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Wegner Estimate for Indefinite Anderson Potentials: Some Recent Results and Applications

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Abstract

We review recent and give some new results on the spectral properties of Schrödinger operators with a random potential of alloy type. Our point of interest is the so called Wegner estimate in the case where the single site potentials change sign. The indefinitness of the single site potential poses certain difficulties for the proof of the Wegner estimate which are still not fully understood.

The Wegner estimate is a key ingredient in an existence proof of pure point spectrum of the considered random Schrödinger operators. Under certain assumptions on the considered models additionally the existence of the density of states can be proven.

Keywords: density of states, random Schrödinger operators, Wegner estimate, multi scale analysis, localization, indefinite single site potential

1 Introduction and statement of results: Alloy type models and Wegner's estimate

The subject matter of this work are families of Schrödinger operators $\{H_\omega\}_{\omega\in\Omega}$ acting on $L^2(\mathbb{R}^d)$. They have been introduced as quantum mechanical models for disordered media in solid state physics. The random Schrödinger operator we consider is of *Anderson* or *alloy* type and given by the following:

Assumption 1.1 (Alloy type model) *Let* *(i)* V_0 *be a* $\mathbb{Z}^d$*-periodic potential, which is a infinitesimally small perturbation of* $-\Delta$ *on* $L^2(\mathbb{R}^d)$*, and* $H_0 := -\Delta + V_0$ *a periodic Schrödinger operator.*

(ii) $\omega := \{\omega_k\}_{k\in\mathbb{Z}^d} \in (\Omega, \mathbb{P})$ *be a random vector composed of the coordinates* ω_k*. Here* $\Omega = \times_{\mathbb{Z}^d}\mathbb{R}, \mathbb{P} := \otimes_{\mathbb{Z}^d}\mu$*, where* μ *is the normalized Lebesgue measure on* $[\omega_-, \omega_+]$.

(iii) the coupling constants $\alpha_k : \Omega \to \mathbb{R}$ *be given by the projection* $\alpha_k(\omega) := \omega_k$, $\forall k \in \mathbb{Z}^d$*. Then* $\{\alpha_k = \omega_k\}_{k\in\mathbb{Z}^d}$ *forms an iid sequence of random variables.*

(iv) the single site potential u *be in* $l^1(L^p) = \{f \in L^p_{loc}(\mathbb{R}^d) | \; \|f\|_{l^1(L^p)} < \infty\}$ *where*

$$\|f\|_{l^1(L^p)} := \sum_{k\in\mathbb{Z}^d} \left(\int_{\|x\|_\infty<1/2} |f(x-k)|^p dx \right)^{1/p}$$

(v) the alloy type potential be given by the stochastic process

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} \alpha_k(\omega)\, u(x-k) = \sum_{k\in\mathbb{Z}^d} \omega_k\, u(x-k). \tag{1}$$

(vi) a family of Schrödinger operators be given by

$$H_\omega := H_0 + V_\omega, \; \omega \in \Omega. \tag{2}$$

The above assumptions ensure by the Kato-Rellich theorem that each H_ω is a selfadjoint operator on the domain of the Laplacian.

Assumption 1.2 (Assumptions for the Wegner estimate) *Let additionally:*

(i) $\kappa > 0$ *and* $\kappa\chi_{[0,1]^d} \leq w \in l^1(L^p(\mathbb{R}^d))$*, where* $p := p(d) = 2$ *for* $d \leq 3$ *and* $p(d) > d/2$ *for* $d \geq 4$.

(ii) a partial ordering on $\mathbb{R}^d \ni j, k$ *be given by* $j \succ k \Leftrightarrow j_i \geq k_i \; \forall \; i = 1, \ldots, d$.

(iii) $\Gamma \subset \{k \in \mathbb{Z}^d | \; k \succ 0\}$ *be a finite set,* $a = \{a_k\}_{k\in\mathbb{Z}^d}$ *be a so called* convolution vector *with* $a_k \neq 0 \Rightarrow k \in \Gamma$ *and* $a^* := \sum_{k\neq 0} |a_k| < a_0$.

(iv) the single site potential be a generalized step function

$$u(x) = \sum_{l\in\mathbb{Z}^d} a_l w(x-l).$$

For any cube $\Lambda_l = \Lambda_l(0) = [0, l[^d$ we can restrict H_ω to $L^2(\Lambda_l)$ with appropriate boundary conditions (b.c.). The results and proofs in this paper are equally valid if we chose for the restriction H_ω^l Dirichlet, Neumann or periodic b.c. We denote the spectral projection of H_ω^l on the energy interval $I =]E_1, E_2[$ by $P_\omega^l(I)$ and the characteristic function of the unit cube $\Lambda_1(j) = [0,1[^d + j$ at the lattice site $j \in \mathbb{Z}^d$ by χ_j. The expectation w.r.t. $\mathbb{P}$ is denoted by $\mathbb{E}$. Our Wegner estimate [Weg81] reads:

Theorem 1.3 *For all $E_2 \in \mathbb{R}$ there exist a constant $C = C(E_2)$ such that for all $l \in \mathbb{N}$ and $E_1 \leq E_2$ we have*

$$\mathbb{E}\left[\operatorname{Tr} P_\omega^l(]E_1, E_2[)\right] \leq C\,(\omega_+ - \omega_-)^{-1}\,(E_2 - E_1)\,l^d. \tag{3}$$

Remark 1.4 By replacing the convolution vector a with κa we may assume $\kappa = 1$ in Assumption 1.1 (i). Furthermore, by rescaling the support of μ we may assume $a_0 = 1$. Note that by adding a part of the periodic potential to V_ω we can assume without loss of generality that the support of μ starts at 0, i.e. $\operatorname{supp}\mu = [0, \omega_+]$ for some $\omega_+ > 0$. Our results are also true, if we have $a_0 = -1$ and $a^* < 1$ in our model. In this case, in the proofs everywhere where positivity is used, negativity has to be used instead.

In the next section we deduce the existence of the density of states from the Wegner estimate in Theorem 1.3 and discuss its role for the proof of localization. Section 3 contains the proof of Theorem 1.3 and Section 4 reviews earlier results for indefinite alloy type models.

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2 Density of states and localization

Under our assumptions the family $H_\omega, \omega \in \Omega$ fits into the general theory of ergodic random Schrödinger operators [Kir89, CL90, PF92]. We infer two central results from this theory.

(A) The spectrum of the family $H_\omega, \omega \in \Omega$ is non-random in the following sense. There exists a subset Σ of the real line and an $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$ one has $\sigma(H_\omega) = \Sigma$. The analogous statement holds true for the essential, discrete, continuous, absolutely continuous, singular continuous, and pure point part of the spectrum. Note that the pure point spectrum σ_{pp} is the closure of the set of eigenvalues of H_ω.

(B) There exists a *self averaging* integrated density of states associated with the family $H_\omega, \omega \in \Omega$. This means that the normalized eigenvalue counting functions

$$N^l_\omega(E) = l^{-d}\#\{i|\ \lambda_i(H^l_\omega) < E\} = l^{-d}\,\mathrm{Tr}\,P^l_\omega(]-\infty, E[) \qquad (4)$$

of H^l_ω converge for almost all ω to a limit $N := \lim_{l\to\infty} N^l_\omega$ which is ω-independent. For definiteness we use periodic b.c. in the construction of H^l_ω.

