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Renormalization Group Flow of Two-Dimensional $O(N)$ Spin Model

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We develope a new block spin transformation and apply it to the 2D $O(N)$ spin model. The transformation does not yield complicated non-local terms and then the transformation recursion formula seems to be controlable for any initial inverse tepmarature $\beta > 0$. The main part of the block spin transformation of the model with large N converges to a massive state, no matter how low the initial temperature $1/\beta$ is, and is close to the flow of the hierarchical model advocated by Dyson and Wilson several decades ago.

I. INTRODUCTION

Though quark confinement in four-dimensional (4D) non-abelian lattice gauge theories and spontaneous mass generations in two-dimensional (2D) non-abelian sigma models are widely believed [17, 18], any rigorous proof of them is still not available except for some hierarchical models [6, 8, 10]. One difficulty in solving these problems is that field variables form some compact manifolds and then block spin transformations break the structures of these manifolds.

Some of these difficulties can be bypassed by introducing an auxiliary field [1]. Using this trick, we recently proved [14, 15] that the critical inverse temperature $\beta(N)$ in 2D $O(N)$ spin model satisfies the bound $\beta(N) > \text{const}N \log N$. Extending the methods used in [14], we can appply a new block spin transformation [5, 19] to the model, which yields, no matter how large β is, only small controllable non-local terms.

Though we leave the rigorous control of these terms for the near future [13], we discuss the main part of the non-linear recursion formulas in this Letter, and show [12] that

Main Theorem. *Within the approximate block spin transformations,*

(1) there exists no phase transition in the 2D O(N) spin model for all β if N is large enough.

(2) The renormalization group flow of the model is close to that of the hierarchical model proposed by Dyson and Wilson several decades ago [3, 6, 11, 19].

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(3) Corrections to the correlation length by the conventional renormalization group method are small.

To begin with, we scale the inverse temperature β by N [16] to obtain the Gibbs expectation values of the ν dimensional $O(N)$ spin model at the inverse temperature $N\beta$:

$$< P >\equiv \frac{1}{Z_\Lambda(\beta)}\int P(\phi)e^{-H_\Lambda(\phi)}\prod_i \delta(\phi_i^2 - N\beta)d\phi_i \tag{1}$$

where $\Lambda = [-(L/2)^M, (L/2)^M)^\nu \subset \mathbf{Z}^\nu$ is the large square with center at the origin, where L is a positive interger (with L around 3 or 4) and M is an arbitrarily large integer. Moreover $\phi(x) = (\phi(x)^{(1)}, \cdots, \phi(x)^{(N)}) \in R^N$ is the vector valued spin at $x \in \Lambda$, Z_Λ is the partition function defined so that $< 1 >= 1$, and H_Λ is the Hamiltonian given by

$$H_\Lambda \equiv -\frac{1}{2}\sum_{|x-y|=1}\phi(x)\phi(y). \tag{2}$$

We first substitute the identity $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)]da/2\pi$ into eq.(1) with the condition [1] that $\mathrm{Im} a_i < -\nu$. We set

$$\mathrm{Im}\, a_i = -(\nu + \frac{m^2}{2}), \quad \mathrm{Re}\, a_i = \frac{1}{\sqrt{N}}\psi_i \tag{3}$$

where $m > 0$. Thus we have

$$\begin{aligned} Z_\Lambda &= c^{|\Lambda|}\int\cdots\int\prod_j e^{i\sqrt{N}\beta\psi_j}\frac{d\phi_j d\psi_j}{2\pi} \\ &\quad \times \exp[-\frac{1}{2} < \phi, (m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)\phi >] \\ &= c^{|\Lambda|}\int\cdots\int F(\psi)\prod\frac{d\psi_j}{2\pi}, & (4) \\ F(\psi) &= \det(1 + \frac{2iG_0}{\sqrt{N}}\psi)^{-N/2}\exp[i\sqrt{N}\beta\sum_j\psi_j]. & (5) \end{aligned}$$

where c's are constants, $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|,1}$ is the lattice laplacian and $G_0 = (m^2 - \Delta)^{-1}$. In the dimension $\nu \le 2$, we can choose $m > 0$ so that $G_0(0) = \beta$ for any $\beta > 0$, where

$$G_0(x) = \int\frac{e^{ipx}}{m^2 + 2\sum(1 - \cos p_i)}\prod_{i=1}^{\nu}\frac{dp_i}{2\pi}. \tag{6}$$

