数理解析研究所講究録 1275 巻 2002 年 112-118

A family of processes interpolating the Brownian motion and the self-avoiding process on the Sierpiński gasket and $\mathbb{R}$

Ben HAMBLY
University of Oxford

Kumiko HATTORI
Shinshu University

Tetsuya HATTORI
Nagoya University

Abstract

We construct a one-parameter family of self-repelling processes on the Sierpiński gasket, by taking scaling limits of self-repelling walks on the pre-Sierpiński gaskets. We prove that our model interpolates between the Brownian motion and the self-avoiding process on the Sierpiński gasket. Namely, we prove that the process is continuous in the parameter in the sense of convergence in law, and that the order of Hölder continuity of the sample paths is also continuous in the parameter. We also establish a law of the iterated logarithm for the self-repelling process. Finally we show that this approach yields a new class of one-dimensional self-repelling processes.

1. Our question

To illustrate our questions, first let us consider the Euclidean lattice, $\mathbb{Z}^d$ and a random walk on it. The **simple random walk** (RW) is a walk that jumps to one of its nearest neighbor points with equal probability. On the other hand, a **self-avoiding walk** (SAW) is a walk that is not allowed to visit any point more than once.

If you take the **scaling limit**, that is, the limit as the lattice spacing (bond length) tends to 0, the RW converges to the **Brownian motion** (BM) in $\mathbb{R}^d$.

The scaling limit of a SAW is far more difficult. It is because a SAW must remember all the points it has once visited. In short, it lacks Markov property. For the 1-dimensional lattice, that is, a line, it is trivial – the scaling limit is a constant speed motion to the right or to the left. For 4 or more dimensions, the scaling limit is the Brownian motion. Since the space is large enough, the RW is not much different from the SAW. However, for the 2 and 3-dimensional lattice, the scaling limit is not known.

¿From this viewpoint, the Sierpiński gasket is a rare example of a low dimensional space, where the scaling limit of a SAW is known. The SAW on the pre-Sierpiński gasket converges to a non-trivial **self-avoiding process**, which is not a straight motion along an edge, nor deterministic, and moreover, whose path Hausdorff dimension is greater than 1. It implies that the path spreads in the Sierpiński gasket, has infinitely fine creases and is self-avoiding. Let us emphasize here that in a low-dimensional space the existence of a non-trivial self-avoiding process itself is "something."

On the other hand, the Brownian motion on the Sierpiński gasket has been constructed by Barlow, Perkins and Kusuoka as the scaling limit of the simple random walk on the pre-Sierpiński gasket. (See [4], [5].)

Our question is : Now that we have two completely different processes on the Sierpiński gasket– the Brownian motion and the self-avoiding process, can we construct a family of processes that interpolates continuously these two?

We construct the interpolating process as the limit of a self-repelling walk. A **self-repelling walk** is something between the RW and a SAW. Visiting the same points more than once is not prohibited, but suppressed compared with the RW. We want to construct a one-parameter family of self-repelling walks such that at one end of the parameter it corresponds to the RW, at the other end the SAW. And we take the scaling limit.

$$\begin{tikzcd}
\text{RW} \arrow[d, "\text{scaling limit}"'] & \overset{\text{self-repelling walk}}{\Longleftrightarrow} \arrow[d] & \text{SAW} \arrow[d] \\
\text{BM} & \Longleftrightarrow & \text{SA process}
\end{tikzcd}$$

Here we will further explain what is meant by interpolation. There is a very important exponent that characterizes walks and their scaling limits. The most well-known scene where it appears is the mean square displacement of the walk on an infinite lattice (graph). For a walk starting at O (the origin), let us assume

$$E[|X_n|^2] \sim n^{2\gamma}, \qquad n \to \infty,$$

where X_n is the walker's location after n steps, and $|X_n|$ denotes the Euclidean distance from the starting point. γ is our exponent. If you take the scaling limit, this exponent governs the short-time behavior,

$$E[|X(t)|^2] \sim t^{2\gamma}, \qquad t \downarrow 0.$$

The same γ determines also other path properties of the scaling limit such as Hölder continuity and the law of the iterated logarithm.

For comparison, in the case of the one-dimensional integer lattice, $\mathbb{Z}$, for the RW, γ is known to be $1/2$ (the well-known exponent for the BM), and $\gamma = 1$, for the SAW, obviously, because it is a straight motion in one direction. In general, exponents are very resistent to changes. Bolthausen proved for a model of self-repelling walk on $\mathbb{Z}$, that γ is always 1 regardless of the strength of self-repulsion. Tóth constructed a different model such that γ varies from $1/2$ to $2/3$. There are a few other models, but none of them connects $1/2$ to 1. (See [6, 7, 8, 9, 10, 11].)