We call N the *integrated density of states* (IDS) of H_ω and N^l_ω the *finite volume* IDS of H^l_ω.

Remark 2.1 While the two above facts (A) and (B) follow from the general theory, one is interested in more detailed spectral properties of specific models $H_\omega, \omega \in \Omega$, e.g.:

- Which spectral types can occur in $\sigma(H_\omega)$?
- Can something be said about the regularity of the IDS N as a function of the energy E? Is it Hölder continuous or does even its derivative, the *density of states* exist.

Our result on the regularity of the IDS is strong enough to imply the existence of the density of states:

Theorem 2.2 (Density of states) *Under the assumptions of Theorem 1.3 the IDS of the alloy type model $\{H_\omega\}_{\omega\in\Omega}$ is Lipschitz continuous: for all $E \in \mathbb{R}$ there exists a constant C such that*

$$N(E) - N(E-\epsilon) \leq C\epsilon, \quad \forall\, \epsilon \geq 0\,. \qquad (5)$$

It follows that the derivative $\frac{dN}{dE}$ exists for almost all E.

Remark 2.3 The theorem follows directly from (3) and the self averaging property $N(\cdot) = \mathbb{E}\, N(\cdot)$.

The second question of Remark 2.1 is related to the transport properties of the medium modelled by H_ω. A perfect crystal is described by a Schrödinger operator with periodic potential. It has purely absolutely continuous spectrum, which reflects its good electric transport properties. In contrast to this, it has been proven that random perturbations of this regular structure give rise to energy intervals with pure point spectrum. This corresponds to the less effective transport properties of random media. The existence of pure point spectrum in this context is called *localization*.

Now we indicate the general scheme of the proof of localization and where the Wegner estimate enters. An intermediary step in the proof of localization is the establishing of the exponential decay of the resolvent

$$\sup_{\epsilon\neq 0} \|\chi_x R(\epsilon)\chi_y\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq const\, e^{-c|x-y|} \text{ for almost all } \omega\ , \tag{6}$$

where $R(\epsilon) := (H_\omega - E - i\epsilon)^{-1}$ is the resolvent of H_ω near an energy value E in the energy interval $I \subset \mathbb{R}$ (typically near a boundary of $\sigma(H_\omega)$) for which we want to prove localization. The χ_x and χ_y are characteristic functions of unit cubes centered at x, respectively at y. This bound can be used to rule out absolutely continuous spectrum [MS85] and is interpreted as absence of diffusion [FS83, MH84] in the energy region I if (6) holds for all $E \in I$.

It turns out that the finite size resolvent $R_\Lambda(\epsilon) := (H_\omega^\Lambda - E - i\epsilon)^{-1}$ is easier approachable than $R(\epsilon)$ on the whole space. Here H_ω^Λ is the restriction of H_ω to $L^2(\Lambda)$ with some appropriate boundary conditions; the use of Dirichlet or periodic b.c. is most common. However the operator H_ω^Λ is not ergodic and for its resolvent an estimate like (6) can be expected to hold only with a probability strictly smaller than one. This is the place where multi scale analysis (MSA) enters. It is an induction argument over increasing length scales l_j. They are defined recursively by $l_{j+1} := [l_j^\zeta]_3$, where $[l_j^\zeta]_3$ is the greatest multiple of 3 smaller than l_j^ζ. The scaling exponent ζ has to be from the interval $]1,2[$. On each scale one considers the box resolvent $R_j(\epsilon) := R_{\Lambda_{l_j}}(\epsilon)$ and proves its exponential decay with a probability which tends to 1 as $j \to \infty$. We outline briefly the ingredients of the MSA as it is given in [CH94, KSS98] or [CL90].

First we explain some notation which is used afterwards. Let $\delta > 0$ be a small constant independent of the length scale l_j and $\phi_j(x) \in C^2$ a function which is identically equal to 0 for x with $\|x\|_\infty > l_j - \delta$ and identically equal to one for x with $\|x\|_\infty < l_j - 2\delta$. The commutator $W(\phi_j) := [-\Delta, \phi_j] := -(\Delta\phi_j) - 2(\nabla\phi_j)\nabla$ is a local operator acting on functions which live on a ring of width δ near the boundary of $\Lambda_j := \Lambda_{l_j}$. We say that a pair $(\omega, \Lambda_j) \in \Omega \times \mathcal{B}(\mathbb{R}^d)$ is *m-regular*, if

$$\sup_{\epsilon\neq 0} \|W(\phi_j) R_j(\epsilon) \chi_{l_j/3}\|_{\mathcal{L}} \leq e^{-ml_j}\ . \tag{7}$$

Here $\|\cdot\|_{\mathcal{L}}$ is the operator norm on $L^2(\Lambda_j)$ and $\chi_{l_j/3}$ the characteristic function of $\Lambda_{l_j/3} := \{y |\, \|y\|_\infty \leq l_j/6\}$. Thus the distance of the supports of $\nabla\phi_j$ and $\chi_{l/3}$ is at least $l_j/3 - 2\delta \geq l_j/4$.

Let $q_0 > 0$ and $m_0 \geq const\, l_0^{-1/4}$. The starting point of the MSA is the estimate

$$(\mathrm{H1})(l_0, m_0, q_0) \qquad\qquad \mathbb{P}\{\omega |\, (\omega, \Lambda_0) \text{ is } m_0\text{-regular}\} \geq 1 - l_0^{q_0}$$

which serves as the base clause of the induction. The induction step consists in proving

$$(\mathrm{H1})(l_j, m_j, q_j) \Longrightarrow (\mathrm{H1})(l_{j+1}, m_{j+1}, q_{j+1}) \tag{8}$$

For the mass of decay m_{j+1} and the probability exponent q_{j+1} on the scale l_{j+1} the following estimates are valid

$$\forall \xi > 0\ \exists c_1, c_2, c_3 \text{ independent of } j \text{ such that}$$

$$m_{j+1} \geq m_j \left(1 - \frac{4l_j}{l_{j+1}}\right) - \frac{c_1}{l_j} - c_2 \frac{\log l_{j+1}}{l_{j+1}} \tag{9}$$

$$l_{j+1}^{q_{j+1}} \leq c_3 \left(\frac{l_{j+1}}{l_j}\right)^{2d} l_j^{2q_j} + \frac{1}{2} l_{j+1}^{-\xi} . \tag{10}$$

For the recursion clause (8) a Wegner estimate as in (3) is needed:

$$\text{(H2)} \qquad \mathbb{P}\{\omega |\, d(\sigma(H_\omega^\Lambda), E) \leq \epsilon\} \leq C_W\, \epsilon |\Lambda|^2$$

for all boxes $\Lambda \subset \mathbb{R}^d$ and all $\epsilon > 0$, such that $[E - \epsilon, E + \epsilon]$ is contained in neighbourhood of I. Here $|\Lambda|$ stands for the Lebesgue measure of the cube Λ.

