedure it is hoped that through passing to ever larger scales, the carefully prepared system will eventually attain a critical state whose features can be computed recursively. This operation assumes that such an effective potential exists from one step to the other. As it turns out, however, this is a highly non-trivial issue: In general a renormalization transformation not only will generate many-body and long-range terms in the effective potential even when the one to start with was possibly a simple nearest neighbour pair potential, but even the existence of any "reasonable" effective potential might be in doubt after just one renormalization step.

Here, therefore, I address some problems related with the mathematical definition and properties of such transformations and sketch the possible solutions we presently think of. In its first part I briefly recall the "a priori" framework of renormalization transformations, in which Gibbs probability measures stand centre-stage. Next I will shortly describe the mathematical difficulties accompanying these transformations and the natural ideas to cope with them. In the concluding part I explain in its main lines how by suitable modifications of the notion of Gibbs measure an "a posteriori" framework can be developed which may be hoped to accomodate renormalization transformations in a mathematically coherent way. At the end I present a list of relevant (though heavily selected) references.

2 Gibbs measures and renormalization transformations

The interacting spin system will be realized on a lattice $\mathcal{L}$ (such as $\mathbb{Z}^d$ etc) at each of whose points a 'spin' will be placed. We think for simplicity of a chemically one-component system by allowing each spin to take its values from the same state space S assumed here to be a finite set. The configuration space is then $\Omega = S^{\mathcal{L}}$. We denote by ω_x the value of $\omega \in \Omega$ at site x, and $\omega_\Lambda \times \xi_{\Lambda^c}$ stands for a configuration agreeing with ω inside $\Lambda \in \mathcal{P}(\mathcal{L})$ and with ξ outside of Λ (here $\mathcal{P}(\mathcal{L})$ denotes the set of finite subsets of the lattice); in this context it is useful to think of ξ as a 'boundary condition'. Ω is further equipped with its Borel σ-field $\mathcal{F}$, and thus turned into a measurable space; $\mathcal{F}_\Lambda$ stands for the field generated by S^Λ. Moreover we take the counting measure assigning the equal chance $1/|S|$ for each spin value at each site independently from one another, and define the product measure χ arising from it by multiplying over all sites of the lattice. The measure space $(\Omega, \mathcal{F}, \chi)$ will thus describe the non-interacting spin system.

Interactions are introduced by potentials $\Phi : \mathcal{P}(\mathcal{L}) \times \Omega \to \mathbb{R}$, $(\Lambda, \omega) \mapsto \Phi_\Lambda(\omega)$, with putting $\Phi_\emptyset(\cdot) \equiv 0$ and assuming that Φ_Λ are $\mathcal{F}_\Lambda$-measurable. For convenience, throughout we assume that Φ_Λ are invariant under shifts on the lattice. The energy associated with a configuration $\omega_\Lambda \times \xi_{\Lambda^c}$ is given in terms of the Hamiltonian

$$\mathcal{H}_\Lambda^\Phi(\omega|\xi) = \sum_{X \cap \Lambda \neq \emptyset} \Phi_X(\omega_{X\cap\Lambda} \times \xi_{X \cap \Lambda^c}) = \sum_{X \subset \Lambda} \Phi_X(\omega) + \sum_{\substack{X \subset \Lambda \\ Y \subset \Lambda^c}} \Phi_{X \cup Y}(\omega_X \times \xi_Y). \tag{2.1}$$

Since the range of the interaction may be infinite, the sum above may diverge; to rule this possibility out we require that the interaction energy of each spin with all others is uniformly bounded:

$$\sum_{\substack{X \in \mathcal{P}(\mathcal{L}) \\ X \ni 0}} \|\Phi_X\|_\infty < \infty, \tag{2.2}$$

where $\|\cdot\|_\infty$ is the usual sup-norm. Using the l.h.s. of the above as a norm, we define the Banach space $\mathcal{B}(\Omega)$ of potentials.

The states of the system are described by suitable probability measures. A compatible and proper family $\Gamma = \{\gamma_\Lambda\}_{\Lambda \in \mathcal{P}(\mathcal{L})}$ of conditional probability kernels $\gamma_\Lambda : S^{\Lambda^c} \times \mathcal{F} \to \mathbb{R}$ is called a specification (see [9] for terminology). A Gibbs specification with respect to Φ is the special choice Γ^Φ given by

$$\gamma_\Lambda^\Phi(\xi_{\Lambda^c}, E) = \frac{1}{Z_\Lambda^\Phi(\xi_{\Lambda^c})} \int e^{-\beta H_\Lambda^\Phi(\xi_\Lambda|\xi_{\Lambda^c})} 1_E(\xi_\Lambda) d\chi_\Lambda(\xi_\Lambda) \tag{2.3}$$

where $Z_\Lambda^\Phi(\xi_{\Lambda^c}) = \int e^{-\beta H_\Lambda^\Phi(\xi_\Lambda|\xi_{\Lambda^c})} d\chi_\Lambda(\xi_\Lambda)$ is the partition function and β is the inverse temperature. Fix now Γ^Φ for a given $\Phi \in \mathcal{B}(\Omega)$. A Gibbs measure for interaction Φ is any probability measure ϱ^Φ on $(\Omega, \mathcal{F}, \chi)$ consistent with Γ^Φ, i.e., if a version of the family of its conditional probabilities with respect to the sub-σ-fields $\mathcal{F}_{\Lambda^c}$ coincides with Γ^Φ.