It is interesting enough if we can connect the BM and the self-avoiding process on the Sierpiński gasket continuously in the sense of weak convergence of path measures. But can we ask for more? So, our question is rephrased as : can we construct an interpolating family of processes that connects the exponent γ for the RW/BM continuously **all the way** to γ for the SAW/SA process? As we have seen above, it's not easy even on the line – the simplest lattice.

However, on the Sierpiński gasket, we give an affirmative answer and the same method works also on the line, $\mathbb{R}$.

2. Our Model

The pre-Sierpiński gaskets and the Sierpiński gasket are defined as follows. Let $O = (0,0)$, $a = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $b = (1,0)$, and let F_0' be the set of all the points on the vertices and edges of $\triangle Oab$. We define a sequence of sets F'_0, F'_1, F'_2, … , inductively by

$$F'_{n+1} = \frac{1}{2}F'_n \cup \frac{1}{2}(F'_n + a) \cup \frac{1}{2}(F'_n + b), \qquad n = 0, 1, 2, \dots ,$$

where $A + a = \{x + a \ : \ x \in A\}$ and $kA = \{kx \ : \ x \in A\}$. Let

$$F_n = F'_n \cup (F'_n - b).$$

We call F_n's the (finite) pre-Sierpiński gaskets. As n increases, the lattice (graph) gets finer. If we superpose all the F_n's and take the closure, we get the (finite) **Sierpiński gasket**, F.

$$F = cl(\bigcup_{n=0}^{\infty} F_n).$$

We denote the set of vertices in F_n by G_n.

Let us denote by W_n the set of continuous functions $w : [0, \infty) \to F_n$ such that there exists $L(w) \in \mathbb{N}$ for which

$$\begin{array}{ll}
w(0) = O, & \\
w(t) = a, & t \geq L(w), \\
w(t) \notin G_0 \setminus \{O\}, & t < L(w), \\
|w(i) - w(i+1)| = 1, & i = 0, \cdots, L(w) - 1, \\
\overline{w(i)w(i+1)} \subset F_n, & i = 0, \cdots, L(w) - 1, \\
w(t) = (i + 1 - t)w(i) + (t - i)w(i + 1), & i \leq t < i + 1, \quad i = 0, 1, 2, \cdots .
\end{array}$$

W_n is the set of paths on F_n that go from O to a without hitting b or c or d. $L(w)$ denotes the steps needed to get to a. (Between integer times, we interpolate by constant speed motion. We've made the path continuous just for later convenience.)

To define a "self-repelling walk," we assign weight to each path. Our model is unique in the way of realizing self-repulsion. In other models on $\mathbb{Z}$, they count the numbers of returns to the same points or bonds, and define a repulsion factor using these numbers. But we count turns at a sharp angle and U-turns as shown below.

For $w \in W_1$, let $M_1(w)$ be the number of returns to the starting point, O. Let $N_1(w)$ be the number of U-turns and sharp turns that occur at points other than O. Here U-turns and sharp turns occur when $\overrightarrow{w(i-1)w(i)} \cdot \overrightarrow{w(i)w(i+1)} < 0$, where $\vec{a} \cdot \vec{b}$ denotes the inner product of $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$.

Let $0 \leq u \leq 1$ and $x > 0$ be parameters. For each path in W_1, we assign the following weight.

$$P_1^u(x)[w] = \frac{x^{L(w)} \ u^{M_1(w)+N_1(w)}}{\Phi(x, u)},$$

where

$$\Phi(x, u) = \sum_{w \in W_1} x^{L(w)} \ u^{M_1(w)+N_1(w)}.$$

The factor involving u is the **repulsion factor**.

Next, we go on to define P_2^u on W_2. In defining a probability on W_2, we note that we get a path in W_2 by adding finer structures to a path in W_1. First consider a path v of W_1. Let us add to the first step of v a finer structure on F_2 that goes from $v(0) = O$ to $v(1)$ without hitting any F_1 vertices other than $v(0)$. The part of F_2 inside the equilateral triangle with $v(0)$ and $v(1)$ as two of the vertices is similar to F_1. Thus, we see that this finer structure between the start and the first step of v corresponds to some element of W_1. We give finer structures to each step of v in a similar way. This way we get a path in W_2, patching up small W_1 paths, $w_1, \cdots, w_{L(v)}$, on a rough path v. Actually, each path in W_2 can be constructed in this way, adding finer structures. Thus, for finer structures between each step, M_1 and N_1 are defined. We define the weight for $w \in W_2$ by

$$\begin{aligned} P_2^u(x)[w] &= \frac{1}{\Phi_2(x,u)}\, x^{L(w)} \quad \underset{\text{[base path on } F_1\text{]}}{u^{M_1(v)+N_1(v)}} \quad \underset{\text{[finer structures]}}{\cdot \prod_i^{L(v)} u^{M_1(w_i)+N_1(w_i)}} \\ &= \frac{1}{\Phi_2(x,u)}\, x^{L(w)} \quad u^{M_2(v)+N_2(v)} \end{aligned}$$