The deterministic part of the induction step uses the *geometric resolvent formula* [CH94, HS96]

$$\phi_\Lambda (H_{\Lambda'} - z)^{-1} = (H_\Lambda - z)^{-1} \phi_\Lambda + (H_\Lambda - z)^{-1} W(\phi_\Lambda)(H_{\Lambda'} - z)^{-1} \tag{11}$$

for $z \in \rho(H_{\Lambda'}) \cap \rho(H_\Lambda)$ and $\phi_\Lambda \in C^2$ with support in $\Lambda \subset \Lambda'$. It gives the estimate

$$\|\chi_{l/3}(\cdot - x) R_{3l'}(\epsilon) \chi_{l/3}(\cdot - y)\|_{\mathcal{L}} \leq (3^d e^{-ml})^{3|x-y|l^{-1}-4} \|R_{3l'}(\epsilon)\|_{\mathcal{L}} \tag{12}$$

if no two disjoint non-regular boxes $\Lambda_l \subset \Lambda_{l'}$ with center in $\frac{l}{3}\mathbb{Z}^d \cap \Lambda_{3l'}$ exist for ω. In our case $l := l_j$ is the length scale on which the exponential decay of the resolvent is already known and $l' := l_{j+1}$ the scale on which we want to prove it. By the estimates (H1),(H2) we have with probability $1 - l_{j+1}^{q_{j+1}}$ (bounded by the inequality (10)) exponential decay on the length scale l_{j+1} with mass m_{j+1} (bounded as in (9)).

We stated above the ingredients of the MSA as they are valid if u is compactly supported. If the single site potential is of long range type (as in (13) below) one has to use the adapted MSA from the papers [KSS98, Zen99].

Once the estimate (H1) is established on all length scales $l_j, j \in \mathbb{N}$, one infers an exponential decay estimate for the resolvent on the whole of $\mathbb{R}^d$. Afterwards one uses a spectral averaging technique (cf.[CH94]) based on ideas of Kotani, Simon, Wolf and Howland to conclude localization [KS87, SW86, How87]. An alternative version of the MSA can be found in the monograph [Sto01] (see also [GK01a, GK01b].

Recent papers concentrate on proofs for the Wegner estimate and the initial length scale decay of the resolvent. At the same time adaptations of the MSA for various random Schrödinger operators, as well as Hamiltonians governing the motion in classical physics appeared [FK96, FK97, CHT99, Sto98].

We discuss briefly some results for quantum mechanical Hamiltonians. For V_ω a Gaussian random field a Wegner estimate was shown in [FHLM97]. Its main feature is that no underlying lattice structure of V_ω is needed. This result allows one to conclude localization for the corresponding Schrödinger operator at low energies [FLM00]. Kirsch, Stollmann and Stolz proved in [KSS98] (cf. also [Zen99]) a Wegner estimate with only polynomial decay conditions on the single site potential u and deduced a localization result for Hamiltonians with long range interactions. They require

$$|u(x)| \leq const\,(|x|+1)^{-m} \text{ for some } m > 4d\,. \tag{13}$$

The resolvent decay estimate (H1) for some initial length scale can be proved with semiclassical techniques. Using the Agmon metric one can achieve rigorously decay bounds with what is called among physicists WKB-method [CH94, HS96]. However this reasoning is only applicable for energies near the bottom of the spectrum.

The so-called *Combes-Thomas argument* [CT73] allows one to infer the following inequality

$$\|\chi_x(H-z)^{-1}\chi_y\|_{\mathcal{L}} \leq \left[const\, d(\sigma(H),z)\right]^{-1} e^{-const\, d(\sigma(H),z)\,|x-y|} \tag{14}$$

where H is a self-adjoint Schrödinger operator on $L^2(\mathbb{R}^d)$ and $z \in \rho(H)$. It was first applied to multiparticle Hamiltonians [CT73], but it is also useful in our case, as soon as we get a lower bound on $d(\sigma(H_\omega^\Lambda), z)$. Thus it is sufficient to prove an estimate like

$$\mathbb{P}\{\omega |\ d(\sigma(H_\omega^l, I) < l^{-\alpha}/2\,\} \leq l^{-q} \tag{15}$$

for some $\alpha \in]0,1/4]$. Now Inequality (14) implies the initial scale estimate (H1) with $m_0 \geq const\, l^{-1/4}$ for l large and $E \in I$, cf. [KSS98, Lemma 5.5]. The constant depends on the energy and the potential, but not on l and m_0.

Two possibilities were used to deduce (15). The first is to assume a special disorder regime, more precisely to demand a sufficiently fast decay of the density g of the distribution of ω near the endpoints 0 and ω_+ of $\operatorname{supp} g$:

$$\exists \tau > d/2\ :\ \forall \text{ small } \epsilon > 0$$
$$\int_0^\epsilon g(s)ds \leq \epsilon^\tau, \text{ respectively } \int_{\omega_+-\epsilon}^{\omega_+} g(s)ds \leq \epsilon^\tau$$

depending on whether one wants to consider an energy interval I at a lower or upper spectral edge. This approach was used in [CH94, KSS98]. Its shortcoming is that it excludes quite a few distributions, e.g. the uniform distribution on $[0, \omega_+]$.

The other way to prove (15), is to use the existence of Lifshitz tails of the integrated density of states at the edges of the spectrum: One can show that for a variety of types of random Schrödinger operators, including ours, the IDS does not change, if one replaces the periodic b.c. in its definition by Dirichlet b.c.:

$$N(E) = \lim_{\Lambda \nearrow \mathbb{R}^d} |\Lambda|^{-1} \#\{ \text{ eigenvalues of } H_\omega^{\Lambda,D} \text{ below } E\} \ , \tag{16}$$

i.e. one considers the IDS as the limit of the normalized counting function of eigenvalues of the Dirichlet Hamiltonian $H_\omega^{\Lambda,D}$ on $L^2(\Lambda)$. The use of Dirichlet b.c. in in the above formula for the IDS implies [KM82]

$$N(E) = \sup_{\Lambda \nearrow \mathbb{R}^d} N(H_\omega^{\Lambda,D}, E) \ . \tag{17}$$

One says that $N(\cdot)$ exhibits *Lifshitz tails* at some spectral edge $\mathcal{E}$ if

$$\lim_{E \to \mathcal{E}} \frac{\log|\log|N(E) - N(\mathcal{E})||}{\log|E - \mathcal{E}|} = -\frac{d}{2} \ . \tag{18}$$

At the infimum of the spectrum, i.e. for $\mathcal{E} = \inf \sigma(H_\omega)$, (17) and (18) imply

$$\#\{\text{eigenvalues of } H_\omega^{\Lambda,D} \text{ in } [\mathcal{E}, E]\} \leq |\Lambda| N(E) \leq |\Lambda| \exp(-cE^{-d/4})$$

since $N(\mathcal{E}) = 0$. This estimate was used in [Klo95] together with a Čebišev inequality to prove (H1) at the bottom of the spectrum, see also [MH84]. For internal spectral edges the situation is similar, however one needs to know some additional properties of the unperturbed periodic operator $H_0 = -\Delta + V_0$, see [Klo99, Ves98].

If one considers the situation where the single site potential changes sign the initial scale estimate has been established only under restrictive hypotheses [Ves00, HK01].

3 Proof of Theorem 1.3

Let $\tilde{\Lambda} := \Lambda \cap \mathbb{Z}^d$ be the lattice points in $\Lambda = \Lambda_l$. As in [CH94] we estimate

$$\mathbb{E}\left[\operatorname{Tr} P_\omega^l(I)\right] \leq e^{E_2} C_V \sum_{j \in \tilde{\Lambda}} \left\| \mathbb{E}\left[\chi_j P_\omega^l(I) \chi_j\right] \right\| \cdot \tag{19}$$

where the constant C_V is an uniform upper bound on $\operatorname{Tr}(\chi_j e^{-H_\omega^{\Lambda+j}} \chi_j)$, cf. proof of Theorem 76 in [RS78]. Thus for the proof of Theorem 1.3 it is sufficient to prove the following proposition dealing with the expectation of a quadratic form.

