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On a strongly starlikeness criteria

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Abstract

H.Silverman [Internat. J. Math. Math. Sci. 22(1999), 75-79] investigated and obtained some results for the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In this paper, we obtain a sufficient condition of functions for strongly starlikeness of order β.

1 Introduction

Let $\mathcal{S}$ denote the class of functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$ that are analytic and univalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{S}$ is said to be starlike of order α if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$$

for some $\alpha\,(0 \leqq \alpha < 1)$, and for all $z \in \mathbb{U}$. The class of starlike functions of order α is denoted by $\mathcal{S}^*(\alpha)$. Further, a function $f(z) \in \mathcal{S}$ is said to be convex of order α if and only if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for some $\alpha\,(0 \leqq \alpha < 1)$, and for all $z \in \mathbb{U}$. Also we denote by $\mathcal{C}(\alpha)$ the subclass of $\mathcal{S}$ consisting of all convex functions of order α in $\mathbb{U}$.
On the other hand, a function $f(z)$ in $\mathcal{S}$ is said to be strongly starlike of order β if it satisfies

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\beta$$

for some $\beta\,(0 < \beta \leqq 1)$, and for all $z \in \mathbb{U}$. We say that $f(z) \in \mathcal{SS}^*(\beta)$ if $f(z)$ is strongly starlike of order β in $\mathbb{U}$.
Silverman [2] investigated the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. Let $\mathcal{G}_b$ be the subclass of $\mathcal{S}$ consisting of functions $f(z) \in \mathcal{S}$ which satisfy

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$$\left| \left\{ \frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \right\} - 1 \right| < b \qquad (z \in \mathbb{U})$$

for some real b.

For this class $\mathcal{G}_b$, Silverman obtained the following rsult.

Theorem A ([2]). *If $0 < b \leqq 1$, then*

$$\mathcal{G}_b \subset \mathcal{S}^*\left(\frac{2}{1+\sqrt{1+8b}}\right).$$

The result is sharp for all b.

In this paper, we consider the strongly starlikeness for functions $f(z)$ belonging to $\mathcal{G}_b$.

2 Strongly starlikeness

To discuss the strongly starlikeness of functions $f(z)$ in $\mathcal{G}_b$, we have to recall here the following result by Nunokawa [1].

Lemma. *Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ $(z \in \mathbb{U})$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg(p(z))| < \frac{\pi}{2}\beta \qquad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\beta,$$

where $\beta > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = iK\beta,$$

where

$$K \geqq \frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi}{2}\beta$$

and

$$K \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi}{2}\beta,$$

where $p(z_0)^{\frac{1}{\beta}} = \pm ia$ and $a > 0$.

Our main result is contained in

Theorem 1. *If $f(z)$ belongs to the class $\mathcal{G}_{b(\beta)}$ with*

$$b(\beta) = \frac{\beta}{\sqrt{(1-\beta)^{1-\beta}(1+\beta)^{1+\beta}}} \qquad (0 < \beta \leqq 1),$$

then $f(z) \in \mathcal{SS}^*(\beta)$.

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{f(z)}.$$

Then it follows that

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 = \frac{zp'(z)}{p(z)^2}.$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\beta \quad for \quad |z| < |z_0|$$

and

$$|\arg(p(z_0))| = \frac{\pi}{2}\beta,$$

then , applying Lemma, we have that

$$\left|\frac{z_0p'(z_0)}{p(z_0)^2}\right| = \left|iK\beta\frac{1}{(\pm ia)^\beta}\right| = \beta|K|a^{-\beta}$$
$$\geqq \frac{\beta}{2}\left(a^{1-\beta} + \frac{1}{a^{1+\beta}}\right).$$

Define the function $g(a)$ by

$$g(a) = a^{1-\beta} + \frac{1}{a^{1+\beta}} \quad (a > 0; 0 < \beta \leqq 1).$$

Since

$$g'(a) = \frac{1}{a^{2+\beta}}((1-\beta)a^2 - (1+\beta)),$$

$g(a)$ takes its minimum value at $a = \sqrt{\frac{1+\beta}{1-\beta}}$. This implies that

$$\left|\frac{z_0p'(z_0)}{p(z_0)^2}\right| \geqq \frac{\beta}{2}g\left(\sqrt{\frac{1+\beta}{1-\beta}}\right)$$
$$= \frac{\beta}{2}\left\{\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1-\beta}{2}} + \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}\right\}$$
$$= \frac{\beta}{\sqrt{(1-\beta)^{1-\beta}(1+\beta)^{1+\beta}}},$$

which contradicts our condition $f(z) \in \mathcal{G}_{b(\beta)}$ of the theorem. Thus we complete the proof of the theorem.

Considering the case of $\beta = 1$ in the proof of Theorem 1, we have

Corollary 1. *If $f(z) \in \mathcal{G}_b$ with $b = \frac{1}{2}$, then $f(z) \in \mathcal{SS}^*(1)$, or $f(z)$ is strongly starlike in $\mathbb{U}$.*

Taking $\beta = \frac{1}{2}$ in Theorem 1, we have

Corollary 2. *If $f(z) \in \mathcal{G}_b$ with $b = \dfrac{1}{\sqrt{3\sqrt{3}}} = 0.438691\cdots$, then $f(z) \in \mathcal{SS}^*(\frac{1}{2})$.*

References

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