Since S is a finite set, compactness arguments guarantee that at least one Gibbs measure exists. The possibility of multiple Gibbs measures for a given potential (selected by different boundary conditions) is also of great interest for it corresponds to situations when a first-order phase transition occurs. Conversely, there is a procedure to reconstruct a potential for a given Gibbs measure, moreover whenever this potential is in $\mathcal{B}$, then it is unique modulo minor details. As it is well known, Gibbs measures minimize the free energy of the system, and therefore provide a natural description of thermodynamic (classical) equilibrium states; for details and proofs we refer to [9].

The following is a useful fact providing an actual way of checking whether or not a probability measure is a Gibbs measure.

Theorem 2.1 (Characterization Theorem) *Let Γ be a specification on $(\Omega, \mathcal{F}, \chi)$. The following statements are equivalent:*

1. *There is a potential $\Phi \in \mathcal{B}(\Omega)$ such that Γ is a Gibbs specification with respect to it.*

2. *Γ is quasilocal, i.e.,*

$$\lim_{\tilde{\Lambda} \to \mathcal{L}} \sup_{\substack{\omega,\xi,\eta \in \Omega \\ \xi_{\tilde{\Lambda} \setminus \Lambda} = \eta_{\tilde{\Lambda} \setminus \Lambda}}} |\gamma_\Lambda(\omega_\Lambda, \xi_{\Lambda^c}) - \gamma_\Lambda(\omega_\Lambda, \eta_{\Lambda^c})| = 0, \ \forall \Lambda \subset \tilde{\Lambda} \in \mathcal{P}(\mathcal{L}) \tag{2.4}$$

and uniformly non-null, i.e., $\exists \varepsilon > 0$ such that for $\forall E \in \mathcal{F}$, $\chi(E) > 0$ implies $\gamma_\Lambda(\xi, E) > \varepsilon$, for all $\Lambda \in \mathcal{P}(\mathcal{L})$ and $\xi \in S^{\Lambda^c}$.

Quasilocality is actually an extension of the usual Markov property.

A renormalization transformation is a probability kernel between in general two distinct probability spaces mapping one probability measure into another, i.e., $T : \Omega \times \mathcal{F}' \to \mathbb{R}$ with $\Omega' = S'^{\mathcal{L}'}$, the image state space, and $\mathcal{F}'$, its associated Borel field; the image measure is

$$(T\mu)(d\omega) = \int_\Omega T(\xi, d\omega)\mu(d\xi). \tag{2.5}$$

In usual practice these are block-spin transformations in the sense that the lattice is divided into non-overlapping blocks (e.g., d-cubes), and T is a product of kernels defined on blocks of "internal" spins:

$$T(\xi, d\omega) = \prod_{x \in \mathcal{L}'} \hat{T}(\xi_{B_x}, d\omega_x) \tag{2.6}$$

where B_x is a block associated with site x in a specific way (e.g. it is the first site of the block in some ordering), and $\hat{T}$ is defined for blocks. Examples include

decimation : $\hat{T}(\xi_{B_x}, d\omega_x) = \delta(\xi_{B_x} - \omega_x) d\omega_x$

Kadanoff transformation :

$$\hat{K}_p(\xi_{B_x}, d\omega_x) = \frac{\exp(p\omega_x \sum_{y \in B_x} \xi_y)}{2\cosh(p \sum_{y \in B_x} \xi_y)} \frac{\delta(\omega_x - 1) + \delta(\omega_x + 1)}{2} d\omega_x, \quad p > 0$$

The first case is an example of a deterministic, the second of a stochastic renormalization transformation, however for $p \to \infty$ the Kadanoff transform becomes the (deterministic) so called majority-rule transformation. In both cases $d\omega_x$ is a shorthand for the counting measure. A useful compendium of mathematical material on RG-transformations is [4].

3 Renormalization pathologies

In 1978-79 Griffiths and Pearce and then in 1981 Israel were the first to signal in their groundbreaking work on the mathematics of renormalization transformations that maps between potentials are not always well defined. The natural way was applying such a transformation to a Gibbs measure and identifying the renormalized potential as the potential associated with the image measure, i.e., studying the map $\mathcal{B}(\Omega) \to \mathcal{B}(\Omega')$ induced by the renormalization transformation. However, as it turned out by looking at specific examples, the image measure is not necessarily a Gibbs measure for any $\mathcal{B}$-type potential, and thus this induced map would not always exist. Changing this space of potentials for a larger one would introduce a number of "unphysical" features for Gibbs measures, hence this is not a clear remedy to the problem. The issue has been taken up once again and clarified to a great extent in the monumental work by van Enter, Fernández and Sokal which appeared in 1993. They produced a number of further "pathological" examples and developed a systematic insight into their nature. As it happened, the specific cases fell into two groups according to the failure of quasilocality or non-nullness of the image measures. For a detailed analysis of examples in the context of renormalizations in a variety of models (Ising, Potts, fuzzy Potts, random cluster, voter, SOS, massless Gaussians etc) we refer to [4, 22, 7, 13] and references therein. Work gathered more momentum when non-Gibbsian measures challengingly appeared also from other quarters [23, 20, 6].