Proposition 3.1 *Let $\Lambda = \Lambda_l$ for some $l \in \mathbb{N}$. For $f \in L^2(\Lambda_l)$ there exists a constant C such that for all $j \in \tilde{\Lambda}$*

$$\mathbb{E}\, \langle f, \chi_j P_\bullet^l(I) \chi_j f \rangle \leq C\, \omega_+^{-1}\, |I|\, \|f\|^2 . \tag{20}$$

It suffices to consider the case $\|f\| = 1$. Assume first $w = \chi_0$. Denote by Λ^+ the set $\tilde{\Lambda} - \Gamma := \{k - \gamma | \, k \in \tilde{\Lambda}, \gamma \in \Gamma\}$ of lattice sites in $\mathbb{Z}^d$ which influence the value of the potential in the cube Λ and by $L = \#\Lambda^+$ its cardinality. The convolution vector a defines a (block) Toeplitz matrix $A := \{A_{j,k}\}_{j,k\in\Lambda^+}$, $A_{j,k} := a_{j-k}, \forall j, k \in \Lambda^+$. Note that the coupling constants with index outside Λ^+ do not influence the random variable P_ω^l in (20). So we may pass on to a "smaller" probability space $\Omega = \mathbb{R}^L$ and consider the linear transformation $A : \mathbb{R}^L \to \mathbb{R}^L$, $A\omega = \eta$ for vectors $\omega := \{\omega_k\}_{k\in\Lambda^+}$ and $\eta := \{\eta_k\}_{k\in\Lambda^+}$. By Assumption 1.1 (iii) the inverse B of A exists and has its column sum norm $|||B|||_1$ bounded by $\frac{1}{1-a^*}$, cf. [Ves01, Sec. 4.4].

The random variable ω_0 has the density $g(x) = \frac{1}{\omega_+}\chi_{[0,\omega_+]}(x)$. Thus $G(\omega) := \prod_{j\in\Lambda^+} g(\omega_j)$ is the common density of ω and $K(\eta) := |\det B| \, G(B\eta)$ the one of η.

We calculate the representation of the alloy type potential in the new coordinates η. For $x \in \Lambda$

$$V_{B\eta}(x) = V_\omega(x) = \sum_{k\in\Lambda^+} \omega_k \sum_{l\in\Gamma} a_l \chi_{k+l}(x) = \sum_{j\in\tilde{\Lambda}} \eta_j \chi_j(x). \tag{21}$$

This representation particularly shows that for any fixed $j \in \tilde{\Lambda}$ we have a one parameter family of potentials, cf. [FHLM97]

$$\eta_j \mapsto \left(\sum_{j\neq k\in\tilde{\Lambda}} \eta_k \chi_k \right) + \eta_j \chi_j \tag{22}$$

which is linearly increasing locally on $\Lambda_1(j)$. This fact will later enable us to apply results from [CH94, Sec. 4]. Using the abbreviation

$$\mathcal{P}(\eta) := \langle f, \chi_j P_{B\eta}^l(I) \chi_j f \rangle. \tag{23}$$

the integral transformation of (20) reads

$$\mathbb{E}\,\langle f, \chi_j P_\bullet^l(I) \chi_j f \rangle = \int_{\mathbb{R}^L} d\eta \, k(\eta) \, \langle f, \chi_j P_{B\eta}^l(I) \chi_j f \rangle = \int_{\mathbb{R}^L} d\eta \, k(\eta) \, \mathcal{P}(\eta). \tag{24}$$

The integration domain $M := A([0, \omega_+]^L)$ in (24) is a compact set, thus for $t > 0$

$$(24) \leq \sup_{\eta\in M} \left[k(\eta)(1 + t\eta_j^2) \right] \int_M d\eta \frac{\mathcal{P}(\eta)}{1 + t\eta_j^2}. \tag{25}$$

The achievement of the last inequality is that we introduced an artificial density $\frac{1}{1+t\eta_j^2}$ with which we can deal better analytically and, more important, that we decoupled the dependence of the density on η_j and on the other components of η. Now

$$\sup_{\eta\in M}\left[k(\eta)(1+t\eta_j^2)\right] \le |\det B|\,\omega_+^{-L}\,(1+t\|A\|_1^2\eta_j^2) \tag{26}$$

leaves us with the analysis of the integral on the rhs of (25). In the next step we will decouple the dependence of the integration domain M on η_j from the dependence on the other components of η. For this aim we will factorize M similarly as in [Ves01, Lem. 4.5.11].

Lemma 3.3 below tells us that B inherits from A the triangular property

$$B_{kk}=1 \text{ and } B_{lk}\neq 0 \Rightarrow l\succ k, \quad \forall\, l,k\in\Lambda^+. \tag{27}$$

For a pair $l,k\in\mathbb{Z}^d$ which does not satisfy $l\succ k$ let us write $l\not\succ k$. We will need the following decomposition of Λ^+ and η adapted to the lattice site $j\in\mathbb{Z}^d$.

$$\Lambda^+ = \Lambda_<\cup\{j\}\cup\Lambda_>,\quad \Lambda_<=\{k\in|\,k\not\succ j\},\quad \Lambda_>=\{n\in|\,n\succ j, n\neq j\} \tag{28}$$
$$\eta=(\eta_<,\eta_j,\eta_>),\quad \eta_<=\{\eta_k|\,k\in\Lambda_<\},\qquad \eta_>=\{\eta_n|\,n\in\Lambda_>\}. \tag{29}$$

Then:

$$\begin{aligned} M &= \{\eta|\,B\eta\in[0,\omega_+]^L\} \\ &= \left\{\eta\left|\begin{array}{lll} \sum_{l\in\Lambda_<}B_{kl}\eta_l & \in[0,\omega_+] & \forall\, k\in\Lambda_< \\ \eta_j & \in[0,\omega_+] & -\sum_{l\in\Lambda_<}B_{jl}\eta_l \\ \sum_{l\in\Lambda_>}B_{nl}\eta_l & \in[0,\omega_+] & -\sum_{l\in\Lambda_<}B_{nj}\eta_j-\sum_{l\in\Lambda_<}B_{nl}\eta_l \quad \forall\, k\in\Lambda_> \end{array}\right.\right\}\end{aligned} \tag{30}$$

Set now

$$\xi=\xi(\eta_>)=-\sum_{l\in\Lambda_<}B_{j,l}\eta_l \quad\text{and}\quad \Xi=\Xi(\eta_<)=-\sum_{l\in\Lambda_<}b_l\eta_l$$

where $b_l := \{B_{nl}\}_{n\in\Lambda_>}$ is a column vector, and

$$\begin{aligned} M_< &:= \left\{\eta_<|\sum_{l\in\Lambda_<}B_{kl}\eta_l\in[0,\omega_+]\forall\, k\in\Lambda_<\right\} \\ M_j(\eta_<) &:= \{\eta_j|\,\eta_j\in[0,\omega]+\xi\}=[\xi,\xi\omega_+] \\ M_>(\eta_<,\eta_j) &:= \left\{\eta_>|\sum_{l\in\Lambda_>}B_{nl}\eta_l\in[0,\omega_+]-\sum_{l\in\Lambda_<}B_{nj}\eta_j-\sum_{l\in\Lambda_<}B_{nl}\eta_l\forall\, k\in\Lambda_>\right\}.\end{aligned} \tag{31}$$

Write the integral in (25) as:

$$\int_{M_<}d\eta_<\int_{M_j(\eta_<)}d\eta_j\int_{M_>(\eta_<,\eta_j)}d\eta_>\frac{\mathcal{P}(\eta)}{1+t\eta_j^2}.$$

Note that we can write the integral in this "successive" form only because property (27) holds.

