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Two-fold ground states of the Pauli-Fierz Hamiltonian including spin

Fumio Hiroshima

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Abstract

The Pauli-Fierz Hamiltonian describes an interaction between a low energy electron and photons. Existence of ground states has been established. The purpose of this talk is to show that its ground states is *exactly* two-fold in a weak coupling region.

1 The Pauli-Fierz Hamiltonian

This is a joint work[1] with Herbert Spohn[2] . The Hamiltonian in question is the Pauli-Fierz Hamiltonian in nonrelativisitic QED with spin, which will be denoted by H acting on the Hilbert space

$$\mathcal{H}=L^2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathcal{F}.$$

Here $L^2(\mathbb{R}^3;\mathbb{C}^2)$ denotes the Hilbert space for the electron with spin σ, where $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ denotes the Pauli spin 1/2 matrices,

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

$\mathcal{F}$ is the symmetric Fock space for the photons given by $\mathcal{F}=\oplus_{n=0}^{\infty}(L^2(\mathbb{R}^3\times\{1,2\}))^n_{\rm sym}$. Here $(\cdots)^n_{\rm sym}$ denotes the n-fold symmetric tensor product of $(\cdots)$ with $(\cdots)^0_{\rm sym}=\mathbb{C}$.

The photons live in $\mathbb{R}^3$ and have helicity ±1. The Fock vacuum is denoted by Ω. The photon field is represented in $\mathcal{F}$ by the two-component Bose field $a(k,j), j=1,2$, with commutation relations

$$[a(k,j),a^*(k',j')]=\delta_{jj'}\delta(k-k'),$$

[1] [12].

[2] Zentrum Mathematik, Technische Universität München, D80290, München, Germany.

$$[a(k,j),a(k',j')]=0,\quad [a^*(k,j),a^*(k',j')]=0.$$

The energy of the photons is given by

$$H_f=\sum_{j=1,2}\int \omega(k)a^*(k,j)a(k,j)dk,$$

i.e., H_f restricted to $(L^2(\mathbf{R}^3\times\{1,2\}))^n_{\mathrm{symm}}$ is the multiplication by $\sum_{j=1}^n \omega(k_j)$, and the momentum of the photons is

$$P_f=\sum_{j=1,2}\int k a^*(k,j)a(k,j)dk.$$

Throughout units are such that $\hbar=1$, $c=1$. Physically $\omega(k)=|k|$. The case is somewhat singular and we assume that ω is continuous, rotation invariant, and that (1) $\inf_{k\in\mathbf{R}^3}\omega(k)\ge\omega_0>0$, (2) $\omega(k_1)+\omega(k_2)\ge\omega(k_1+k_2)$, (3) $\lim_{|k|\to\infty}\omega(k)=\infty$. A typical example is

$$\omega(k)=\sqrt{|k|^2+m_{\mathrm{ph}}^2},\quad m_{\mathrm{ph}}>0.$$

For a recent result of the massless case see [3]. The quantized transverse vector potential is defined through

$$A_\varphi(x)=\sum_{j=1,2}\int \frac{\widehat{\varphi}(k)}{\sqrt{2\omega(k)}}e_j(k)\left(a^*(k,j)e^{-ikx}+a(k,j)e^{ikx}\right)dk.$$

Here e_1 and e_2 are polarization vectors which together with $\widehat{k}=k/|k|$ form a standard basis in $\mathbf{R}^3$. $\varphi:\mathbf{R}^3\to\mathbf{R}$ is a form factor which ensures an ultraviolet cutoff. It is assumed to be $\varphi(Rx)=\varphi(x)$ for an arbitrary rotation R, continuous, bounded with some decay at infinity, and normalized as $\int\varphi(x)dx=1$. We will work with the Fourier transform $\widehat{\varphi}(k)=(2\pi)^{-3/2}\int\varphi(x)e^{-ikx}dx$. It satisfies (1) $\widehat{\varphi}(Rk)=\widehat{\varphi}(k)$, (2) $\overline{\widehat{\varphi}}=\widehat{\varphi}$ for notational simplicity, (3) $\widehat{\varphi}(0)=(2\pi)^{-3/2}$, and (4) the decay

$$\int\left(\omega(k)^{-2}+\omega(k)^{-1}+1+\omega(k)\right)|\widehat{\varphi}(k)|^2dk<\infty.$$

The quantized magnetic field is correspondingly

$$B_\varphi(x)=i\sum_{j=1,2}\int \frac{\widehat{\varphi}(k)}{\sqrt{2\omega(k)}}(k\times e_j(k))\left(a^*(k,j)e^{-ikx}-a(k,j)e^{ikx}\right)dk.$$

With these preparation the Pauli-Fierz Hamiltonian, including spin, is defined by

$$H=\frac{1}{2}(-i\nabla_x\otimes 1-eA_\varphi(x))^2+1\otimes H_f-\frac{e}{2}\sigma\otimes B_\varphi(x). \tag{1.1}$$

Since obvious from the context we will drop the tensor notation $\otimes$.