Having a notion of the occurrence of pathologies one first step was mapping them out in function of the parameter space. Pathologies first seemed to appear only in certain parameter regions (like the low temperature regime in the Ising model), but later developments revealed that by no means are there safe-havens where some general principle would rule them out [5, 3]. Contrasting the picture, cases of no pathologies have been reported first in [12, 10], and more general results have been obtained in [8].

A decisive influence in dealing with these pathologies was exercised by the late Professor Dobrushin. His papers in this direction [2] appeared late in time but his ideas became common currency at a much earlier stage for most of the people involved in this research. One natural reaction to pathologies was that perhaps the notion of Gibbs measure is too strong in the sense that it supposes both quasilocality and non-nullness *uniformly* in configurations, and that the potential with which it is constructed is also *uniformly* summable. Two possible way-outs have been suggested: Perhaps configurations for which quasilocality breaks down are untypical and form only a subset of measure zero, which once removed would leave a sufficiently large ground on which to construct some generalized Gibbs measures following the usual DLR way. Or perhaps uniform summability of potentials can be replaced by a pointwise summability on a full-measure subset of configurations and thus again some generalized Gibbs measures can be arrived at.

The first scenario led to what are called today *almost Gibbsian* measures [17, 19, 2], and the second led to *weakly Gibbsian* measures [21, 14, 1, 2]. It turned out that almost Gibbsian measures are weakly Gibbsian but this is not true the other way round [19, 11]. Also, presently we have an understanding of when some classes of transformations map certain (generalized) Gibbs measures in other (generalized) Gibbs measures [15, 1]. Here we will not touch upon further questions about the nature of such generalized Gibbs measures, however it is worth noting that these problems have grown into a new and stimulating field of research pinpointing a class of probability measures that can be expected to describe physically interesting equilibrium states though not being as strong as usual Gibbs measures. As it happens, however, in some cases Gibbs measures transform into measures which are not even weakly Gibbsian [15], going thus beyond the likely limit of the range of thermodynamically sensible notions of equilibrium state.

Since there is no single general principle of how to choose a specific renormalization scheme for studying a specific model system, another possibility to obtain RG-maps transforming Gibbs measures into other Gibbs measures is that of combining them in certain ways [18]. In the next section I will discuss cases when combined RG-maps indeed preserve Gibbsianness; in general this may lead to results depending on the measures to transform [16]. There is no clear relationship between this way and the other described above, and our present-day understanding is that for practical purposes the two might be taken in some combination.

4 Generalized Gibbs measures: Is this the right framework?

In the light of the previous discussion the central questions are: *What are conditions a Gibbs measure and a renormalization transformation should satisfy for a renormalized potential to exist in $\mathcal{B}(\Omega')$? Furthermore, how can the concept of Gibbs measure be generalized such that the so obtained object is a useful description of thermodynamical equilibrium states and a more stable class under renormalization transformations?*

Here are the new concepts presently in use:

Definition 4.1 *A probability measure ϱ on $(\Omega, \mathcal{F}, \chi)$ is* almost Gibbsian *if there exists a uniformly non-null specification Γ on $(\Omega, \mathcal{F})$ such that ϱ is consistent with it and the subset*

$$\Omega_\Gamma = \{\xi \in \Omega : \lim_{\tilde{\Lambda} \to \mathcal{L}} \sup_{\substack{\omega, \eta \in \Omega \\ \xi_{\tilde{\Lambda} \setminus \Lambda} = \eta_{\tilde{\Lambda} \setminus \Lambda}}} |\gamma_\Lambda(\omega_\Lambda, \xi_{\Lambda^c}) - \gamma_\Lambda(\omega_\Lambda, \eta_{\Lambda^c})| = 0, \ \forall \Lambda \subset \tilde{\Lambda} \in \mathcal{P}(\mathcal{L})\} \tag{4.1}$$

carries full measure, i.e., $\varrho(\Omega_\Gamma) = \varrho(\Omega) = 1$.

Definition 4.2 *A probability measure ϱ on $(\Omega, \mathcal{F}, \chi)$ is* weakly Gibbsian *with respect to a potential $\Phi : \mathcal{L} \times \Omega \to \mathbb{R}$ if there exists a function $b : \Omega \to \mathbb{R}$ such that the subset*

$$\Omega_\Phi = \{\omega \in \Omega : \sum_{\Lambda \ni 0} |\Phi_\Lambda(\omega)| < b(\omega)\} \tag{4.2}$$

carries full measure, i.e., $\varrho(\Omega_\Phi) = \varrho(\Omega) = 1$, and ϱ is consistent on this subset with Γ^Φ.