In fact $m^2 \sim 32e^{-4\pi\beta}$ for $\nu = 2$ as $\beta \to \infty$, which is consistent with the renormalizaiton group (RG) analysis, see e.g. [2]. Thus for $\nu = 2$, we can rewrite

$$F(\psi) = \det{}_3(1 + \frac{2iG_0}{\sqrt{N}}\psi)^{-N/2}\exp[-\mathrm{Tr}(G_0\psi)^2] \tag{7}$$

for all β, where $\det{}_3(1 + A) = \det[(1 + A)e^{-A+A^2/2}]$.

If N is sufficiently large for given β, $\det_3(1 + 2iG_0\psi/\sqrt{N}) \sim 1$ is a small perturbation to the Gaussian measure $\exp[-\mathrm{Tr}(G_0\psi)^2]\prod d\psi$ and exponential cluster decay follows [14]. It is also argued in [14] that the correlation functions decay exponetially fast if $0 \leq F(\psi)$. However if β is large, then $G_0(x,y) \sim \beta - (2\pi)^{-1}\log(1+|x-y|)$ for $|x-y| << m^{-1}$ and the previous argument fails.

But the previous argument may survive if the main contribution to the integral comes from $|\psi| < \beta^{-\alpha}$, $\alpha > 0$ so that the expansion of the determinat can be justified. How can we check this ? We decompose the determinant into the product of determinants each of which is expandable and easy to analyze. The block spin transformation (BST) is most convenient for this purpose and each determinant comes from the integration over the fluctuation field ξ_n of ϕ. Since the fluctuation fields have short correlations, the resultant determinants are expandable.

We organize the paper as follows: in Section 2, we introduce the block spin transformation with the auxiliary field ψ and calculate the first step transformation. In section 3, we extract the main part of the transformation and solve it. The effective interactions V_n and conclusions are given in Section 4.

II. BLOCK SPIN TRANSFORMATION WITH THE AUXILIARY FIELD

To realize our scenario, we decompose $\Lambda \subset Z^2$ into blocks $\square_{Lx}$ of size $L \times L$, centered at $Lx \in LZ^2$, and repeat the following steps ($\phi_0 \equiv \phi$, $\psi_0 \equiv \psi$):

(1) integrate by ϕ_{n-1} keeping their block averages at ϕ_n,

(2) integrate by ψ_{n-1} keeping their block sums at ψ_n.

To start with, we note that $G(0) = \beta$ and set

$$W_0(\phi,\psi) = \frac{1}{2} < \phi, G_0^{-1}\phi > -i < J_0, \psi >, \tag{8}$$

$$J_0(x) = -\frac{1}{\sqrt{N}} : \phi^2(x) :_{G_0} = \sqrt{N}\beta - \frac{1}{\sqrt{N}}\phi^2(x), \tag{9}$$

where $: A :_{G_0}$ is the Wick product of A with respect to the Gaussian probability measure $d\mu_0(\phi)$ of mean zero and covariance G_0^{-1}, and $< f, g >= \sum_x f(x)g(x)$ (if $f(x)$, $g(x) \in R^N$, the inner product in R^N is also taken.)