2 Invariances

2.1 Total momentum

Let us define the total momentum by $P_{\rm total} = -i\nabla_x + P_{\rm f}$. We see that

$$[P_{\rm total}, H] = 0. \tag{2.1}$$

(2.1) immediately implies that H has no ground state. Instead of H we consider the Hamiltonian with a fixed total momentum as follows. By (2.1), we see that (1.1) is decomposable with respect to the spectrum of $P_{\rm total}$,

$$H = \int_{\mathbb{R}^3}^{\oplus} H_p dp,$$

where

$$H_p = \frac{1}{2}\left(p - P_{\rm f} - eA_\varphi\right)^2 - \frac{e}{2}\sigma B_\varphi + H_{\rm f}, \tag{2.2}$$

acting on $\mathbb{C}^2 \otimes \mathcal{F}$. Here $A_\varphi = A_\varphi(0)$ and $B_\varphi = B_\varphi(0)$. The total momentum $p \in \mathbb{R}^3$ is regarded as a parameter. Recently an adiabatic perturbation of the Hamiltonian (2.2) has been studied in [16]. We define

$$H_{p0} = \frac{1}{2}(p - P_{\rm f})^2 + H_{\rm f},$$

and $H_{{\rm I}p} = H_p - H_{p0}$. We have $\|H_{{\rm I}p}\psi\| \le c_*(e)\|(H_{p0}+1)\psi\|$, where

$$c_*(e) = c_*\left\{|e|\left\{\int\left(\frac{1}{\omega(k)^2} + \omega(k)\right)|\hat{\varphi}(k)|^2 dk\right\}^{1/2} + e^2\int\left(\frac{1}{\omega(k)^2} + 1\right)|\hat{\varphi}(k)|^2 dk\right\}$$

with some constant c_*. Then $|e| < e_*$ with a certain $e_* > 0$ implies $c_*(e) < 1$. In particular H_p is self-adjoint on $D(H_{\rm f}) \cap D({P_{\rm f}}^2)$ for all $p \in \mathbb{R}^3$ and bounded from below, for $|e| < e_*$. The ground state energy of H_p is

$$E(p) = \inf \sigma(H_p) = \inf_{\psi \in D(H_p), \|\psi\|=1} (\psi, H_p\psi).$$

If $E(p)$ is an eigenvalue, the corresponding spectral projection is denoted by P_p. ${\rm Tr}P_p$ is identical with the multiplicity of ground states. The bottom of the continuous spectrum is denoted by $E_{\rm c}(p)$. Under our assumptions one knows that

$$E_{\rm c}(p) = \inf_{k \in \mathbb{R}^3}(E(p-k) + \omega(k)).$$

See [4, 5, 17]. Thus it is natural to set

$$\Delta(p) = E_{\rm c}(p) - E(p) = \inf_{k \in \mathbb{R}^3}\left(E(p-k) + \omega(k) - E(p)\right).$$

2.2 Total angular momentum

Let $\vec{n} \in \mathbb{R}^3$ be a unit vector. It follows that, for $\theta \in \mathbb{R}$,

$$e^{i(\theta/2)\vec{n}\cdot\theta}\sigma_\mu e^{-i(\theta/2)\vec{n}\cdot\theta} = (R\sigma)_\mu, \quad \mu = 1,2,3,$$

where $R = (R_{\mu\nu})_{1\le\mu,\nu\le3} = R(\vec{n},\theta) \in \mathrm{SO}(3)$ presents the rotation around $\vec{n}$ through an angle θ, and $(R\sigma)_\mu = \sum_{\mu=1,2,3} R_{\mu\nu}\sigma_\nu$. We define the field angular momentum relative to the origin by

$$J_{\mathrm{f}} = \sum_{j=1,2} \int (k \times (-i\nabla_k))a^*(k,j)a(k,j)dk$$

and the helicity by

$$S_{\mathrm{f}} = i\int \widehat{k}\,\{a^*(k,2)a(k,1) - a^*(k,1)a(k,2)\}\,dk.$$

Let $a^\sharp(f,j) = \int a^\sharp(k,j)f(k)dk$. It holds that

$$[a(f,1),S_{\mathrm{f}}] = -ia(\widehat{k}f,2), \quad [a(f,2),S_{\mathrm{f}}] = ia(\widehat{k}f,1),$$

$$[a^*(f,1),S_{\mathrm{f}}] = -ia^*(\widehat{k}f,2), \quad [a^*(f,2),S_{\mathrm{f}}] = ia^*(\widehat{k}f,1).$$