where $L(w)$ is the number of the steps on F_2, and $\Phi_2(x,u)$ is the normalization factor

$$\Phi_2(x,u) = \sum_{w \in W_2} x^{L(w)} \; u^{M_2(w)+N_2(w)}.$$

¿From the fact that we constructed a path on F_2 by adding finer structures to a path on F_1, it is easy to see

$$\Phi_2(x,u) = \Phi(\Phi(x,u),u).$$

We go on to define P_n^u on W_n **recursively**.
First, we consider a path on F_{n-1} and patch up small W_1 paths on it. For general n, we have the **recursion relation**

$$\Phi_n(x,u) = \Phi_{n-1}(\Phi(x,u),u).$$

This is one of the key properties of our model. We can see the meaning of the recursion in this way. Consider a self-repelling walk on F_n with propability $P_n^u(x)$. Pick up all the F_{n-1} -points the walk visits. Then we get a self-repelling walk on F_{n-1} with renormalized probability $P_{n-1}^u(\Phi(x,u))$.

Let us choose $x = x_u$ to be the unique positive solution to the equation,

$$x_u = \Phi(x_u, u).$$

This choice of x makes the measure self-similar. $u = 1$ corresponds to the simple random walk with the first exit at a. In this case, u-factor is absent and $x_u = 1/4$. It shows the walker chooses one of its four nearest neighbors with equal probability. For $u = 0$ only self-avoiding paths survive. (For more details of our model, see [1].)

3. Results

We study the function $\Phi(x,u)$ (this corresponds to the partition function, or the generating function) closely and get the following results.

Let

$$\lambda_u \overset{\text{def}}{=} E^{P_1^u}[L].$$

λ_u is the average steps from O to on F_1. It is continuous in u, and

$$\lambda_1 = 5 \;\; (RW), \;\; \lambda_0 = \frac{7-\sqrt{5}}{2} \quad \text{(SAW)} \quad 2 < \lambda_0 < 3.$$

Now we are going to take the continuum limit. It corresponds to the limit as $n \to \infty$.

We defined P_n^u as a probability measure on W_n. We can re-consider it as a probability measure defined on a space of continuous functions C on the Sierpiński gasket supported on W_n. Thus the base space is common to all n's. Let us consider an accelerated process by the factor of $(\lambda_u)^n$. Recall that for our path, it takes time 1 to go to a nearest neighbor vertex. As the lattice gets finer, our walk gets slower. So we need a proper acceleration to get a non-trivial limit. Let $X_n(\cdot)$ be a process that obeys P_n^u, and denote the destribution of time-scaled process, $X_n((\lambda_u)^n \cdot\,)$ by $\tilde{P}_n^u$.

Our first theorem states the existence of the scaling limit.

Theorem 1 *$\tilde{P}_n^u$ converges weakly to a probability measure P^u on C as $n \to \infty$.*

P^1 corresponds to the Brownian motion conditioned that it hits a before b, c, d, (and is stopped at a). P^0 corresponds to the non-trivial self-avoiding process mentioned in Section 1.

Remark

In [2, 3], a different model of self-avoiding walk on the Sierpiński gasket has been studied. In this model, for each self-avoiding path w that goes from O to a, a positive weight propotional to $e^{-\beta L(w)}$ is assigned, where $\beta > 0$ is a parameter. It has been proved that there exisits a unique $\beta_c > 0$ for which the scaling limit is a self-avoiding process with path Hausdorff dimension greater than 1 almost surely. Our scaling limit process coincides with this limit process. Our model at $u = 0$ is more restricted than usual SAW because sharp turns are prohibited as well as returning to the same points. But it produces the same scaling limit as the 'standard SAW.'

Our second theorem shows that our limit process is continuous in u and does connect the BM and the self-avoiding process continuously.

Theorem 2 *(Continuity in u) For all $u_0 \in [0, 1]$,*

$$P^u \longrightarrow P^{u_0} \quad \textit{weakly as } u \to u_0$$

The following theorems concern path properties of the limit process.

Theorem 3 *For all $p > 0$, there exist $C_i = C_i(p, u) > 0$, $i = 1, 2$ such that*

$$C_1 \leq \liminf_{t \to 0} \frac{E^u[|X(t)|^p]}{t^{\gamma_u p}} \leq \limsup_{t \to 0} \frac{E^u[|X(t)|^p]}{t^{\gamma_u p}} \leq C_2,$$

where

$$\gamma_u = \frac{\log 2}{\log \lambda_u}$$

and is continuous in u.