Definition 4.3 *A non-Gibbsian probability measure ϱ on $(\Omega, \mathcal{F}, \chi)$ is* robustly non-Gibbsian *if for every decimation transformation T on Ω the measure $T\varrho$ is non-Gibbsian. If there is a decimation transformation $T : \Omega \to \Omega'$ for which $T\varrho$ is Gibbsian for some potential in $\mathcal{B}(\Omega')$ then we call ϱ* non-robustly non-Gibbsian.

Comments:

(1) Clearly, the first two generalizations relax the uniformity in configurations occurring in the quasilocality property, respectively summability of the potential. Almost Gibbsian measures arise by requiring pointwise quasilocality almost surely and using Theorem 2.1, while weakly Gibbsian measures arise by requiring the potential to be almost surely pointwise absolutely summable.

(2) For a Gibbs measure ϱ consistent with a specification Γ we have $\Omega_\Gamma = \Omega$, respectively b can be chosen to be a constant so that $\Omega_\Phi = \Omega$.

(3) The potential (unique up to some details, inessential here) with respect to which we speak of an almost Gibbsian measure is one which can be reconstructed from Γ by formally taking its "logarithm"; the main idea will be sketched below. Also, it will be shown below for a class of transformations how to obtain from the full set of configurations the subset of allowed ones on which to construct a weakly Gibbsian measure.

(4) We know of examples of probability measures for which $\Omega_\Gamma = \emptyset$ [7, 16]. This is an extreme form of non-Gibbsianness in the sense that we believe that no sensible weak form of Gibbs measure can be defined in this case. Though there is no rigorous evidence of it, it may be conjectured that in this case the measure is not even weakly Gibbsian.

(5) There is no clear relationship between either of the classes defined by Defs. 4.1 and 4.2 and the class defined by Def. 4.3. Indeed, it is possible that a measure is non-robustly non-Gibbsian but is strongly non-Gibbsian in the sense discussed in point (4) above [7].

In this survey we first give a general result on the Gibbsianness of renormalized measures. For simplicity we choose here $\mathcal{L} = \mathbb{Z}^d$, and write $\nu = T\mu$ for the renormalized measure; also, we suppose that μ is a Gibbs measure for a given finite range potential.

Take finite volumes $\Lambda \subset \Lambda' \subset \Lambda'' \subset \mathcal{L}'$, where $\mathcal{L}' \subset \mathbb{Z}^d$ is the "renormalized lattice", and write $\Lambda_1 = \Lambda' \setminus \Lambda$, $\Lambda_2 = \Lambda'' \setminus \Lambda'$. Also, pick $\xi, \bar{\xi} \in \Omega$, such that $\xi_{\Lambda_1} = \bar{\xi}_{\Lambda_1}$. For an $\mathcal{F}_{\Lambda^c}$-measurable function f we write the conditional expectations

$$\mu^{\xi_\Lambda}(f) \equiv \frac{\int f(\omega) \prod_{x\in\Lambda} t_x(\xi_x|\omega)\mu(d\omega)}{\int \prod_{x\in\Lambda} t_x(\xi_x|\omega)\mu(d\omega)} = \mu(f|\xi_x = T_x(\omega), x \in \Lambda). \tag{4.3}$$

A computation yields that

$$\nu(f|\bar{\xi}_{\Lambda_1\cup\Lambda_2}) - \nu(f|\xi_{\Lambda_1\cup\Lambda_2}) = \frac{\mu^{\xi_{\Lambda_1\cup\Lambda_2}}(Tf; p_y^{\xi,\bar{\xi}})}{\mu^{\xi_{\Lambda_1\cup\Lambda_2}}(p_y^{\xi,\bar{\xi}})} \tag{4.4}$$

where $p_y^{\xi,\bar{\xi}} = t_y(\bar{\xi}_y|\omega)/t_y(\xi_y|\omega)$ and Tf is a function obtained from f by the transformation induced by T on the space of measurable functions. In the numerator at the right hand side above we have the truncated pair correlation function for the functions that appear there. The denominator can be bounded uniformly; so if it can be shown that the specific correlations of $\mu^{\xi_{\Lambda_1\cup\Lambda_2}}$ decay well enough, then quasilocality of ν will follow.

For our argument in the present set up it suffices to look at

$$\mu_x(\cdot \mid \eta) = \mu(\omega_x = \cdot \mid \omega_y = \eta_y, |x-y| \le R), \tag{4.5}$$

where R is the range of the potential for μ. For every $x \in \mathbb{Z}^d$ define the parameters

$$q_x = \max_{\eta,\bar{\eta}} \operatorname{var}\left(\mu_x(\cdot \mid \eta), \mu_x(\cdot \mid \bar{\eta})\right) \tag{4.6}$$

ranging from 0 to 1, "var" denoting the variational distance of probability measures. Our main point here is to give a condition on quasilocality of the image measure.