One sees that

$$e^{i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})}H_{\mathrm{f}}e^{-i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})} = H_{\mathrm{f}},$$

$$e^{i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})}P_{\mathrm{f}}e^{-i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})} = RP_{\mathrm{f}},$$

$$e^{i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})}A_\varphi e^{-i\theta\vec{n}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})} = RA_\varphi.$$

Define the total angular momentum by

$$J_{\mathrm{total}} = J_{\mathrm{f}} + S_{\mathrm{f}} + \frac{1}{2}\sigma.$$

It follows that

$$e^{i\theta\vec{n}\cdot J_{\mathrm{total}}}H_{Rp}e^{-i\theta\vec{n}\cdot J_{\mathrm{total}}} = \frac{1}{2}\{(R\sigma)\cdot(Rp - RP_{\mathrm{f}} - eRA_\varphi)\}^2 + H_{\mathrm{f}} = H_p.$$

In particular $E(p) = E(Rp)$. Moreover taking $\vec{n} = \hat{p} = p/|p|$ we have

$$e^{i\theta\hat{p}\cdot J_{\mathrm{total}}}H_p e^{-i\theta\hat{p}\cdot J_{\mathrm{total}}} = H_p.$$

Formally wa may say that H_p has a "field angular momentum+helicity+SU(2)" symmetry. It is easily seen that $\sigma(\hat{p}\cdot(J_{\mathrm{f}}+S_{\mathrm{f}})) = \mathbb{Z}$ and $\sigma(\hat{p}\cdot\sigma) = \{-1,1\}$. Thus

$$\sigma(\hat{p}\cdot J_{\mathrm{total}}) = \mathbb{Z} + \frac{1}{2},$$

which is independent of p. Thus $\mathbb{C}^2 \otimes L^2(\mathbb{R}^3)$ and H_p are decomposable as

$$\mathbb{C}^2 \otimes \mathcal{F} = \bigoplus_{z \in \mathbb{Z}+\frac{1}{2}} \mathcal{H}(z),$$

and

$$H_p = \bigoplus_{z \in \mathbb{Z}+\frac{1}{2}} H_p(z).$$

As our main result we state

Theorem 2.1 *Suppose* $|e| < e_0$ *with some constant* e_0 *given in (3.3), and* $\Delta(p) > 0$. *Then* H_p *has two orthogonal ground states,* $\psi_\pm$, *with* $\psi_\pm \in \mathcal{H}(\pm 1/2)$.

We emphasize that all our estimates on the allowed ranges for p and e do *not* depend on $m_{\rm ph}$ if we take $\omega(k) = \sqrt{|k|^2 + m_{\rm ph}^2}$.

3 A proof of Theorem 2.1

In what follows $\psi_p = \begin{pmatrix} \psi_{p+} \\ \psi_{p-} \end{pmatrix}$ denotes an *arbitrary* ground state of H_p. The number operator is defined by

$$N_{\rm f} = \sum_{j=1,2} \int a^*(k,j)a(k,j)dk.$$

The following lemma is shown in [15]

Lemma 3.1 *Suppose* $\Delta(p) > 0$. *Then*

$$(\psi_p, N_{\rm f}\psi_p) \leq 2e^2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k)+\omega(k)-E(p))^2} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} dk \|\psi_p\|^2.$$

We set

$$\theta(p) = 2 \int \frac{|k|^2/4 + 6E(p)}{(E(p-k)+\omega(k)-E(p))^2} \frac{|\hat{\varphi}(k)|^2}{\omega(k)} dk.$$

Let P_Ω be the projection onto $\{\mathbb{C}\Omega\}$.

Lemma 3.2 *Suppose that* $\Delta(p) > 0$ *and* $e^2 < 1/\theta(p)$. *Then* $(\psi_p, P_\Omega\psi_p) > 0$.

Proof: Since $P_\Omega + N_{\rm f} \geq 1$, we have

$$(\psi_p, P_\Omega\psi_p) \geq \|\psi_p\|^2 - \|N_{\rm f}{}^{1/2}\psi_p\|^2 > (1 - e^2\theta(p))\|\psi_p\|^2.$$

Thus the lemma follows. □

Let $\varphi_+ = \begin{pmatrix} \Omega \\ 0 \end{pmatrix}$ and $\varphi_- = \begin{pmatrix} 0 \\ \Omega \end{pmatrix}$, which are the ground states of H_{p0} with $p = (0,0,1)$ and $\varphi_\pm \in \mathcal{H}(\pm 1/2)$. Let us denote by P the projection onto $\{c_1\varphi_1 + c_2\varphi_2, c_1, c_2 \in \mathbb{C}\}$.