We represent $\phi(x) \equiv \phi_0(x)$ and $\psi(x) \equiv \psi_0(x)$ in terms of block spins $\phi_1(x) = (C\phi_0)(x)$ and $\psi_1(x) = (C'\psi_0)(x)$, and fluctuations $\xi_0(\zeta)$ of ϕ_0 and $\tilde{\psi}_0(\zeta)$ of ψ_0, where $x \in \Lambda_1$, $\Lambda_n \equiv Z^2 \cap L^{-n}\Lambda$ and $\zeta \in \Lambda - L\Lambda_1$. The operator C takes the arithmatic averages of $\phi(x)$ over the blocks and the operator C' takes sums of $\psi(x)$ over the blocks, and the both subsequently scale the coordinates

$$(C\phi)(x) = L^{-2}\sum_{\zeta\in\Box}\phi(Lx+\zeta), \tag{10}$$

$$(C'\psi)(x) = L^2(C\psi)(x) = \sum_{\zeta\in\Box}\psi(Lx+\zeta) \tag{11}$$

where $x \in \Lambda_1$ and $\Box$ is the box of size $L\times L$ center at the origin. These transformation rules mean that we assume that the boson fields ϕ_n (as well as ϕ_n^2) are relevant, but the auxiliary fields ψ_n (as well as $\phi_n^2(x)\psi_n(x)$) are marginal. The latter reflects the fact that the ψ field interacts almost antiferromagnetically, see (7).

The covariance matrix $G_n(x,y)$ of $\phi_n = C\phi_{n-1}$ $(n=1,2,\cdots)$ is given by

$$CG_{n-1}C^+(x,y) = L^{-4}\sum_{\zeta_1,\zeta_2\in\Box} G_{n-1}(Lx+\zeta_1, Ly+\zeta_2).$$

We introduce the transformation matrices A_n and the operator Q by

$$A_n(x,y) = G_{n-1}C^+G_n^{-1}(x,y), \tag{12}$$

$$(Q\xi)(x) = \begin{cases} \xi(x) & \text{if } x\notin LZ^2 \\ -\sum_{y\in\Box(x)}\xi(y) & \text{if } x\in LZ^2 \end{cases} \tag{13}$$

Then the substitution

$$\phi_n(x) = (A_{n+1}\phi_{n+1})(x) + (Q\xi_n)(x) \tag{14}$$

yields the decomposition

$$<\phi_n, G_n^{-1}\phi_n> = <\phi_{n+1}, G_{n+1}^{-1}\phi_{n+1}> + <\xi_n, Q^+G_n^{-1}Q\xi_n>$$

where $(Q\xi_n)(x)$ are the zero-average fluctuations made from $\xi(\zeta)$ $(x\in\Lambda_n, \zeta\in\Lambda_n - L\Lambda_{n+1})$. Since

$$(Q^+f)(x) = f(x) - f(x_0) \tag{15}$$

with $x_0 \in L\Lambda_1$ being the nearest point to x, $Q^+ : R^\Lambda \to R^{\Lambda\setminus L\Lambda_1}$ acts as a differentiation.

Let us see what happens in the fluctuation integral:

$$e^{-W_1(\phi_1,\psi_1)} = \int\prod d\tilde\psi(x)\left\{\int e^{-W_0(A_1\phi_1+Q\xi_0,\tilde A_1\psi_1+Q\tilde\psi)}\prod d\xi_0(x)\right\} \tag{16}$$

where $\tilde A_1$ is determined later. We see that

$$\begin{aligned}&\{\cdots\} \text{ of } (16) = \det{}^{-\frac{N}{2}}(1+K_0)\\ &\times\exp[-\frac{1}{2}<\phi_1, G_1^{-1}\phi_1> + i\sqrt{N}\sum_x(\beta - \frac{1}{N}(\varphi_1)_x^2)\psi_x]\\ &\times\exp[-\frac{2}{N}<Q^+(\varphi_1\cdot\psi), \frac{1}{P}Q^+(\varphi_1\cdot\psi)>]\end{aligned} \tag{17}$$

except for the trivial coefficient, where $\varphi_1(x) = (A_1\phi_1)(x)$, $x \in \Lambda$ and

$$\begin{aligned}
\Gamma_0 &\equiv [Q^+(-\Delta + m^2)Q]^{-1}, & (18)\\
K_0 &= \frac{2i}{\sqrt{N}}\Gamma_0 Q^+ \psi Q, & (19)\\
P &\equiv \Gamma_0^{-1} + \frac{2i}{\sqrt{N}} Q^+\psi Q. & (20)
\end{aligned}$$