Theorem 4.4 (Quasilocality of transformed measures) *Suppose T is a renormalization transformation. Then there is $q^* = q^*(\beta)$ such that if $q_x < q^*$ for all $x \in \mathbb{Z}^d$, then $T\mu$ is quasilocal. If $d = 1$, we have $q^* = 1$.*

As a direct consequence we have

Corollary 4.5 *If μ is a Gibbs measure for a finite range potential, then there is a $\beta^* > 0$ such that $T\mu$ is a Gibbs measure for all $\beta < \beta^*$. If $d = 1$, then $T\mu$ is Gibbsian for all β.*

The idea of proof goes like this (for more details see [15]). Denote by η a configuration on $\mathcal{L}'$; it is an element of $S'^{\mathcal{L}'}$. Look at the conditional probabilities $\mu_x^{\xi_{\Lambda_1\cup\Lambda_2}}(\cdot \mid \eta) \equiv \mu_x^{\xi_{\Lambda_1\cup\Lambda_2}}(\cdot \mid \eta_y, y \sim x)$, where η_y denotes the configuration restricted to the sites that are adjacent to x (denoted $y \sim x$) in the sense that $\operatorname{dist}(B_x, B_y) = 1$, where B_x, B_y are the blocks assigned by the renormalization transformation to sites x and y, respectively. Next, look at

$$q_x^{\xi_{\Lambda_1\cup\Lambda_2}} = \max_{\eta,\bar{\eta}} \operatorname{var}\left(\mu_x^{\xi_{\Lambda_1\cup\Lambda_2}}(\cdot \mid \eta), \mu_x^{\xi_{\Lambda_1\cup\Lambda_2}}(\cdot \mid \bar{\eta})\right). \tag{4.7}$$

It is easily seen that if q_x, then $q_x^{\xi_{\Lambda_1\cup\Lambda_2}} \to 0$. Choose $q \equiv \sup_x q_x \ge q_x \ge q_x^{\xi_{\Lambda_1\cup\Lambda_2}}$. In a further step construct the graph with vertex set $\mathbb{Z}^d$, and edges connecting those pairs of vertices (x, y) that are adjacent in the sense that $x \sim y$. Connected edges take the value 1, the others 0 on the graph. Put Bernoulli measure λ_q on $\{0,1\}^{\mathbb{Z}^d}$ with density q as defined above. From Bernoulli percolation we know that if p_c is the threshold percolation density for the given graph then for $q < p_c$ there are constants $c, m > 0$ such that $\lambda_q(A \rightsquigarrow B) \le c\exp(-m \operatorname{dist}(A, B))$. Here $A \rightsquigarrow B$ is a shorthand for the event that a volume A on the lattice is connected with a disjoint volume B through a random path formed by connected edges. By a somewhat involved argument, which we skip in this presentation, it turns out that the two point correlations appearing in (4.4) can be bounded from above by such Bernoulli path probabilities. Putting all this together we are led to the estimate

$$|\nu(f|\bar{\xi}_{\Lambda_1\cup\Lambda_2}) - \nu(f|\xi_{\Lambda_1\cup\Lambda_2})| \le 2\|Tf\|_\infty \, \lambda_q(\bar{\Lambda} \rightsquigarrow \Lambda_2) \le \text{const} \, \exp(-m \operatorname{dist}(\bar{\Lambda}, \Lambda_2)) \tag{4.8}$$

where $\bar{\Lambda} = \Lambda \cup (\cup_{z\in\Lambda} B_z)$, which on taking thermodynamic limit brings about quasilocality of ν. The whole idea underlying the argument was thus to investigate the effect in the state ν of far-out spins on the spin at the origin by some two point correlation functions in the constrained state $\mu^{\xi_{\Lambda_1\cup\Lambda_2}}$; these correlations were at their turn shown to be decaying exponentially by comparing with a specially constructed Bernoulli percolating system.

Next, to pin down the other reasonable end of scenarios we want to give an idea, once again using methods of stochastic geometry, of how far renormalizations can be expected to lead to at least weakly Gibbsian measures. For simplicity, let us look only at decimation from $\mathcal{L} = \mathbb{Z}^d$ to the sublattice $\mathcal{L}' = b\mathbb{Z}^d \equiv \{x \in \mathbb{Z}^d : x \bmod b = 0\}$ with some positive integer number $b > 1$, and start from a Gibbs measure μ given for a pair potential Φ. Here the constrained measure with boundary condition τ is

$$\mu_\Lambda^{\tau,\xi} \equiv \mu_\Lambda(\cdot \mid \omega_{\Lambda\cap b\mathbb{Z}^d} = \xi, \omega_{\Lambda^c} = \tau). \tag{4.9}$$

We denote by $\mu_\Lambda^\tau = \mu_\Lambda(\cdot \mid \omega_{\Lambda^c} = \tau)$ the usual conditional measure, i.e., when no constraint inside Λ is imposed. The decimated measure obtained from μ is denoted by ν.

We will argue here, again without going into details of proof (for that see [14]), that for the decimated measure those are "good" configurations ξ that in some sense resemble τ, which is chosen to be a typical configuration of μ; putting them together will make a subset of configurations on which $\mu^{\tau,\xi}$ is a weakly Gibbsian measure. Such a situation would occur, for instance, in what is called the Pirogov-Sinai regime of Ising systems.