Let $\{\phi_i\}$ be a base of the space spanned by ground states of H_p and $\{\psi_j\}$ that of the complement.

Lemma 3.3 *Suppose* $e^2 < 1/(3\theta(p))$. *Then* $\mathrm{Tr}P_p \le 2$.

Proof: For $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, since $(\psi, P\psi) = |(\Omega, \psi_+)|^2 + |(\Omega, \psi_-)|^2 = (\psi, (1 \otimes P_\Omega)\psi)$, we have $(\psi, (P + 1 \otimes N_f)\psi) = (\psi, 1 \otimes (P_\Omega + N_f)\psi) \ge \|\psi\|^2$. Hence $P + N_f \ge 1$. Then

$$\mathrm{Tr}(P_p(1-P)) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_p(1-P)\phi) = \sum_{\phi \in \{\phi_i\}} (\phi, (1-P)\phi)$$

$$\le \sum_{\phi \in \{\phi_i\}} (\phi, N_f\phi) = \sum_{\phi \in \{\phi_i\}} (\phi, P_p N_f \phi) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_p N_f \phi) = \mathrm{Tr}(P_p N_f).$$

Thus $\mathrm{Tr}(P_p(1-P)) \le \mathrm{Tr}(P_p N_f)$. It follows that

$$\mathrm{Tr}(P_p P) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_p P \phi) = \sum_{\phi \in \{\phi_i\}} (\phi, P_p \phi) \le 2.$$

Thus $\mathrm{Tr}(P_p P) \le 2$. Moreover we have $\mathrm{Tr}(P_p N_f) \le e^2\theta(p)\mathrm{Tr}P_p$, since

$$\mathrm{Tr}(P_p N_f) = \sum_{\phi \in \{\phi_i\} \oplus \{\psi_j\}} (\phi, P_p N_f \phi) = \sum_{\phi \in \{\phi_i\}} (\phi, N_f \phi)$$

$$\le e^2\theta(p) \sum_{\phi \in \{\phi_i\}} (\phi, \phi) = e^2\theta(p)\mathrm{Tr}P_p.$$

Then $\mathrm{Tr}P_p - \mathrm{Tr}(P_p P) = \mathrm{Tr}P_p(1-P) \le \mathrm{Tr}(P_p N_f) \le e^2\theta(p)\mathrm{Tr}P_p$. Hence it follows that $(1 - e^2\theta(p))\mathrm{Tr}P_p \le \mathrm{Tr}(P_p P) \le 2$. We have

$$\mathrm{Tr}P_p \le \frac{2}{1 - e^2\theta(p)} < 3.$$

Thus the lemma follows. □

We say that $\psi \in \mathcal{F}$ is real, if $\psi^{(n)}(k_1, j_1, \cdots, k_n, j_n)$ is a real-valued function on $L^2(\mathbb{R}^{3n} \times \{1,2\}^n)$ for all $n \ge 0$. The set of real ψ is denoted by $\mathcal{F}_{\mathrm{real}}$. We define the set of reality-preserving operators $\mathcal{O}_{\mathrm{real}}(\mathcal{F})$ as follows:

$$\mathcal{O}_{\mathrm{real}}(\mathcal{F}) = \{A | A : \mathcal{F}_{\mathrm{real}} \cap D(A) \longrightarrow \mathcal{F}_{\mathrm{real}}\}.$$

It is seen that H_f and P_f are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Since, for all $k \in \mathbb{R}$ and $z \in \mathbb{R}^3$,

$$((H_{p0}+z)^k\psi)^{(n)}(k_1, j_1, \cdots, k_n, j_n)$$
$$= \left(\frac{1}{2}\left(p - \sum_{i=1}^n k_i\right)^2 + \sum_{i=1}^n \omega(k_i) + z\right)^k \psi^{(n)}(k_1, j_1, \cdots, k_n, j_n),$$

$(H_{p0}+z)^k$ is also in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Moreover A_φ and iB_φ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$.

Lemma 3.4 *Suppose $|e| < e_*$. Let $x \in \mathbb{C}^2$. Then there exists $a(t) \in \mathbb{R}$ independent of x such that for $t \geq 0$*

$$(x \otimes \Omega, e^{-t(H_p - E(p))} x \otimes \Omega)_{\mathcal{H}} = a(t)(x, x)_{\mathbb{C}^2}. \tag{3.1}$$

Proof: Note that $\|H_{\text{I}p}(1+H_{p0})^{-1}\| < 1$ for $|e| < e_*$. Then, by spectral theory, one has

$$e^{-t(H_p - E(p))} = \lim_{n\to\infty}\left(1 + \frac{t}{n}(H_p - E(p))\right)^{-n}$$
$$= \lim_{n\to\infty}\lim_{k\to\infty}\left\{\left(1+\frac{t}{n}H_{0p}\right)^{-1/2}\left(\sum_{k=0}^m\left(-\frac{t}{n}\widetilde{H_{\text{I}p}}\right)^k\right)\left(1+\frac{t}{n}H_{0p}\right)^{-1/2}\right\}^n.$$