No matter how small m^2 is, Γ_0, the propagator of the fluctuations has the mass of order $(m^2 + L^{-2})^{1/2}$ [5], and then the determinant has locality even if $m = 0$. $P^{-1} \sim \Gamma_0$ also exhibits uniform exponential decay, uniformly in ψ [14]. Note that

$$\begin{aligned}
\det{}^{-\frac{N}{2}}(1 + K_0) &= \exp[-i\sqrt{N} < \mathrm{diag}\mathcal{T}_0, \psi > - < \psi, \mathcal{T}_0^{\circ 2}\psi >]\eta(\psi),\\
\eta(\psi) &\equiv \det{}_3^{-\frac{N}{2}}(1 + K_0),
\end{aligned}$$

where $\mathcal{T}_0 = Q\Gamma_0 Q^+$ and we set $(A \circ B)(x, y) \equiv A(x, y)B(x, y)$ and $A^{\circ 2} \equiv A \circ A$ for matrices A and B (Hadamard product). Thus our integrand is written

$$\exp\left[- < \psi, \hat{H}_0^{-1}\psi > +i < J_1, \psi >\right] \eta(\psi) \tag{21}$$

except for $e^{-\frac{1}{2}<\phi_1, G_1^{-1}\phi_1>}$, where

$$\begin{aligned}
\hat{H}_0^{-1} &= \mathcal{T}_0^{\circ 2} + \frac{2}{N}[(Q\frac{1}{P}Q^+) \circ (\varphi_1 \cdot \varphi_1)], & (22)\\
J_1(x) &= \sqrt{N}\beta - \sqrt{N}\mathcal{T}_0(x, x) - \frac{1}{\sqrt{N}}(\varphi_1)_x^2 & \\
&= -\frac{1}{\sqrt{N}} : \varphi_1^2(x) :_{G_1} & (23)
\end{aligned}$$

and $: \varphi(x)\varphi(y) :_{G_1} \equiv \varphi(x)\varphi(y) - N(A_1G_1A_1^+)(x, y)$ denotes the Wick product with respect to $d\mu_{G_1}$, and we have used $Q\Gamma_0Q^+ = G_0 - G_0C^+G_1^{-1}CG_0$.

This expansion must be carefully treated. We define the small field of $\psi(x)$ by

$$|\frac{2}{\sqrt{N}}\Gamma_0 Q^+\psi Q| = o(1). \tag{24}$$

Γ_0 is strictly positive and bounded, so this means $|Q^+\psi Q| < N^\delta$, where $\delta < \frac{1}{2}$. To integrate over ψ in this way, we assume that $Q^+\psi Q$ are small, that is, $||\Gamma_0 Q^+\psi Q|| < O(N^\delta)$, where $0 < \delta < 1/2$. We take N large (independet of β.) If there are blocks $\square_i$ where ψ takes large values which prohibit the expansion of the determinant, we extract the large field regions and we estimate directly. The small field region is a collection of blocks where the expansion of the determinant converges absolutely