Take finite volumes $\Lambda_k \subset \mathbb{Z}^d$ and construct $V_k(x) = (\Lambda_k + x) \cap b\mathbb{Z}^d$, by shifting $V_k \equiv V_k(0)$ with $x \in b\mathbb{Z}^d$. We fix a boundary condition to be, for simplicity, a constant $\tau_x = a$, $\forall x$, and use it as a reference configuration. In all these boxes we measure the degree of agreement between τ and the ξ configuration on which the decimated measure lives by the counting measure

$$\mathrm{agr}_{k,x}(\xi,\tau) \equiv \frac{1}{|V_k(x)|} \sum_{y\in V_k(x)} 1_{\{\xi_y = a\}}. \tag{4.10}$$

Define the subsets of configurations

$$\Omega_l^\tau(x) = \{\xi \in S^{b\mathbb{Z}} : \ \forall k > l, \ \mathrm{agr}_{k,x}(\xi,\tau) > 1-\epsilon\} \tag{4.11}$$

with some suitable $\epsilon > 0$, and

$$\Omega^\tau = \bigcup_{l\geq 1} \Omega_l^\tau(x). \tag{4.12}$$

Ω^τ is tail measurable and does actually not depend on x; it can be shown actually to carry full measure. Moreover, for any of its elements ξ and every position $x \in b\mathbb{Z}^d$ a characteristic length $l(\xi, x) < \infty$ exists such that $\mathrm{agr}_{k,x}(\xi, a) > 1 - \epsilon$. Look now at the joint space $S^{\Lambda_n} \times S^{\Lambda_n}$, and define the set

$$\mathcal{D}_{\Lambda_n}(\omega,\omega';a) = \{x \in \Lambda_n : S^{\Lambda_n} \times S^{\Lambda_n} \ni (\omega_x, \omega_x') \neq (a,a)\}, \tag{4.13}$$

i.e., the positions in Λ_n where both configurations ω, ω' disagree with a. On a similar joint space we define the event

$$\Pi_{\Lambda_n}^x(A,B) = \{(\omega,\omega') \in S^{\Lambda_n+x} \times S^{\Lambda_n+x} : A \leftrightarrow B\} \tag{4.14}$$

where $A \leftrightarrow B$ stands as a shorthand for "$\exists\{x_0, x_1, ..., x_n\} \subset \mathcal{D}_{\Lambda_n+x}(\omega, \omega'; a) : A \cap B = \emptyset, A \ni x_0, B \ni x_n, |x_i - x_{i+1}| = 1, \forall i = 0, ..., n-1$"; $\Pi^x_{\Lambda_n}(A, B)$ corresponds thus to the existence of a "path of disagreement" in the above sense linking a volume A with a disjoint volume B. We need one more notation: $\xi^{k,x}$ will be the configuration that agrees with ξ on $V_k(x)$ and with τ on $b\mathbb{Z}^d \setminus V_k(x)$.

Take now the independent coupling $\mu^{\tau,\xi^{k,x}}_{\Lambda_n(x)} \times \mu^{\tau,\xi^{k,x}}_{\Lambda_n(x)}$ with factors as defined in (4.9). We say that μ^τ is a *stable low temperature phase with respect to boundary condition* τ if there exist constants $C(\beta), m(\beta) > 0$, $\lim_{\beta\to\infty} m(\beta) = \infty$, such that

$$\mu^{\tau,\xi^{k,x}}_{\Lambda_n(x)} \times \mu^{\tau,\xi^{k,x}}_{\Lambda_n(x)} \left(\Pi^x_{\Lambda_n(x)}(\mathcal{O}, B_{k,n}(x))\right) \leq C(\beta)e^{-m(\beta)k} \tag{4.15}$$

uniformly in n whenever $n > k > l(\xi, x)$, $\xi \in \Omega^\tau$, where $\mathcal{O} = \{x \in \mathbb{Z}^d : |x| = 1\}$ and $B_{k,n}(x) = (\Lambda_n + x) \setminus (\Lambda_k + x)$. Roughly speaking, this means that disagreement probabilities between the "reference" configuration and "good" configurations become exponentially small as soon as one looks at the constrained system from beyond the scale of the characteristic length for the picked configuration, or in other words, disagreement is localized in relatively small pockets and $\mu^{\tau,\xi}$ looks pretty much the same as μ^τ on large scales. This "pretty much" will imply on taking the thermodynamic limit that if μ was a Gibbs measure, then $T\mu$ will be a weakly Gibbsian measure (or possibly more regular).