We would like to apply the spectral averaging result of [CH94, Section 4] to the integral $\int_{M_j(\eta_<)} d\eta_j$. The integration over $\eta_<$ causes no problem because it stands outside the $d\eta_j$-integral. However, the integration domain $M_>(\eta_<, \eta_j)$ of the "inner" integral is a function of η_j, so we cannot pull this integral out of $\int_{M_j(\eta_<)} d\eta_j$. To solve this problem we will carefully enlarge the domain $M_>(\eta_<, \eta_j)$ so that it becomes η_j-independent. In doing so we have to make sure that the enlargement is not too "generous". More precisely, the factor by which the volume of the domain increases has to remain bounded as Λ tends to $\mathbb{R}^d$. If one enlarges $M_>(\eta_<.\eta_j)$ too naively one can incur a factor growing exponentially in $L = \#\Lambda^+$, cf. [Ves01, Remark 4.5.8.].

Fix $\eta_< \in M_<$ and thus ξ and Ξ. Now $M_>(\eta_<, \eta_j)$ is for all values of $\eta_j \in [\xi, \xi + \omega_+]$ contained in

$$M_>^+(\eta_<) := \bigcup_{s \in [\xi, \xi+\omega_+]} \left\{ \eta_> \mid \sum_{l \in \Lambda_>} B_{nl}\eta_l \in [0, \omega_+] - sB_{nj} + \Xi_n, \ n \in \Lambda_> \right\}.$$

Thus

$$\int_{M_j(\eta_<)} d\eta_j \int_{M_>(\eta_<, \eta_j)} d\eta_> \frac{\mathcal{P}(\eta)}{1 + t\eta_j^2} \le \int_{M_>^+(\eta_<)} d\eta_> \int_{M_j(\eta_<)} d\eta_j \frac{\mathcal{P}(\eta)}{1 + t\eta_j^2}.$$

Now by inequality (4.) of [CH94] we have

$$\int d\eta_j \frac{\mathcal{P}(\eta)}{1 + t\eta_j^2} \le |I|. \tag{32}$$

Denote by $A_> = \{A_{lk}\}_{l,k \in \Lambda_>}, A_< = \{A_{lk}\}_{l,k \in \Lambda_<}$ "blocks" of the linear map A. From Lemmata 3.4 and 3.5 below we infer

$$\mathrm{vol}(M_>^+(\eta_<)) = |\det A_>| \, \omega^{|\Lambda_>|} \sum_{n \in \Lambda_> \cup \{j\}} |B_{nj}|.$$

Since $\mathrm{vol}(M_<) = |\det A_<| \, \omega_+^{|\Lambda_<|}$ and Lemma 3.3 tells us

$$\det A = A_{jj} \det A_< \det A_>$$

we arrive at

$$\begin{aligned} \int_{\mathbb{R}^L} d\eta \, k(\eta) \, \mathcal{P}(\eta) &\le |\det B| \, \omega_+^{-L} \, (1 + t|||A|||_1^2 \omega_+^2) |\det A| \, \omega_+^{|\Lambda_>|+|\Lambda_<|} \sum_{n \in \Lambda_> \cup \{j\}} |B_{nj}| \, |I| \\ &\le \omega_+^{-1} \, (1 + t|||A|||_1^2 \omega_+^2) \, |||B|||_1 \, |I|. \end{aligned}$$

Taking the limit $t \searrow 0$ we get

$$\int_{\mathbb{R}^L} d\eta\, k(\eta)\, \mathcal{P}(\eta) \leq \frac{\omega_+}{1-a^*}\, |I|$$

which proves the proposition for the case $w = \chi_0$.

Now consider general w. We have $V_{B\eta} = \sum_{j\in\tilde{\Lambda}} \eta_j w(\cdot - j)$ on Λ and the spectral averaging applies as in inequality (32). By independence of the coupling constants $\omega_k, k \in \mathbb{Z}^d$ we have

$$\mathbb{E}\, \langle f, \chi_j P^l_\bullet(I)\chi_j f\rangle \leq \mathbb{E}\left[\int_{\mathbb{R}^L} d\eta\, k(\eta)\, \mathcal{P}(\eta)\right]$$

and now the proof proceed as in the special case $w = \chi_0$.

q.e.d.

Remark 3.2 Since a_j may be 0 for a $j \in \Gamma$ we can assume (by enlargement) that Γ is a discrete cube. It follows that Λ^+ is a cube, too. If $\Gamma := \{\gamma \in \mathbb{Z}^d | \, \gamma_i \in [0, g]\, \forall i = 1, \ldots, d\}$ and $\Lambda = \Lambda_l, l \in \mathbb{N}$, then $\Lambda^+ = \{k \in \mathbb{Z}^d | \, k_i \in [-g, l]\, \forall i = 1, \ldots, d\}$.

The following lemma is trivial in the case $\mathbb{Z}^d = \mathbb{Z}$. In the higher dimensional case it depends on the definition of the relation "$\succ$".

Lemma 3.3 *(1) Assume w.l.o.g. that Λ^+ is a discrete cube, cf. Remark 3.2. Let $L = \#\Lambda^+$ and $A : \mathbb{R}^L \to \mathbb{R}^L, (A\omega)_j := \sum_{k\in\Lambda^+} A_{jk}\omega_k$ be a linear map as before such that for all $j, k \in \Lambda^+$*

$$A_{jk} \neq 0 \Rightarrow j \succ k \tag{33}$$

$$A_{jj} = 1. \tag{34}$$

Then A is invertible and the coefficients of $A^{-1} = B = \{B_{jk}\}_{j,k\in\Lambda^+}$ satisfy (33) and (34) for all $j, k \in \Lambda^+$.

(2) Let $O \subset \mathbb{Z}^d$ be finite and $A : \mathbb{R}^{|O|} \to \mathbb{R}^{|O|}$ be given by $(A\omega)_j := \sum_{k\in O} A_{jk}\omega_k$ with (33) for all $j, k \in O$. Then

$$\det A = \prod_{j\in O} A_{jj}. \tag{35}$$

(1) By part (2) $\det A = \prod_{j\in\Lambda^+} A_{jj} = 1$ and A^{-1} exists. We prove by induction over $j \in \Lambda^+$

$$B_{jj} = 1 \text{ and } B_{jk} = 0, \ \forall\, k \in \Lambda^+, j \not\succ k \tag{36}$$

for all $j \in \Lambda^+$. Without loss of generality we may assume by translation $\Lambda^+ = [0,\lambda]^d \cap \mathbb{Z}^d$. The induction anchor is:

$$j = 0, k \succ 0 : \delta_{0,k} = \sum_{l\in\Lambda^+} A_{0l} B_{lk} = A_{00} B_{0,k} = B_{0,k}.$$

Induction step: Let $m \in \Lambda^+$. If (36) is true for all $j \in \Lambda^+, j \prec m, j \neq m$ then (36) is true also for $j = m$. Proof:

$$\delta_{mk} = \sum_{j\in\Lambda^+, j\prec m} A_{mj} B_{jk} = \sum_{j\in\Lambda^+, k\prec j\prec m, j\neq m} A_{mj} B_{jk} + A_{mm} B_{mk} = B_{mk}$$

for $k \not\prec i$ or $k = i$.