We obtain the main term $\tilde{H}_0^{-1}$ of $\hat{H}_0^{-1}$ by replacing $\varphi_1(x)\varphi(y)$ by $N\mathcal{G}_1(x,y)$, $\mathcal{G}_1 \equiv A_1 G_1 A_1^+$:

$$\hat{H}_0^{-1} \equiv \tilde{H}_0^{-1} + \delta\tilde{H}_0^{-1}, \tag{25}$$

$$\tilde{H}_0^{-1} \equiv \mathcal{T}_0^{\circ 2} + 2\mathcal{T}_0 \circ \mathcal{G}_1 = \mathcal{G}_0^{\circ 2} - \mathcal{G}_1^{\circ 2}, \tag{26}$$

$$\begin{aligned} \delta\tilde{H}_0^{-1} &= \frac{2}{N}[\mathcal{T}_0 \circ (: \varphi_1 \cdot \varphi_1 :_{G_1})] \\ &+ \frac{2}{N}[(Q(\frac{1}{P} - \Gamma_0)Q^+) \circ (\varphi_1 \cdot \varphi_1)], \end{aligned} \tag{27}$$

Note that $\tilde{H}_0^{-1}$ is strictly positive and $\tilde{H}_0^{-1} \geq O(\beta)$ on the zero-average field $\{Q\tilde{\psi}\} = QR^{\Lambda\backslash L\Lambda_1}$. Moreover, $\tilde{H}_0 \delta H_0^{-1} = O(1/N)$ on $\mathcal{K}_1$ defined below. Thus we treat $\tilde{H}_0\delta\tilde{H}_0^{-1}$ by perturbation. Thus, we should set

$$\tilde{A}_1 = \tilde{H}_0(C')^+ H_1^{-1}, \ H_1 = C'\tilde{H}_0(C')^+, \tag{28}$$

so that

$$\begin{aligned} <\psi, \tilde{H}_0^{-1}\psi> &= <\psi_1, H_1^{-1}\psi_1> + <\tilde{\psi}, Q^+\tilde{H}_0^{-1}Q\tilde{\psi}>, \\ <J_0, \psi> &= <J_1, \tilde{A}_1\psi_1> + <Q^+J_1, \tilde{\psi}>. \end{aligned}$$

The operator $\tilde{A}_1$ is almost diagonal for large β. In fact $\tilde{H}_0^{-1} \sim 2\beta Q\Gamma_0 Q^+$ is a differential operator restricted to the blocks $\square_{Lx}$, $x \in \Lambda_1$ and thus zero on the set of blockwise constant functions and is large on the set of (blockwise) average-zero functions. Thus one finds [12] that

$$\begin{aligned} \tilde{A}_1(x,y) &= \frac{1}{L^2}\delta_{[x/L],y} + \frac{1}{\beta}\delta\tilde{A}_1(x,y), && (29) \\ \delta\tilde{A}_1(x,y) &= O(\exp[-|x/L - y|]), \quad (x \in \Lambda, \ y \in \Lambda_1), \\ H_1^{-1}(x,y) &= \frac{1}{L^4}\sum_{\zeta,\xi\in\square}(Q\Gamma_0 Q^+)(Lx+\zeta, Ly+\xi)^2 + O(1) \\ &\sim \delta_{x,y}, \quad x, y \in \Lambda_1 && (30) \end{aligned}$$

where for $x \in \Lambda$, $[x/L] \in \Lambda_1$ is the lattice point nearest from x/L.

The small-smooth field $\mathcal{K}_n$ is the set of φ_n and ψ_n which dominates the integrals by ξ_n and $\tilde{\psi}_n$. $\mathcal{K}_1(X)$ is a collection of $\{\varphi_1(x), (\tilde{A}_1\psi_1)(x); x \in X \subset \Lambda\}$ such that

$$(1) \quad ||\varphi_1(x)| - (N\mathcal{G}_1(x,x))^{1/2}| < \beta^{-1/2}N^{\varepsilon}, \tag{31}$$

$$(2) \quad |\partial_\mu\varphi_1(x)| < (N)^{1/2+\varepsilon}, \tag{32}$$

$$(3) \quad |\partial_\mu(\tilde{A}_1\psi_1)(x)| < \beta^{-1/2}N^c \tag{33}$$

for all $x \in X$, where $\partial\mu$ is the lattice differential operator on the lattice space $L^{-1}\Lambda$ [5], $0 < \varepsilon < 1/2$ and $0 < c$ are small positive constants. The first condition means that φ_1 stays around at the bottom of the potential, and the second means that there exist no strong domain walls in $\mathcal{K}_1(X)$. The sets $\mathcal{K}_n(X)$, $n = 2, 3\cdots$ are defined in the same way. The "large" and/or "irregular" field configurations which do not obey the above, have small probabilities to exist.