Here

$$\widetilde{H_{\text{I}p}} = \widetilde{H}_{\text{II}p} + i\sigma \cdot \widetilde{B},$$
$$\widetilde{B} = \left(1+\frac{t}{n}H_{0p}\right)^{-1/2}(iB_\varphi)\left(1+\frac{t}{n}H_{0p}\right)^{-1/2},$$
$$\widetilde{H}_{\text{II}p} = \left(1+\frac{t}{n}H_{0p}\right)^{-1/2}(H_{\text{II}p} - E(p))\left(1+\frac{t}{n}H_{0p}\right)^{-1/2},$$
$$H_{\text{II}p} = -e(p - P_f)\cdot A_\varphi + \frac{e^2}{2}A_\varphi^2.$$

It is seen that

$$\widetilde{H_{\text{I}p}}^2 = \widetilde{H}_{\text{II}p}\widetilde{H}_{\text{II}p} - \widetilde{B}\cdot\widetilde{B} + i\sigma\cdot(\widetilde{H}_{\text{II}p}\widetilde{B} + \widetilde{B}\widetilde{H}_{\text{II}p} - \widetilde{B}\wedge\widetilde{B}) = M + i\sigma\cdot L.$$

Here both of $M = \widetilde{H}_{\text{II}p}\widetilde{H}_{\text{II}p} - \widetilde{B}\cdot\widetilde{B}$ and $L = \widetilde{H}_{\text{II}p}\widetilde{B} + \widetilde{B}\widetilde{H}_{\text{II}p} - \widetilde{B}\wedge\widetilde{B}$ are in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Moreover

$$\widetilde{H_{\text{I}p}}^3 = \widetilde{H}_{\text{II}p}M - \widetilde{B}L + i\sigma\cdot(\widetilde{B}M + \widetilde{H}_{\text{II}p}L - \widetilde{B}\wedge L),$$

where both of $\widetilde{H}_{\text{II}p}M - \widetilde{B}L$ and $\widetilde{B}M + \widetilde{H}_{\text{II}p}L - \widetilde{B}\wedge L$ are also in $\mathcal{O}_{\text{real}}(\mathcal{F})$. Thus, repeating above procedure, one obtains

$$\sum_{k=0}^m\left(-\frac{t}{n}\widetilde{H_{\text{I}p}}\right)^k = a_m + i\sigma\cdot b_m,$$

where a_m and b_m are in $\mathcal{O}_{\mathrm{real}}(\mathcal{F})$. Hence there exist $a_{nm} \in \mathcal{O}_{\mathrm{real}}(\mathcal{F})$ and $b_{nm} \in \mathcal{O}_{\mathrm{real}}(\mathcal{F})$ such that

$$\left\{\left(1+\frac{t}{n}H_{0p}\right)^{-1/2}\left(\sum_{k=0}^{m}\left(-\frac{t}{n}\widetilde{H_{\mathrm{I}p}}\right)^k\right)\left(1+\frac{t}{n}H_{0p}\right)^{-1/2}\right\}^n = a_{nm} + i\sigma\cdot b_{nm}.$$

Finally

$$(x\otimes\Omega, e^{-t(H_p-E(p))}x\otimes\Omega) = \lim_{n\to\infty}\lim_{k\to\infty}(x,x)(\Omega, a_{nm}\Omega) + i\lim_{n\to\infty}\lim_{k\to\infty}(x,\sigma x)(\Omega, b_{nm}\Omega).$$

Since the left-hand side is real, the second term of the right-hand side vanishes and $a(t) = \lim_{n\to\infty}\lim_{k\to\infty}(\Omega, a_{nm}\Omega)$ exists, which establishes the desired result. □

Lemma 3.5 *Suppose* $|e| < e_*$ *and* $|e| < 1/\sqrt{\theta(p)}$. *Then there exists* $a > 0$ *such that*

$$PP_pP = aP.$$

Proof: Note that $P_p = s - \lim_{t\to\infty} e^{-t(H_p-E(p))}$. Thus by Lemma 3.4,

$$(x\otimes\Omega, P_px\otimes\Omega) = \lim_{t\to\infty}(x\otimes\Omega, e^{-t(H_p-E(p))}x\otimes\Omega) = \lim_{t\to\infty}a(t)(x,x)$$

for all $x\in\mathbb{C}^2$. Since by Lemma 3.2, $(x\otimes\Omega, P_px\otimes\Omega)\neq 0$ for some $x\in\mathbb{C}^2$, $\lim_{t\to\infty}a(t)$ exists and it does not vanish. For arbitrary $\phi_1,\phi_2\in\mathcal{H}$, the polarization identity leads to $(\phi_1, PP_pP\phi_2) = a(\phi_1, P\phi_2)$. The lemma follows. □

Lemma 3.6 *Suppose* $|e| < e_*$ *and* $|e| < 1/\sqrt{\theta(p)}$. *Then* $\mathrm{Tr}P_p \geq 2$.

