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Coefficient Estimates for Certain Classes of Analytic Functions

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Abstract

For some real $\alpha(\alpha>1)$, two subclasses $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ of analytic fuctions $f(z)$ with $f(0)=0$ and $f'(0)=1$ in $\mathbb{U}$ are introduced. The object of the present paper is to discuss the coefficient estimates for functions $f(z)$ belonging to the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{M}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<\alpha \qquad (z\in\mathbb{U})$$

for some $\alpha(\alpha>1)$. And let $\mathcal{N}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}<\alpha \qquad (z\in\mathbb{U})$$

for some $\alpha(\alpha>1)$. Then, we see that $f(z)\in\mathcal{N}(\alpha)$ if and only if $zf'(z)\in\mathcal{M}(\alpha)$.

Remark 1.1. *For $1<\alpha\leq\frac{4}{3}$, the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were introduced by Uralegaddi, Ganigi and Sarangi* [2].

We easily see that

Example 1.1. (i) $f(z)=z(1-z)^{2(\alpha-1)}\in\mathcal{M}(\alpha)$.

(ii) $g(z)=\dfrac{1}{2\alpha-1}\{1-(1-z)^{2\alpha-1}\}\in\mathcal{N}(\alpha)$.

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2 Coefficient estimates for functions

We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leqq 2(\alpha - 1)$$

for some k $(0 \leqq k \leqq 1)$ and some α $(\alpha > 1)$, then $f(z) \in \mathcal{M}(\alpha)$.

Proof. Let us suppose that

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leqq 2(\alpha - 1) \tag{1}$$

for $f(z) \in \mathcal{A}$.
It sufficies to show that

$$\left| \frac{\frac{zf'(z)}{f(z)} - k}{\frac{zf'(z)}{f(z)} - (2\alpha - k)} \right| < 1 \qquad (z \in \mathbb{U}).$$

We note that

$$\left| \frac{\frac{zf'(z)}{f(z)} - k}{\frac{zf'(z)}{f(z)} - (2\alpha - k)} \right| = \left| \frac{1 - k + \sum_{n=2}^{\infty}(n-k)a_n z^{n-1}}{1 + k - 2\alpha + \sum_{n=2}^{\infty}(n+k-2\alpha)a_n z^{n-1}} \right|$$

$$\leqq \frac{1 - k + \sum_{n=2}^{\infty}(n-k)|a_n||z|^{n-1}}{2\alpha - 1 - k - \sum_{n=2}^{\infty}|n+k-2\alpha|\,|a_n||z|^{n-1}}$$

$$< \frac{1 - k + \sum_{n=2}^{\infty}(n-k)|a_n|}{2\alpha - 1 - k - \sum_{n=2}^{\infty}|n+k-2\alpha|\,|a_n|}.$$

The last expression is bounded above by 1 if

$$1 - k + \sum_{n=2}^{\infty}(n-k)|a_n| \leqq 2\alpha - 1 - k - \sum_{n=2}^{\infty}|n+k-2\alpha|\,|a_n|$$

which is equivalent to our condition

$$\sum_{n=2}^{\infty}\{(n-k) + |n+k-2\alpha|\}|a_n| \leqq 2(\alpha - 1)$$

of the theorem. This completes the proof of the theorem.

If we take $k = 1$ and some $\alpha \left(1 < \alpha \leqq \frac{3}{2}\right)$ in Theorem 2.1, then we have

Corollary 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}(n - \alpha)|a_n| \leqq \alpha - 1$$

for some $\alpha \left(1 < \alpha \leqq \frac{3}{2}\right)$, then $f(z) \in \mathcal{M}(\alpha)$.

Example 2.1. The function $f(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha - 1)}{n(n+1)(n - k + |n + k - 2\alpha|)} z^n$$

belongs to the class $\mathcal{M}(\alpha)$.

For the class $\mathcal{N}(\alpha)$, we have

Theorem 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n(n - k + 1 + |n + k - 2\alpha|)|a_n| \leqq 2(\alpha - 1) \tag{2}$$

for some $k\,(0 \leqq k \leqq 1)$ and some $\alpha\,(\alpha > 1)$, then $f(z)$ belongs to the class $\mathcal{N}(\alpha)$.

Corollary 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n(n - \alpha)|a_n| \leqq \alpha - 1$$

for some $\alpha \left(1 < \alpha \leqq \frac{3}{2}\right)$, then $f(z) \in \mathcal{N}(\alpha)$.

Example 2.2. The function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{4(\alpha - 1)}{n^2(n+1)(n - k + |n + k - 2\alpha|)} z^n$$

belongs to the class $\mathcal{N}(\alpha)$.

Further, denoting by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of $\mathcal{A}$ consisting of all starlike functions of order α, and of all convex functions of order α, respectively, we derive

Theorem 2.3. *If $f(z) \in \mathcal{A}$ satisfies the coefficient inequality* (1) *for some* α $\left(1 < \alpha \leqq \frac{k+2}{2} \leqq\right.$ *then* $f(z) \in \mathcal{S}^*\left(\frac{4-3\alpha}{3-2\alpha}\right)$. *If $f(z) \in \mathcal{A}$ satisfies the coefficient inequality* (2) *for some* α $\left(1 < \alpha \leqq \frac{k-2}{2} \leqq \frac{3}{2}\right)$ *then* $f(z) \in \mathcal{K}\left(\frac{4-3\alpha}{3-2\alpha}\right)$.