Theorem 4.6 (Weakly Gibbsian low temperature renormalized measures) *If μ^τ is a stable low temperature phase with respect to configuration τ, then the decimation to any sublattice $b\mathbb{Z}^d$, $b = 2, 3, ...$, of μ is weakly Gibbsian on Ω^τ.*

In what follows we outline a method showing how to construct a potential. Denote by ξ^0 the configuration agreeing with ξ everywhere except the origin, and set equal to a at the origin. We look at the quantities [1]

$$h^\tau_n(\xi) = \log \frac{\nu^\tau_{\Lambda_n}(\xi)}{\nu^\tau_{\Lambda_n}(\xi^0)}. \tag{4.16}$$

Take now the sequence of volumes $U_k = U_{k-1} \cup \{u_k\}$ constructed inductively with $|u_k| \geq |u_{k-1}|$, $u_1 = 0$ and $U_0 = \emptyset$, with some sequence u_k such that $|u_k| \leq x$, $|x| \geq |u_{k-1}|$, $x \in b\mathbb{Z}^d$. The configuration ${}^k\xi$ is set to agree with ξ on U_k and with τ on $b\mathbb{Z}^d \setminus U_k$. By rewriting (4.16) in the manner of a telescopic sequence, we arrive at

$$h^\tau_n(\xi) = \sum_{k=1}^{n^*} \left(h^\tau_n({}^k\xi) - h^\tau_n({}^{k-1}\xi)\right) \tag{4.17}$$

n^* being a number fixed by the equality $U_{n^*} = V_n$. As it is easily seen, ${}^0\xi = \tau$ and $h^\tau_n(\tau) = 0$.

[1] This is inspired by the fact that for μ^Φ, or more generally for any Gibbs measure, $h^\tau_\Lambda(\xi) \equiv \mathcal{H}^\Phi_\Lambda(\xi|\tau) - \mathcal{H}^\Phi_\Lambda(\xi^0|\tau) = \log[\mu^\Phi_\Lambda(\xi|\tau)/\mu^\Phi_\Lambda(\xi^0|\tau)]$ is a formula (interpreted as a relative energy) that makes the inverse relationship between measure and Hamiltonian. Once having these relative Hamiltonians at hand, a potential can be computed by inverse Möbius transform which is essentially described in the following.

Define

$$\Psi_n^k(\xi) = h_n^\tau({}^k\xi) - h_n^\tau({}^{k-1}\xi) \tag{4.18}$$

and

$$f_x^{\tau,\xi}(\omega) = \exp\left(-\beta \sum_{y:|y-x|=1} [\Phi(a,\omega_y) - \Phi(\xi_x,\omega_y)]\right) \tag{4.19}$$

where remember that Φ is the potential for the Gibbs measure μ. It can be checked that the sequence of functions Ψ_n^k is a potential. Moreover, the following properties can be proven:

1. for all $k > 2$
$$\Psi_n^k(\xi) = \log\left(1 + \frac{\mu_{\Lambda_n}^{\tau,{}^k\xi}\left(f_0^{\tau,\xi}; f_{u_k}^{\tau,\xi}\right)}{\mu_{\Lambda_n}^{\tau,{}^k\xi}\left(f_0^{\tau,\xi}\right)\mu_{\Lambda_n}^{\tau,{}^k\xi}\left(f_{u_k}^{\tau,\xi}\right)}\right); \tag{4.20}$$

2. for all $\xi \in \Omega^\tau$
$$|\mu_{\Lambda_n}^{\tau,{}^k\xi}(f_0^{\tau,\xi}; f_{u_k}^{\tau,\xi})| \leq 2e^{4\beta\|\Phi\|_\infty}\left(\mu_{\Lambda_n}^{\tau,{}^k\xi} \times \mu_{\Lambda_n}^{\tau,{}^k\xi}\right)\left(\Pi_{\Lambda_n}^0(\mathcal{O}, \Lambda_n \setminus \Lambda_k)\right). \tag{4.21}$$

As seen in the first statement, Ψ_n^k can be controlled by specific two-point correlation functions of the constrained measure $\mu_{\Lambda_n}^{\tau,{}^k\xi}$. The second statement says that these correlations are at their turn controlled by disagreement probabilities in the independently coupled copies of these measures. Putting these two facts together we conclude that whenever μ^τ is a stable low temperature phase, some constants $0 < C(\beta) < \infty$, $\delta(\beta) > 0$ ($\delta(\beta) \to \infty$ as $\beta \to \infty$) can be found such that

$$|\Psi_n^k(\xi)| \leq C(\beta)\, e^{-\delta(\beta)k} \tag{4.22}$$

for every $\xi \in \Omega_l^a(0)$ whenever $l(\xi,k) < k$. This then means that on the full-measure subset Ω^a Ψ is an absolutely summable potential, i.e., ν is a weakly Gibbsian measure on this subset.

Finally we turn to an example showing how by decimating a non-Gibbsian measure the resulting measure can be Gibbsian. This point will indicate that various RG-maps combined between them can be well behaving in the sense of keeping the initial measure Gibbsian. However, we diverge from the usual renormalization transformations defined above for the sake of illustrating a whole range of possible phenomena; one should note, though, that there are examples of decimations combined with genuine RG-transformations behaving similarly well, see e.g. [18]. In the concluding part of this report we talk thus of lower dimensional projections of the pure phases in the Ising model. This example was presented first by Schonmann, who showed that the plus-phase of the 2D Ising model projected to a line of the square lattice is non-Gibbsian in the entire subcritical region.