Since $|Q^+J(x)| = O(N^\epsilon)$ for $\phi_1 \in \mathcal{K}_1$, we can integrate by $\tilde{\psi}$ to obtain

$$\begin{aligned} &\det{}^{-1/2}(Q^+\tilde{H}_0^{-1}Q)\exp[-\mathcal{F}_1], \\ \mathcal{F}_1 &= \frac{1}{4} < Q^+J_1, (Q^+\tilde{H}_0^{-1}Q)^{-1}Q^+J_1 > . \end{aligned}$$

Since $\mathcal{T}_0 = Q\Gamma_0 Q^+$ is a strictly positive operator of short range on the set $\{Q\tilde{\psi}\}$, so is $\mathcal{T}_0 \circ \mathcal{G}_1 \sim \beta\mathcal{T}_0 > O(\beta)$. Then $\tilde{H}_0^{-1}$ and $Q^+\tilde{H}_0^{-1}Q > O(\beta)$ are positive operators of short range, and the contribution of $\tilde{\psi}$ comes from $|\tilde{\psi}(x)| < \text{const.}\beta^{-1/2}$. Since $Q^+\tilde{H}_0^{-1}Q$ is bounded below by const.β, $\mathcal{F}_1 \leq O(N^{2\epsilon}/\beta)$ on $\mathcal{K}_1$ per unit volume. Therefore the integral over $\tilde{\psi}$ yields small corrections of ψ_1 and $:\phi_1^2:_{G_1}$ only. Thus we have $\exp[-W_1(\phi_1, \psi_1)]$ as follows:

$$\begin{aligned} &\exp[-\frac{1}{2} < \phi_1, G_1^{-1}\phi_1 > - < \psi_1, H_1^{-1}\psi_1 > \\ &\quad +i < J_1, \tilde{A}_1\psi_1 > -\mathcal{F}_1 + \delta W_1] \end{aligned} \tag{34}$$

where δW_1 is the remainder. Comparing this with W_0, (8), we see that the approximate flow is represented by

$$\begin{aligned} J_0 = -\frac{1}{\sqrt{N}} : \phi_0^2(x) :_{G_0} \;\rightarrow\; J_1 = -\frac{1}{\sqrt{N}} : \varphi_1^2(x) :_{G_1}, \\ H_0^{-1} = 0 \;\rightarrow\; H_1^{-1} = (C'\tilde{H}_0 C'^+)^{-1} \end{aligned}$$

or simply by the flow of β_k: $\beta_1 = \beta - \mathcal{T}_0(x,x)$.

What we found here is that the large factor β_1 in $\tilde{H}_0$ is wiped out by the $\tilde{\psi}$ integral with negligible reminiscent, and the coefficient of the block spin ψ_1 does not contain β_1 and is order $O(1)$. This also means that the fluctuation fields $\xi_0 \in R^N$ are almost orthogonal to the block spins $\phi_1 \in R^N$.

In the next section, we show that this is the case for all n. Thus, we can obtain the recursion formulas in a closed form under physically reasonable approximations.