Proof: Suppose $\mathrm{Tr}P_p = 1$. Let $P = |\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|$ and $P_p = |\psi_p\rangle\langle\psi_p|$. Lemma 3.5 yields that

$$\begin{aligned} PP_pP &= (|\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|)|\psi_p\rangle\langle\psi_p|(|\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|) \\ &= |(\varphi_+,\psi_p)|^2|\varphi_+\rangle\langle\varphi_+| + |(\varphi_-,\psi_p)|^2|\varphi_-\rangle\langle\varphi_-| \\ &\quad +(\varphi_+,\psi_p)(\psi_p,\varphi_-)|\varphi_+\rangle\langle\varphi_-| + (\varphi_-,\psi_p)(\psi_p,\varphi_+)|\varphi_-\rangle\langle\varphi_+| \\ &= a(|\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|). \end{aligned} \tag{3.2}$$

It follows that $(\varphi_+,\psi_p)(\psi_p,\varphi_-) = 0$. Let us assume $(\psi_p,\varphi_-) = 0$. It implies in terms of (3.2) that $|(\varphi_+,\psi_p)|^2|\varphi_+\rangle\langle\varphi_+| = a(|\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|)$. This contradicts $(\varphi_+,\psi_p)\neq 0$ and $a\neq 0$. Thus the lemma follows. □

We define

$$e_0 = \inf\left\{|e| \,\middle|\, |e| < 1/\sqrt{3\theta(p)}, |e| < e_*\right\}. \tag{3.3}$$

A proof of Theorem 2.1

By Lemma 3.6, $\mathrm{Tr}P_p \geq 2$, and by Lemma 3.3, $\mathrm{Tr}P_p \leq 2$. Hence $\mathrm{Tr}P_p = 2$ follows. Without loss of generalization we may assume that $p = (0,0,1)$. Then $\varphi_\pm \in \mathcal{H}(\pm 1/2)$. Let $\psi_\pm$ be ground states of H_p such that $\psi_+ \in \mathcal{H}(z)$ and $\psi_- \in \mathcal{H}(z')$ with some $z, z' \in \mathbb{Z} + 1/2$. Since $PP_pP = aP$ we have $(\varphi_\pm, P_p\varphi_\pm) = a > 0$. Let $Q_\pm$ be the projections to $\mathcal{H}(\pm 1/2)$. Then $Q_+P_p\varphi_+ \neq 0$ and $Q_-P_p\varphi_- \neq 0$. The alternative $Q_+\psi_+ \neq 0$ or $Q_+\psi_- \neq 0$ holds, or the altenative $Q_-\psi_+ \neq 0$ or $Q_-\psi_- \neq 0$ holds. We may set $Q_+\psi_+ \neq 0$. Then $\psi_+ \in \mathcal{H}(+1/2)$ and $\psi_- \in \mathcal{H}(-1/2)$. The theorem follows. □

4 Confining potentials

In this section we set $\omega(k) = |k|$ and

$$H = \frac{1}{2}(-i\nabla_x - eA_\varphi(x))^2 + H_{\mathrm{f}} - \frac{e}{2}\sigma B_\varphi(x) + V.$$

Let V be relatively bounded with respect to $-\Delta/2$ with a relative bound strictly smaller than one. It has been established in [10, 11] that H is self-adjoint on $D(-\Delta) \cap D(H_{\mathrm{f}})$ and bounded from below, for *arbitrary* e. A confining potential V breaks the total momentum invariance,

$$[P_{\mathrm{total}}, H] \neq 0. \tag{4.1}$$

Existence of ground states of H is expected by (4.1). Actually by many authors it has been established that H has ground states, e.g., [1, 6, 7, 8, 14, 13], and in a spinless case, the ground state is unique [9].

Let $H_0 = H_{\mathrm{el}} + H_{\mathrm{f}}$ and $H_{\mathrm{el}} = \frac{1}{2}p^2 + V$. We set $E = \inf\sigma(H)$, $E_{\mathrm{el}} = \inf\sigma(H_{\mathrm{el}})$ and $\Sigma_{\mathrm{el}} = \inf\sigma_{\mathrm{ess}}(H_{\mathrm{el}})$.

We define a class of external potentials.