Theorem 4 *(Hölder continuity) For any $M > 0$ and any $0 < \gamma' < \gamma_u$, there exist a.s. $b = b(M, \gamma', \omega) > 0$ and $H = H(M, \gamma', \omega) > 0$ such that*

$$|X(t+h) - X(t)| \leq b|h|^{\gamma'},$$

$$\forall t \in [0, M], |h| \leq H$$

Theorem 5 *(Law of the Iterated Logarithm) There exist* $C_i = C_i(p, u) > 0$, $i = 3, 4$ *such that*

$$C_3 \leq \limsup_{t \to 0} \frac{|X(t)|}{\psi(t)} \leq C_4, \qquad a.s.,$$

where

$$\psi(t) = t^{\gamma_u} (\log\log \frac{1}{t})^{1-\gamma_u}.$$

Thus, in our model, the exponent γ in Section 1 is given by

$$\gamma_u = \frac{\log 2}{\log \lambda_u}$$

and is a continuous function in u connecting $\gamma_1 = \dfrac{\log 2}{\log 5}$ for the simple random walk and $\gamma_0 = \dfrac{\log 2}{\log \frac{7-\sqrt{5}}{2}}$ for the self-avoiding walk.

4. Self-repelling processes on $\mathbb{R}$

We start with a sequence of random walks on $\mathbb{Z}$ (instead of the pre-Sierpiński gasket). The vertex set that we will use for our walks is $G_n = \{k2^{-n} : k = -2^n, -2^n + 1, \cdots, 0, 1, 2, \cdots, 2^n\}$. W_n is the set of continuous functions such that at integer times it takes values in G_n with nearest neighbor jumps from 0 to 1. $N_k(w)$ and $M_k(w)$ can be defined similarly to the case of the Sierpiński gasket.

The generating function $\Phi_1(x, u)$ is given by

$$\Phi_1(x, u) = \frac{x^2}{1 - 2u^2x^2}.$$

In particular, we have $\Phi_n(x, 0) = x^{2^n}$, which implies that when $u = 0$ we have a single path which connects 0 and 2^n by a straight line (i.e., the self-avoiding path on $\mathbb{Z}$), and for $u = 1$ we reproduce the generating function for the simple random walk.

We can give explicit formulas for $x_u > 0$ and $\lambda_u > 0$.

$$x_u = \frac{1}{4u^2}(\sqrt{1 + 8u^2} - 1), \quad \lambda_u = \frac{2}{x_u} = \sqrt{1 + 8u^2} + 1\,.$$

Once we have established these properties of the generating function the subsequent analysis follows quite similar lines to the Sierpiński gasket case. For example, the probability measures on the paths are defined in a similar way to the case of Sierpiński gasket , and the existence of a continuum limit (Theorem 1) and the weak continuity of the path measure P^u in $u \in [0, 1]$ (Theorem 2) hold. The sample path properties such as Theorems 3 through 5 also hold with $\gamma_u = \dfrac{\log 2}{\log \lambda_u}$.

References

[1] B. Hambly, K. Hattori, T. Hattori, *Self-repelling walk on the Sierpiński gasket,* To appear in Probab. Theor. Relat. Fields.

[2] K. Hattori, T. Hattori, *Self-avoiding process on the Sierpiński gasket,* Probab. Theor. Relat. Fields **88** (1991), 405–528.

[3] K. Hattori, T. Hattori, S. Kusuoka, *Self-avoiding paths on the pre-Sierpiński gasket,* Probab. Theor. Relat. Fields **84** (1990) 1–26.

[4] M. T. Barlow, E. A. Perkins, *Brownian Motion on the Sierpiński gasket,* Probab. Theor. Relat. Fields **79** (1988) 543–623.

[5] S. Kusuoka, *A diffusion process on a fractal,* Proc. Taniguchi Symposium (1985) 251–274.

[6] E. Bolthausen, *On self-repellent one dimensional random walks,* Probab. Theor. Relat. Fields **86** (1990) 423–441.

[7] D. C. Bryges, G. Slade, *The diffusive phase of a model of self-interacting walks,* Probab. Theor. Relat. Fields **103** (1995) 285–315.

[8] A. Greven, F. Hollander, *A variational characterization of the speed of a one-dimensional self-repellent random walk,* Ann. Appl. Probab. **3** (1993) 1067–1099.

[9] B. Tóth, *'True' self-avoiding walk with generalized bond repulsion on* **Z**, Journ. Stat. Phys. **77** (1994) 17–33.

[10] B. Tóth, *The 'true' self-avoiding walk with bond repulsion on* **Z**: *Limit theorems,* Ann. Probab. **23** (1995) 1523–1556.

[11] B. Tóth, *Generalized Ray-Knight theory and limit theorems for self-interacting random walks on* $\mathbf{Z}^1$, Ann. Probab. **24** (1996) 1324–1367.