Proof. For some α $\left(1 < \alpha \leqq \frac{k+2}{2} \leqq \frac{3}{2}\right)$, we see that the coefficient inequality (1) implies that

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leqq \alpha - 1.$$

It is well-known that if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty}\frac{n-\beta}{1-\beta}|a_n| \leqq 1$$

for some $\beta\,(0 \leqq \beta < 1)$, then $f(z) \in \mathcal{S}^*(\beta)$ by Silverman [1]. Therefore, we have to find the smallest positive β such that

$$\sum_{n=2}^{\infty}\frac{n-\beta}{1-\beta}|a_n| \leqq \sum_{n=2}^{\infty}\frac{n-\alpha}{\alpha-1}|a_n| \leqq 1.$$

This gives that

$$\beta \leqq \frac{(2-\alpha)n-\alpha}{n-2\alpha+1} \tag{3}$$

for all $n = 2, 3, 4, \cdots$. Noting that the right hand side of the inequality (3) is increasing for n, we conclude that

$$\beta \leqq \frac{4-3\alpha}{3-2\alpha},$$

which proves that $f(z) \in \mathcal{S}^*\left(\frac{4-3\alpha}{3-2\alpha}\right)$. Similarly, we can show that if $f(z) \in \mathcal{A}$ satisfies (2), then $f(z) \in \mathcal{K}\left(\frac{4-3\alpha}{3-2\alpha}\right)$.

□

Our result for the coefficient estimates of functions $f(z) \in \mathcal{M}(\alpha)$ is contained in

Theorem 2.4. *If $f(z) \in \mathcal{M}(\alpha)$, then*

$$|a_n| \leqq \frac{\Pi_{j=2}^{n}(j+2\alpha-4)}{(n-1)!} \qquad (n \geqq 2). \tag{4}$$

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{\alpha - \frac{zf'(z)}{f(z)}}{\alpha - 1}$$

for $f(z) \in \mathcal{M}(\alpha)$. Then $p(z)$ is aslytic in $\mathbb{U}$, $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0\,(z \in \mathbb{U})$. Therefore, if we write

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then $|p_n| \leqq 2\,(n \geqq 1)$. Since

$$\alpha f(z) - zf'(z) = (\alpha - 1)p(z)f(z),$$

we obtain that

$$(1-n)a_n = (\alpha-1)(p_{n-1} + a_2p_{n-2} + a_3p_{n-3} + \cdots + a_{n-1}p_1).$$

If $n = 2$, then $-a_2 = (\alpha - 1)p_1$ implies that

$$|a_2| = (\alpha-1)|p_1| \leqq 2\alpha - 2.$$

Thus the coefficient estimate (4) holds true for $n = 2$. Next, suppose that the coefficient estimate

$$|a_k| \leqq \frac{\Pi_{j=2}^{k}(j + 2\alpha - 4)}{(k-1)!}$$

is true for all $k = 2, 3, 4, \cdots, n$. Then we have that

$$-na_{n+1} = (\alpha-1)(p_n + a_2p_{n-1} + a_3p_{n-2} + \cdots + a_np_1),$$

so that

$$\begin{aligned} n|a_{n+1}| &\leqq (2\alpha-2)(1 + |a_2| + |a_3| + \cdots + |a_n|) \\ &\leqq (2\alpha-2)\left(1 + (2\alpha-2) + \frac{(2\alpha-2)(2\alpha-1)}{2!} + \cdots + \frac{\Pi_{j=2}^{n}(j+2\alpha-4)}{(n-1)!}\right) \\ &= (2\alpha-2)\left(\frac{(2\alpha-1)2\alpha(2\alpha+1)\cdots(2\alpha+n-4)}{(n-2)!} + \frac{(2\alpha-2)(2\alpha-1)2\alpha\cdots(2\alpha+n-4)}{(n-1)!}\right) \\ &= \frac{\Pi_{j=2}^{n+1}(j+2\alpha-4)}{(n-1)!}. \end{aligned}$$

Thus, the coefficient estimate (4) holds true for the case of $k = n + 1$. Applying the mathematical induction for the coefficient estimate (4), we complete the proof of the theorem.

□

For the functions $f(z)$ belonging to the class $\mathcal{N}(\alpha)$, we also have

Theorem 2.5. *If $f(z) \in \mathcal{N}(\alpha)$, then*

$$|a_n| \leqq \frac{\Pi_{j=2}^{n}(j+2\alpha-4)}{n!} \qquad (n \geqq 2).$$

Remark 2.1. We can not show that Theorem 2.4 and Theorem 2.5 are sharp. If we prove that Theorem 2.4 is sharp, then the sharpness of Theorem 2.5 follows.

References

[1] H.Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51**(1975), 109 - 116.

[2] B.A.Uralegaddi, M.D.Ganigi and S.M.Sarangi, *Univalent functions with positive coefficients*, Tamkang J. Math. **25**(1994), 225 - 230.

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