Definition 4.1 (1) *We say $V = Z + W \in V_{\mathrm{exp}}$ if the following (i)–(iv) hold, (i) $Z \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, (ii) $Z > -\infty$, (iii) $W < 0$, (iv) $W \in L^p(\mathbb{R}^3)$ for some $p > 3/2$.*

(2) *We say $V \in V(m)$, $m \geq 1$, if (i) $V \in V_{\mathrm{exp}}$, (ii) $Z(x) \geq \gamma|x|^{2m}$, outside a compact set for some positive constant γ.*

(3) *We say $V \in V(0)$, $m \geq 1$, if (i) $V \in V_{\mathrm{exp}}$, (ii) $\liminf_{|x|\to\infty} Z(x) > \inf\sigma(H)$.*

We assume that V satisfies that (1) $\|Vf\| \le a\|(p^2/2)f\| + b\|f\|$ with some $a < 1$ and some $b \ge 0$, (2) $V \in V(m)$ with some $m \ge 0$, (3) $V(x) = V(-x)$, (4) $\Sigma_{\rm el} - E_{\rm el} > 0$ and the ground state ϕ_0 of $H_{\rm el}$ is unique and real.

(1) guarantees self-adjointness of H, (2) derives a boundedness of $\||x|\psi_0\|$ for ground states ψ_0 of H, and (3) will be needed to estimate a lower bound of the multiplicity of ground states of H. (4) ensures that H has ground states and H_0 has twofold ground states. Actually H_0 has the two ground states, $\phi_+ = \begin{pmatrix} \phi_0 \otimes \Omega, \\ 0 \end{pmatrix}$ and $\phi_- = \begin{pmatrix} 0, \\ \phi_0 \otimes \Omega \end{pmatrix}$.

Let P_{ϕ_0} denote the projection onto $\{\mathbb{C}\phi_0\}$. Define

$$P = P_{\phi_0} \otimes P_\Omega, \quad Q = P_{\phi_0}^\perp \otimes P_\Omega.$$

Furthermore P_e denotes the projection onto the space spanned by ground states of H. Let ψ be arbitrary ground state of H. It is proven in [1] that

$$\|{N_{\rm f}}^{1/2}\psi\|^2 \le \theta_1(e)\||x|\psi\|^2, \tag{4.2}$$

and in [2, 12] that

$$\||x|^k\psi\|^2 \le \theta_2(e)\|\psi\|^2. \tag{4.3}$$

Then together with (4.2) and (4.3), we have

$$\|{N_{\rm f}}^{1/2}\psi\|^2 \le \theta_1(e)\theta_2(e)\|\psi\|^2. \tag{4.4}$$

Suppose $\Sigma_{\rm el} - E > 0$. Then there exists $\theta_3(e)$ such that

$$\|Q\psi\|^2 \le \theta_3(e)\|\psi\|^2. \tag{4.5}$$

Note that $\lim_{|e|\to 0}\theta_j(e) = 0$.

Lemma 4.2 *Suppose* $\theta_1(e)\theta_2(e) + \theta_3(e) < 1$. *Then* $(\psi_0, P\psi_0) > 0$.

Proof: It follows from (4.4), (4.5) and $P \ge 1 - N_{\rm f} - Q$. □

Lemma 4.3 *Suppose* $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. *Then* ${\rm Tr}P_e \le 2$.

Proof: It can be proven in the similar way as Lemma 3.3. □

Next we estimate $\mathrm{Tr}P_e$ from below using the realness argument used in the previous section. Let F denote the Fourier transformation on $L^2(\mathbb{R}^3)$. We define the unitary operator $\mathcal{O}$ on $\mathcal{H}$ by $\mathcal{O} = (F \otimes 1)e^{ix\otimes P_\mathrm{f}}$. Then $\mathcal{O}$ maps $D(-\Delta) \cap D(H_\mathrm{f})$ onto $D(|x|^2) \cap D(H_\mathrm{f})$ with

$$\widetilde{H} = \mathcal{O}H\mathcal{O}^{-1} = \frac{1}{2}(x - P_\mathrm{f} - eA(0))^2 + \widetilde{V} + H_\mathrm{f} - \frac{e}{2}\sigma \cdot B(0).$$

Here $\widetilde{V}$ is defined by

$$\widetilde{V}f = FVF^{-1}f = \widehat{V} * f$$

where $*$ denotes the convolution. By the assumption $V(x) = V(-x)$ we see that $\widetilde{V}$ is a reality preserving operator. Let

$$\widetilde{H}_0 = \frac{1}{2}(x - P_\mathrm{f})^2 + H_\mathrm{f} + \widetilde{V}.$$