(2) Let Π_o denote the permutation group of O. Since $\det A = \sum_{\pi\in\Pi_O} \prod_{k\in O} A_{k\pi(k)}$ it suffices to show $\prod_{k\in O} A_{k\pi(k)} = 0$ for all $\pi \neq \mathrm{Id}_O$. For $\pi \neq \mathrm{Id}_O$ there exists a $k \in O$ such that $\pi(k) \neq k$. We claim that there exists a $n \in \mathbb{N}$ such that

$$\Lambda^+ \ni j := \pi^{n-1}(k) \not\succ \pi^n(k) = \pi(j).$$

This implies $A_{j\pi(j)} = 0$ and we are finished. To prove the claim assume $\pi^{n-1}(k) \succ \pi^n(k)$ for all $n \in \mathbb{N}$. $\pi(k) \neq k$ implies $\pi^{n-1}(k) \neq \pi^n(k)$ for all $n \in \mathbb{N}$. Since O is finite there exist $n, m \in \mathbb{N}$ such that $\pi^n(k) = \pi^{n+m}(k)$. Thus for any $i \in \{1,\ldots,d\}$

$$\pi^n(k)_i \geq \pi^{n+1}(k)_i \geq \cdots \geq \pi^{n+m}(k)_i =\geq \pi^n(k)_i \quad \Rightarrow \pi^n(k)_i = \pi^{n+1}(k)_i.$$

Therefore $\pi^n(k) = \pi^{n+1}(k)$ which is a contradiction.

q.e.d.

Lemma 3.4 *Let $t \in \mathbb{R}^n$ and*

$$S = \bigcup_{s\in[0,\omega_+]} \{x \in \mathbb{R}^n | \, x \in [0,\omega_+]^n + st\}.$$

Then

$$\mathrm{vol}(S) = \left(1 + \sum_{i=1}^n |t_i|\right) \omega_+^n. \tag{37}$$

Proof.

For each $i \in \{1, \ldots, n\}$ define the linear map $\tilde{T}_i : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\tilde{T}_i(e_l) = e_l \quad \text{for } l \neq i \text{ and } \tilde{T}_i(e_i) = t.$$

Then $\det \tilde{T}_i = t_i$. Define an invertible, affine map $T_i : \mathbb{R}^n \to \mathbb{R}^n$ by $T_i(x) = \tilde{T}_i(x) + \omega_+ e_i$. Set

$$\begin{aligned}
Q &:= \{x \in \mathbb{R}^n | \, x_i \in [0, \omega_+] \; \forall \, i = 1, \ldots, n\} \\
K_i &:= \{x \in Q \, | \, x_i = \omega_+\}, \; \forall \, i = 1, \ldots, n \\
S_i &:= \{x \in \mathbb{R}^n | \, x = y + s\,t, \; s \in [0, \omega_+], y \in K_i\}.
\end{aligned}$$

Then we have up to sets of measure zero the disjoint union

$$S = Q \cup \bigcup_{i=1}^{n} S_i.$$

We prove $S_i = T_i(Q)$ for all $i = 1, \ldots, n$. Since $T_i(Q) = \tilde{T}_i(Q) + \omega_+ e_i$ the claim is equivalent to $S_i - \omega_+ e_i = \tilde{T}_i(Q)$. Now $y \in K - i$ is equivalent to $y_i = \omega_+$ and $y_l \in [0, \omega_+]$ for all $l \neq i$. Thus $K_i - \omega_+ e_i = \{x \in Q | \, x_i = 0\} =: K^i$. It follows

$$\begin{aligned}
S_i - \omega_+ e_i &= \{x | \, x = y - \omega_+ e_i + st, y \in K_i, s \in [0, \omega_+]\} \\
&= \{x | \, x = z + st, y \in K^i, s \in [0, \omega_+]\} \\
&= \{x | \, x = \sum_{l=1, l \neq i}^{n} z_l e_l + st, y \in K^i, s, z_l \in [0, \omega_+], \; \forall l = 1, \ldots, n, l \neq i\} \\
&= \tilde{T}_i(Q).
\end{aligned}$$

Thus $S = Q \cup \bigcup_{i=1}^{n} T_i(Q)$ and

$$\mathrm{vol}(s) = \mathrm{vol}(Q) + \sum_{i=1}^{n} |\det \tilde{T}_i| \; \mathrm{vol}(Q) = \left[1 + \sum_{i=1}^{n} |t_i|\right] \mathrm{vol}(Q).$$

q.e.d.

Lemma 3.5 *(1)*

$$M_>^+(\eta_<) = A_>(\Xi - \xi b_j) + A_> \left[\bigcup_{s \in [0, \omega_+]} ([0, \omega_+]^{|\Lambda_>|} - s b_j) \right]$$

(2)

$$\mathrm{vol}[M_>^+(\eta_<)] = |\det A_>| \sum_{n \in \Lambda_> \cup \{j\}} |B_{nj}| \, \omega_+^{|\Lambda_>|}$$

$$\begin{aligned} M_>^+(\eta_<) &= \bigcup_{r\in[\xi,\xi+\omega_+]} \{\eta_>|\sum_{l\in\Lambda_>} B_{nl}\eta_l \in [0,\omega_+] - rB_{nj} + \Xi \forall n \in \Lambda_>\} \\ &= \bigcup_{r\in[\xi,\xi+\omega_+]} \{\eta_>|\,\eta_> \in A_>([0,\omega_+]^{|\Lambda_>|}) - rA_>b_j + A_>\Xi\} \\ &= \ldots \end{aligned}$$

where we used property (37) for the inversion of the bloc matrix $A_>$.

$$\begin{aligned} \ldots &= \bigcup_{s\in[0,\omega_+]} \{A_>([0,\omega_+]^{|\Lambda_>|}) - sA_>b_j - \xi A_>b_j + A_>\Xi\} \\ &= A_>(\Xi - \xi b_j) + \bigcup_{s\in[0,\omega_+]} [A_>([0,\omega_+]^{|\Lambda_>|} - sB_j)] \\ &= A_>(\Xi - \xi b_j) + A_>\left[\bigcup_{s\in[0,\omega_+]} ([0,\omega_+]^{|\Lambda_>|} - sb_j)\right]. \end{aligned}$$

This proves the firs claim and

$$\mathrm{vol}[M_>^+(\eta_<)] = |\det A_>|\ \mathrm{vol}\left[\bigcup_{s\in[0,\omega_+]} ([0,\omega_+]^{|\Lambda_>|} - sb_j)\right].$$

Lemma 3.4 and $\left(1+\sum_{n\in\Lambda_>}|B_{nj}|\right) = \left(\sum_{n\in\Lambda_>\cup\{j\}}|B_{nj}|\right)$ prove the second assertion.

q.e.d.