Schonmann's example can be generalized to projections from $\mathbb{Z}^d$ to $\mathbb{Z}^{d-1}$. Take thus $S = \{-1,+1\}$, and $\mu^{+,\beta}$, the translation invariant plus-phase of the d-dimensional Ising system at inverse temperature β, and look at the measure formally defined by

$$\nu^{+,\beta}(d\xi) = \int_{\{-1,+1\}^{\mathbb{Z}^d} \setminus \{-1,+1\}^{\mathbb{Z}^{d-1}} \times \{0\}} \mu^{+,\beta}(d\sigma \times d\xi). \tag{4.23}$$

$\nu^{+,\beta}$ is thus the marginal distribution of $\mu^{+,\beta}$ over the one dimension less sublattice. Completely similarly one can define $\nu^{-,\beta}$ and $\nu^{h,\beta}$ as the marginals of the translation invariant minus-phase $\mu^{-,\beta}$, respectively the state $\mu^{h,\beta}$ given in the presence of an external magnetic field $h \in \mathbb{R}$ on $\mathbb{Z}^d$. Below we will denote the critical temperature of the d-dimensional Ising system by β_c, and by J its coupling constant. We take furthermore the sublattices $b\mathbb{Z}^{d-1}$, $b = 2, 3, \ldots$, and consider the measures $\nu_b^{+,\beta}, \nu_b^{-,\beta}, \nu_b^{h,\beta}$ arising by taking the marginals of $\nu^{+,\beta}, \nu^{-,\beta}, \nu^{h,\beta}$ to $b\mathbb{Z}^{d-1}$. These last measures can be seen as arising from the corresponding d-dimensional Ising measures through a combined operation of lower dimensional projection and decimation.

Theorem 4.7 *The following statements are true:*

1. [low-temperature non-Gibbsianness] *for $d \geq 2$ and every $\beta > \beta_c$ the measures $\nu^{+,\beta}, \nu^{-,\beta}$ are non-Gibbsian;*

2. [high-temperature analyticity] *for $d \geq 2$ and $\beta J < \pi/4z(d) < \beta_c J$, $\nu^{+,\beta}$ resp. $\nu^{-,\beta}$ are completely analytic (in the sense of Dobrushin-Shlosman theory, i.e., in a sense "very regular"), where $z(d)$ is the coordination number of the lattice $\mathbb{Z}^{d-1}$;*

3. [2D uniqueness regime] *take $d = 2$; for every $\beta > 0$ and $h \neq 0$ the measure $\nu^{h,\beta}$ is Gibbsian; moreover, for every $\beta < \beta_c$ the unique measure $\nu^{+,\beta} = \nu^{-,\beta}$ is Gibbsian;*

4. [high-D uniqueness regime] *take $d \geq 3$; for every $\beta > 0$ and $|h|$ large enough the measure $\nu^{h,\beta}$ is Gibbsian; for too small $|h| \neq 0$ it is conjectured that a surface phase transition appears (so called Basuev states) and then $\nu^{h,\beta}$ is almost Gibbsian but non-Gibbsian; moreover for every $\beta < \beta_c$ the unique measure $\nu^{+,\beta} = \nu^{-,\beta}$ is almost Gibbsian;*

5. [weak Gibbsianness at worst] *for every $\beta > 0$ the measures $\nu^{+,\beta}, \nu^{-,\beta}$ are weakly Gibbsian;*

6. [2D non-robustness] *take $d = 2$; for each $b \geq 3$ there is $\beta_c < \beta_b = \beta_\infty(b+1)/(b-2)$, with some $0 < \beta_\infty < \infty$, such that for $\beta \geq \beta_b$ there exist two (in Ruelle sense) inequivalent potentials $\Phi_b^+, \Phi_b^- \in \mathcal{B}(\{-1,+1\}^{b\mathbb{Z}^{d-1}})$ such that $\nu_b^{+,\beta}$ is a Gibbs measure with respect to Φ_b^+ and $\nu_b^{-,\beta}$ is a Gibbs measure with respect to Φ_b^-.*

7. [strong non-Gibbsianness] *in the last set-up consider a non-trivial mixture $\mu^\beta(\lambda) = \lambda\mu^{+,\beta} + (1-\lambda)\mu^{-,\beta}$, $0 < \lambda < 1$; then for the specification $\Gamma_b^\beta(\lambda)$ corresponding to the measure $\nu_b^\beta(\lambda) = \lambda\nu_b^{+,\beta} + (1-\lambda)\nu_b^{-,\beta}$ we have $\Omega_{\Gamma_b^\beta(\lambda)} = \emptyset$, for every $\beta \geq \beta_b$.*

For the proof of (1) see [23], for (2-4) see [12], for (5) [21], and for (6-7) see [16, 7].

As seen from statement (6) above, the non-robustness result is state dependent; even though $\nu_b^{+,\beta}$ and $\nu_b^{-,\beta}$ originate from the same potential by applying the combined transformations to them, they are Gibbs measures with respect to two distinct potentials, i.e., the potential one ends up with after performing the transformations will depend on via which particular phase one has gone. This seems to be specific for Schonmann's example and is not the case in the known schemes of RG-transformations combined with decimations.

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