Lemma 4.4 *We have $(\widetilde{H}_0 - z)^{-n} \in \mathcal{O}_{\mathrm{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$ for all $z \in \mathbb{R}$ with $z \notin \sigma(\widetilde{H}_0)$ and $n \in \mathbb{R}$.*

Proof: We have

$$(\widetilde{H}_0 - z)^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{-1+n} e^{-t\widetilde{H}_0} e^{tz} dt,$$

where $\Gamma(\cdot)$ denotes the Gamma function. It is enough to prove $e^{-t\widetilde{H}_0} \in \mathcal{O}_{\mathrm{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$. Since by the Trotter product formula,

$$e^{-t\widetilde{H}_0} = s-\lim_{n\to\infty} \left(e^{-(t/n)(P_\mathrm{f}-x)^2/2} F^{-1} e^{-(t/n)V} F\right)^n,$$

$$F^{-1}e^{-sV}F \in \mathcal{O}_{\mathrm{real}}(L^2(\mathbb{R}^3; \mathcal{F})),$$

and

$$e^{-s(P_\mathrm{f}-x)^2} \in \mathcal{O}_{\mathrm{real}}(L^2(\mathbb{R}^3; \mathcal{F})),$$

it follows that $e^{-t\widetilde{H}_0} \in \mathcal{O}_{\mathrm{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$. The lemma follows. □

From this lemma it follows that $(\widetilde{H}_0 - z)^{-1}, (\widetilde{H}_0 - z)^{-1/2} \in \mathcal{O}_{\mathrm{real}}(L^2(\mathbb{R}^3; \mathcal{F}))$. We decompose $\overline{H} = \widetilde{H} - E$ as $\overline{H} = \widetilde{H}_0 + \widetilde{H}_\mathrm{I}$, where

$$\widetilde{H}_\mathrm{I} = -\frac{e}{2}(x - P_\mathrm{f})A_\varphi(0) - \frac{e}{2}A_\varphi(0)(x - P_\mathrm{f}) + \frac{e^2}{2}A_\varphi^2(0) - \frac{e}{2}\sigma B_\varphi(0) - E.$$

Lemma 4.5 *There exists $e_c > 0$ such that for all $|e| < e_c$,* $\mathrm{Tr}P_e \geq 2$.

Proof: First we prove $PP_eP = aP$ with some $a > 0$ in the similar way as Lemma 3.4 with H_p and H_{Ip} replaced by $\widetilde{H}$ and $\widetilde{H}_I$, respectively. Then the lemma follows from the proof of Lemma 3.6. □

Theorem 4.6 *Suppose* $\Sigma_{el} - E > 0$, $|e| < e_c$ *and* $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. *Then* $\mathrm{Tr}P_e = 2$.

Proof: It follows from Lemmas 4.3 and 4.5. □

Suppose that V is rotation invariant. Let

$$\mathcal{J}_{\text{total}} = x \times (-i\nabla_x) + J_f + S_f + \frac{1}{2}\sigma.$$

Then we have for $\theta \in \mathbf{R}$, $\vec{n} \in \mathbf{R}^3$ with $|\vec{n}| = 1$,

$$e^{i\theta\vec{n}\cdot\mathcal{J}_{\text{total}}} H e^{-i\theta\vec{n}\cdot\mathcal{J}_{\text{total}}} = H.$$

Since $\sigma(\vec{n}\cdot\mathcal{J}_{\text{total}}) = \mathbb{Z}+1/2$ for each $\vec{n}$, $\mathcal{H}$ and H are decomposable as $\mathcal{H} = \bigoplus_{z\in\mathbb{Z}+\frac{1}{2}} \mathcal{H}(z)$, and $H = \bigoplus_{z\in\mathbb{Z}+\frac{1}{2}} H(z)$. In the same way as the proof of Theorem 2.1 one can prove the following corollary.

Corollary 4.7 *Suppose that* V *is translation invariant, and* $\Sigma_{el} - E > 0$, $|e| < e_c$ *and* $\theta_1(e)\theta_2(e) + \theta_3(e) < 1/3$. *Then* H *has two orthogonal ground states,* $\psi_\pm$, *with* $\psi_\pm \in \mathcal{H}(\pm 1/2)$.

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References

[1] V. Bach, J. Fröhlich, I. M. Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field, *Commun. Math. Phys.* **207** (1999), 249–290.

[2] V. Betz, F. Hiroshima, J. Lorinczi, R. A. Minlos and H. Spohn, Gibbs measure associated with particle-field system, to be published in *Rev. Math. Phys.*

[3] T. Chen, Operator-theoretic infrared renormalization and construction of dressed 1-particle states in non-relativistic QED, mp-arc 01301, preprint, 2001.

[4] J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons, *Ann. Inst. Henri Poincaré* **19** (1973), 1–103.

[5] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschritte der Physik* **22** (1974), 159–198.