Remark 3.6 Lemma 3.3 and thus Proposition 3.1 holds true if $\Gamma \subset \{k\in\mathbb{Z}^d|\, k \prec 0\}$ or if Γ is a subset of some other d-dimensional "quadrant". Probably we can can allow Γ to be a larger set.

Consider the relation on $\mathbb{R}^d$

$k \succ 0 \Leftrightarrow$

$k_1 \geq 0$ and for all $i = 2,\ldots,d$ we have: $k_i \geq 0$ if $k_\nu = 0\,\forall\,\nu = 1,\ldots,i-1$

For this relation and $\Gamma \subset \{k\in\mathbb{Z}^d|\, k \succ 0\}$ the proof should work, too. The reason ist that $k\succ 0$ and $-k \succ 0$ imply $k = 0$.

4 Discussion of recent results on Wegner estimates for indefinite potentials

Results from [Ves00] concerning differentiable densities

Assume the hypotheses of Theorem 1.3 up to following changes

- the "support" Γ of the convolution vector is an arbitrary finite subset of $\mathbb{Z}^d$.
- the single site measure μ has a density $g \in W^{1,1}(\mathbb{R})$.

Denote as in Section 3 by B the inverse of the matrix $A := \{A_{j,k}\}_{j,k\in\Lambda^+}$, $A_{j,k} := a_{j-k}, \forall\, j,k \in \Lambda^+$. In [Ves00] the following Wegner estimate is proven:

Theorem 4.1 *We have for all $E \in \mathbb{R}$*

$$\mathbb{E}\,\left[\operatorname{Tr} P_\omega^l([E-\epsilon,E])\right] \le const\, |||B|||\epsilon\, l^d \le const\, \frac{\epsilon\, l^d}{1-a^*}, \quad \forall \epsilon \ge 0. \tag{38}$$

The constant depends on E but not on ϵ.

Remark 4.2 In [Ves00] the geometric series is used to deduce $|||B||| \le \frac{1}{1-a^*}$ and thereby the second inequality in (38). Alternatively one can use criteria formulated in terms of the *symbol* S_A of the (block) Toeplitz matrix A which alow one to control the behaviour of the eigenvalue $\nu(l)$ of $A = A_\Lambda^+, \Lambda^+ = \Lambda_l^+$ closest to 0 as $\Lambda = \Lambda_l$ tends to the whole space $\mathbb{R}^d$. If we can show that $|\nu(l)|$ tends to zero not faster than a a inverse power of l we have by (38) a Wegner estimate which can be used for the multi scale analysis.

We discuss first the one dimensional case $d = 1$. There is a series of papers by S. Serra where the assumes that the symbol

$$S_A(\theta) = \sum_{k\in\mathbb{Z}} a_k e^{ik\theta}, \; \theta \in [-\pi,\pi], \; i = \sqrt{-1}$$

is a real function assuming non-negative values. This corresponds to the case that the matrix A is selfadjoint and non negative. In [Ser98a] it is proven that if S_A has one single zero of order n then $|\nu(l)|^{-1} = \mathcal{O}(l^n)$. This means for our situation that we obtain a Wegner estimate with corresponding volume dependence

$$\mathbb{E}\,\left[\operatorname{Tr} P_\omega^l([E-\epsilon,E])\right] \le \text{const}\,\epsilon\, l^{n+1}, \quad \forall \epsilon \ge 0.$$

In the article [Ser96] Serra considers a similar situation, but now S_A is allowed to have several minima, and finally in [Ser94, Ser98b] the block-Toeplitz case is considered. Similar results are obtained by Böttcher and Grudsky in [BG98].

Results from [HK01]

In the paper [HK01] Hislop and Klopp prove a Wegner estimate for indefinite alloy type models. Their proof does not require any condition on the form of the single site potential u as we do in Assumption 1.2 (iv). Their result is not sufficient to imply the existence of the density of states. The results in [HK01] are restricted to energy regions which do not belong to the spectrum of the unperturbed operator H_0. The reason is that they make use of a different type of restriction of the operator H_ω to the finite cube $\Lambda = \Lambda_l$. Namely, the operator associated to Λ is $H^l_\omega := H_0 + V^l_\omega$ where $V^l_\omega(x)$ stands for $\sum_{k\in\tilde\Lambda} \omega_k u(x-k)$.

Hislop and Klopp assume that the single site potential u is continuous and compactly supported and has no zero in a neighbourhood of $0 \in \mathbb{R}^d$. The density g of the single site distribution has to be from $L^\infty_0(\mathbb{R})$ and be locally absolutely continuous.

Theorem 4.3 ([HK01]) *For any $q < 1$ and $E_0 < \inf\sigma(H_0)$ there exists $C \in]0,\infty[$ such that for all $\epsilon > 0$ with $E_0 + \epsilon < \inf\sigma(H_0)$ one has*

$$\mathbb{P}\{\omega \mid \sigma(H^l_\omega) \cap [E_0-\epsilon, E_0+\epsilon] \neq \emptyset\} \leq C\epsilon^q |\Lambda|$$

where the constant depends only on d, q and the distance between E_0 and the unperturbed spectrum $\sigma(H_0)$.

The Wegner estimate is also true at internal spectral edges (away from the unperturbed spectrum $\sigma(H_0)$) if one works in the weak coupling regime. This means that the considered operator is $H_0 + \lambda V_\omega$ with $|\lambda|$ sufficiently small. Moreover their proof applies also to the case where the single site potentials have different shapes instead of being the translates $u_k = u(\cdot - k)$ of a single function u and for certain families of correlated coupling constants $\omega_k, k \in \mathbb{Z}^d$ and also to certain random operators where the randomness enters via an multiplicative perturbation. For details see [HK01].

Remark 4.4 (Birman-Schwinger Principle) In [HK01] actually an auxiliary operator of Birman-Schwinger type is introduced and the behaviour of its eigenvalues is analyzed, rather than the one of H^l_ω itself.

Namely, for an $E \in (\mathbb{R} \setminus \sigma(H^l_\omega)) \cap]-\infty, \inf\sigma(H_0)[$ one defines the selfadjoint and compact operator

$$\Gamma^l_\omega(E) := (H_0 - E)^{-1/2} V^l_\omega (H_0 - E)^{-1/2}$$

and writes now the resolvent of H^l_ω as

$$(H^l_\omega - E)^{-1/2} = (H_0 - E)^{-1/2} [1 + \Gamma^l_\omega(E)]^{-1} (H_0 - E)^{-1/2}$$

whose norm is bounded by

$$\delta \, \|[1 + \Gamma^l_\omega(E)]^{-1}\|, \quad \delta := [\inf\sigma(H_0) - E]^{-1}.$$

Having this in mind one can reformulate the Wegner estimate as follows

$$\begin{aligned}\mathbb{P}\{\omega|\, d(\sigma(H^l_\omega), E) < \epsilon\} &= \mathbb{P}\{\omega|\, \|(H^l_\omega - E)^{-1}\| > \epsilon^{-1}\} \\ &\le \mathbb{P}\{\omega|\, \|[1+\Gamma^l_\omega(E)]^{-1}\| > \delta/\epsilon\} \\ &= \mathbb{P}\{\omega|\, d(\sigma[\Gamma^l_\omega(E)], -1) < \delta/\epsilon\}\end{aligned} \tag{39}$$

use the Čebyšev inequality

$$\mathbb{P}\{\omega|\, d(\sigma[\Gamma^l_\omega(E)], -1) < \delta/\epsilon\} \le \mathbb{E}\left\{\operatorname{Tr} \chi_{]-1-\delta/\epsilon, -1+\delta/\epsilon[}(\Gamma^l_\omega(E))\right\} \tag{40}$$

and proceed with the spectral analysis of the Birman-Schwinger operator $\Gamma^l_\omega(E)$, cf. [Klo95, HK01].