III. APPROXIMATE RENORMALIZATION GROUP FLOW

We introduce

$$\mathcal{A}_n = A_1 \cdots A_n = G_0(C^+)^n G_n^{-1} \tag{35}$$

and set $\varphi_n(x) = (\mathcal{A}_n\phi_n)(x)$, $x \in \Lambda$ so that the transformation rule (14) is written $\varphi_n(x) = \varphi_{n+1}(x) + z_n(x)$, where the covariances of φ_n and $z_n \equiv \mathcal{A}_n Q\xi_n$ are

$$\mathcal{G}_n = \mathcal{A}_n G_n \mathcal{A}_n^+, \tag{36}$$

$$\mathcal{T}_n = \mathcal{A}_n Q\Gamma_n Q^+ \mathcal{A}_n^+. \tag{37}$$

The iteration is easy if we neglect $\delta\tilde{H}_n^{-1}$, the higher order terms coming from the determinants and

$$\mathcal{F}_n = \frac{1}{4} < Q^+\tilde{\mathcal{A}}_{n-1}^+ J_n, (Q^+\tilde{H}_{n-1}^{-1}Q)^{-1}Q^+\tilde{\mathcal{A}}_{n-1}^+ J_n >$$

which comes from the $d\tilde{\psi}_{n-1}$ integral, where

$$\begin{aligned} J_n(x) &= -\frac{1}{\sqrt{N}} : \varphi_n^2(x) :_{\mathcal{G}_n}, & (38)\\ \tilde{H}_{n-1}^{-1} &= H_{n-1}^{-1} + \tilde{\mathcal{A}}_{n-1}^+[\mathcal{T}_{n-1} \circ (\mathcal{T}_{n-1} + 2\mathcal{G}_n)]\tilde{\mathcal{A}}_{n-1}, & (39)\\ H_n &= C'\tilde{H}_{n-1}(C')^+, & (40)\\ \tilde{A}_n &= \tilde{H}_{n-1}(C')^+ H_n^{-1}, & (41)\\ \tilde{\mathcal{A}}_n &= \tilde{A}_1 \cdots \tilde{A}_n. & (42) \end{aligned}$$

In fact the property (15) of Q^+ and the fact that $G_n(0) \sim \beta_n \sim \beta$ imply that $\mathcal{F}_n$ are marginal and of order $O(N^{2\varepsilon}/\beta_n)$ per unit volume. Then the effects of the fluctuations z_n coming from $\mathcal{F}_n$ are small. (Some of them may be absorbed by renormalizations.)

Neglecting all marginal terms of order less than $O(N^{2\varepsilon})$, we have the approximate RG flow:

$$\begin{aligned} W_n(\phi_n, \psi_n) &= \frac{1}{2} < \phi_n, G_n^{-1}\phi_n > + < \psi_n, H_n^{-1}\psi_n > \\ &\quad -i < J_n, \tilde{\mathcal{A}}_n\psi_n >, & (43)\\ J_n(\phi_n) &= J_{n-1}(A_n\phi_n) - \sqrt{N}\mathcal{T}_{n-1} \\ &= \sqrt{N}(\beta - \sum_0^{n-1} \mathcal{T}_i - \frac{1}{N}\varphi_n^2), & (44) \end{aligned}$$

with $H_0^{-1} = 0$. Since $C^n\mathcal{A}_n = (C')^n\tilde{\mathcal{A}}_n = 1$, and since $\mathcal{A}_n(x,y)$ and $\tilde{\mathcal{A}}_n(x,y)$ decay exponentially fast, the approximate diagonality of $\mathcal{A}_n$ and $\tilde{\mathcal{A}}_n$ follows:

$$\mathcal{A}_n(x,y) \sim \delta_{[\frac{x}{L^n}],y}, \quad \tilde{\mathcal{A}}_n(x,y) \sim \frac{1}{L^{2n}}\delta_{[\frac{x}{L^n}],y} \tag{45}$$

where $x \in \Lambda$, $y \in \Lambda_n$ and $[x/L^n] \in \Lambda_n$ is the lattice point nearest from x/L^n. In fact the first follows from the definition (35), see [5], and the second follows as a generalization of (29) which holds for all $\tilde{A}_\ell$, $\ell = 1, \cdots, n$ whenever $\beta_n >> L^2$ [12]. (This notation for $\mathcal{A}_n(x,y)$ is different from that in [5] where x stands for $x/L^n \in L^{-n}\Lambda$.)