References

[BG98] A. Böttcher and S. M. Grudsky. On the condition numbers of large semi-definite Toeplitz matrices. *Linear Algebra Appl.*, 279(1-3):285–301, 1998.

[CH94] J.-M. Combes and P.D. Hislop. Localization for some continuous, random Hamiltionians in d-dimensions. *J. Funct. Anal.*, 124:149–180, 1994.

[CHT99] J. M. Combes, P. D. Hislop, and A. Tip. Band edge localization and the density of states for acoustic and electromagnetic waves in random media. *Ann. Inst. H. Poincaré Phys. Théor.*, 70(4):381–428, 1999.

[CL90] R. Carmona and J. Lacroix. *Spectral Theory of Random Schrödinger Operators.* Birkhäuser, Boston, 1990.

[CT73] J.M. Combes and L. Thomas. Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators. *Commun. Math. Phys.*, 34:251–270, 1973.

[FHLM97] W. Fischer, T. Hupfer, H. Leschke, and P. Müller. Existence of the density of states for multi-dimensional continuum Schrödinger operators with Gaussian random potentials. *Comm. Math. Phys.*, 190(1):133–141, 1997.

[FK96] A. Figotin and A. Klein. Localization of classical waves. I. Acoustic waves. *Comm. Math. Phys.*, 180(2):439–482, 1996.

[FK97] A. Figotin and A. Klein. Localization of classical waves. II. Electromagnetic waves. *Comm. Math. Phys.*, 184(2):411–441, 1997.

[FLM00] W. Fischer, H. Leschke, and P. Müller. Spectral localization by Gaussian random potentials in multi-dimensional continuous space. *J. Statist. Phys.*, 101(5-6):935–985, 2000.

[FS83] J. Fröhlich and T. Spencer. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Commun. Math. Phys.*, 88:151–184, 1983.

[GK01a] F. Germinet and A. Klein. Bootstrap multiscale analysis and localization in random media. *Comm. Math. Phys.*, 222(2):415–448, 2001.

[GK01b] F. Germinet and A. Klein. A characterization of the Anderson metal-insulator transport transition. Preprint, www.ma.utexas.edu/mp_arc/, 2001.

[HK01] P. D. Hislop and F. Klopp. The integrated density of states for some random operators with nonsign definite potentials. www.ma.utexas.edu/mp_arc, preprint no. 01-139, 2001.

[How87] J. S. Howland. Perturbation theory of dense point spectra. *J. Func. Anal.*, 74:52–80, 1987.

[HS96] P. D. Hislop and I.M. Sigal. *Introduction to spectral theory: with Applications to Schrödinger Operators.* Springer, New York, 1996.

[Kir89] W. Kirsch. Random Schrödinger operators. In H. Holden and A. Jensen, editors, *Schrödinger Operators*, Lecture Notes in Physics, **345**, Berlin, 1989. Springer.

[Klo95] F. Klopp. Localization for some continuous random Schrödinger operators. *Commun. Math. Phys.*, 167:553–569, 1995.

[Klo99] F. Klopp. Internal Lifshits tails for random perturbations of periodic Schrödinger operators. *Duke Math. J.*, 98(2):335–396, 1999.

[KM82] W. Kirsch and F. Martinelli. On the density of states of Schrödinger operators with a random potential. *J. Phys. A: Math. Gen.*, 15:2139–2156, 1982.

[KS87] S. Kotani and B. Simon. Localization in general one-dimensional random systems II: continuum Schrödinger operators. *Commun. Math. Phys.*, 112:103–119, 1987.

[KSS98] W. Kirsch, P. Stollmann, and G. Stolz. Localization for random perturbations of periodic Schrödinger operators. *Random Oper. Stochastic Equations*, 6(3):241–268, 1998. available at www.ma.utexas.edu/mp_arc, preprint no. 96-409.

[MH84] F. Martinelli and H. Holden. On absence of diffusion near the bottom of the spectrum for a random Schrödinger operator on $L^2(R^\nu)$. *Commun. Math. Phys.*, 93:197–217, 1984.

[MS85] F. Martinelli and E. Scoppola. Remark on the absence of the absolutely continuous spectrum for d-dimensional Schrödinger operator with random potential for large disorder or low energy. *Commun. Math. Phys.*, 97:465–471, 85.

[PF92] L. A. Pastur and A. L. Figotin. *Spectra of Random and Almost-Periodic Operators.* Springer Verlag, Berlin, 1992.

[RS78] M. Reed and B. Simon. *Methods of Modern Mathematical Physics IV, Analysis of Operators.* Academic Press, San Diego, 1978.

[Ser94] S. Serra. Preconditioning strategies for asymptotically ill-conditioned Toeplitz matrices. *BIT*, 34:579–593, 1994.

[Ser96] S. Serra. On the extreme spectral properties of Toeplitz matrices generated by L^1 functions with several minima/maxima. *BIT*, 36(1):135–142, 1996.

[Ser98a] S. Serra. On the extreme eigenvalues of Hermitian (block) Toeplitz matrices. *Linear Algebra Appl.*, 270:109–129, 1998.

[Ser98b] S. Serra. Asymptotic results on the spectra of block Toeplitz preconditioned matrices. *SIAM J. Matrix Anal.Appl.*, 20:31–44, 1998.

[Sto98] P. Stollmann. Localization for random perturbations of anisotropic periodic media. *Israel J. Math.*, 107:125–139, 1998.

[Sto01] P. Stollmann. *Caught by disorder: A Course on Bound States in Random Media*, volume 20 of *Progress in Mathematical Physics.* Birkhäuser, July 2001.

[SW86] B. Simon and T. Wolff. Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians. *Comm. Pure Appl. Math.*, 39:75–90, 1986.

[Ves98] I. Veselić. Localisation for random perturbations of periodic Schrödinger operators with regular Floquet eigenvalues. to appear in *Ann. Henri Poincaré*, available at www.ma.utexas.edu/mp_arc preprint no. 98-569, 1998.

[Ves00] I. Veselić. Wegner estimate for some indefinite Anderson-type Schrödinger operators with differentiable densities, to appear in *Lett Math. Phys.* preprint 2000, www.ma.utexas.edu/mp_arc/.

[Ves01] I. Veselić. *Indefinite Probleme bei der Anderson-Lokalisierung.* Ph.D thesis, Ruhr-Universität Bochum, 44780 Bochum, January 2001. http://www-brs.ub.ruhr-uni-bochum.de/netahtml/HSS/Diss/VeselicIvan/.

[Weg81] F. Wegner. Bounds on the DOS in disordered systems. *Z. Phys. B*, 44:9–15, 1981.

[Zen99] H. Zenk. Anderson localization for a multidimensional model including long range potentials and displacements. Preprint-Reihe des Fachbereichs Mathematik at the Johannes Guteberg-Universität Mainz, 1999.