Since $Q\Gamma_n Q^+ = G_n - G_n C^+ G_{n+1}^{-1} C G_n$, we see that J_n is given by

$$\sqrt{N}\left(\mathcal{G}_n(x,x) - \frac{\varphi_n^2(x)}{N}\right) = -\frac{1}{\sqrt{N}} :\varphi_n^2(x):_{G_n}. \tag{46}$$

Note that $G_0(x) \sim \beta - (2\pi)^{-1}\log(1+|x|)$ for $|x| << m^{-1}$ and $G_0(x) \sim c_1 \exp[-c_2 m|x|]$ for $|x| > m^{-1}$ ($c_i = \text{const} > 0$). Then

1. $G_n(x,y) \sim \beta - (2\pi)^{-1}\log L^n(1+|x-y|)$, if $L^n|x-y| < m^{-1}$,
2. $G_n(x,y) \sim L^{-2n}m^{-2}\delta_{xy}$, if $L^n m > 1$.

and we have $\beta_n \equiv G_n(x,x) \sim \beta - (n/2\pi)\log L$ for $mL^n << 1$ and $\beta_n \sim m^{-2}L^{-2n}$ for $L^n m > 1$.

IV. EFFECTIVE POTENTIAL AND CONCLUSION

To see the flow of the effective interactions, we substitute (45) into $\mathcal{G}_n$ and $\mathcal{T}_n$. Then the second factor of $\tilde{H}_{n-1}^{-1}$ in (39) is

$$\tilde{\mathcal{A}}_{n-1}^+[\mathcal{T}_{n-1}^{\circ 2} + 2\mathcal{G}_n \circ \mathcal{T}_{n-1}]\tilde{\mathcal{A}}_{n-1} \sim \alpha 1 + 2\beta_n Q\Gamma_{n-1}Q^+, \tag{47}$$

where $\alpha = O(1) > 0$. The effect of H_{n-1}^{-1} is small since ψ_n^2 is irrelevant. In fact see (30). Then we again have

$$H_n^{-1} \sim \delta_{x,y}, \quad x,y \in \Lambda_n \tag{48}$$

$$< J_n, \tilde{\mathcal{A}}_n\psi_n > \sim -\frac{1}{\sqrt{N}} <:\phi_n^2:_{G_n}, \psi_n > . \tag{49}$$

Thus, the ψ_n integral yields the double-well potential approximately of the form

$$V_n \sim \frac{1}{N}(\phi_n^2 - N\beta_n)^2. \tag{50}$$

This is very close to the flow of the hierarchical model advocated by Dyson and Wilson (with large N) [3, 6, 11, 19, 20], rather than to that by Gallavotti [4, 7]. We note that this fact comes from the approximate diagonality of $\tilde{A}_n$ or equivalently from the fact that the fluctuation fields ξ_n are almost orthogonal to the blockspins ϕ_{n+1}. In ref. [10], this is claimed to be the origin of the mass generation in the model.

Since $V_n \sim \beta_n(|\phi_n| - \sqrt{N\beta_n})^2$, $||\phi_n| - \sqrt{N\beta_n}|$ must be less than $\beta_n^{-1/2}$ and the constraint (31) follows. One corollary of our results is that the main contribution of the ψ integral comes from

$|\psi| <$ const.$\beta^{-1/2}$ since $|\tilde{\psi}_n| <$ const.$\beta_n^{-1/2}$ and $\psi(x) \sim \sum L^{-2n}\tilde{\psi}_n([x/L^n])$. Thus, we expect that the determinant is effectively expandable and correction to the correlation length $\xi = m^{-1}$ is small.

It will be possible to make these arguments rigorous by taking the effects of the large fields and the non-local terms into considerations. This will be reported elsewhere [13].

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