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ON A SUBCLASS OF ALPHA-CONVEX FUNCTIONS

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ABSTRACT. Mocanu [4] introduced and studied the class of α-convex functions which is a subclass of analytic functions in the open unit disc. The properties of this class have been obtained. In this paper, we consider the order of strongly starlikeness for a subclass of α-convex functions.

1. INTRODUCTION.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if and only if it satisfies the condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad z \in \mathbb{U} \tag{1}$$

where $0 \leq \alpha < 1$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisiting of all starlike functions of order α in $\mathbb{U}$. A function $f(z) \in \mathcal{A}$ is said to be strongly starlike of order α in $\mathbb{U}$ if and only if it satisfies the condition

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}\alpha \quad z \in \mathbb{U} \tag{2}$$

where $0 < \alpha \leq 1$. We denote by $\mathcal{SS}^*(\alpha)$ the subclass of $\mathcal{A}$ consisiting of all strongly starlike functions of order α in $\mathbb{U}$. A function $f(z) \in \mathcal{A}$ is said to be starlike in $\mathbb{U}$ when $\alpha = 0$ for (1) and $\alpha = 1$ for (2). We denote by $\mathcal{S}^*$ the subclass of $\mathcal{A}$ consisiting of all starlike functions in $\mathbb{U}$. A function $f(z) \in \mathcal{A}$ is said to be convex in $\mathbb{U}$ if and only if it satisfies the condition

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > 0 \quad z \in \mathbb{U}. \tag{3}$$

We denote by $\mathcal{C}$ the subclass of $\mathcal{A}$ consisiting of all convex functions in $\mathbb{U}$. These conditions (1), (2) and (3) are also sufficient conditions for univalence of $f(z) \in \mathcal{A}$. (See, e.g., [1].)

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Mocanu [4] defined a subclass of $\mathcal{A}$ as the following. A function $f(z) \in \mathcal{A}$ is said to be α-convex in $\mathbb{U}$ if and only if it satisfies the condition $f(z)f'(z)/z \neq 0$ and

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0 \quad z \in \mathbb{U} \tag{4}$$

where α is a real number. If the condition (4) is satisfied, then the condition $f(z)f'(z)/z \neq 0$ is always true, so this condition is not needed. We denote by $\mathcal{M}(\alpha)$ the subclass of $\mathcal{A}$ consisiting of all α-convex functions in $\mathbb{U}$.

Miller, Mocanu and Reade [2] obtained the following result.

Theorem A. *If $f(z) \in \mathcal{M}(\alpha)$, then $f(z) \in \mathcal{S}^*$. Moreover, if $\alpha \geq 1$, then $f(z) \in \mathcal{C}$.*

Furthermore, they [3] obtained the following result.

Theorem B. *If $f(z) \in \mathcal{M}(\alpha)$, $\alpha \geq 0$, then $f(z) \in \mathcal{S}^*(\beta(\alpha))$, where*

$$\beta(\alpha) = \begin{cases} 0, & 0 \leq \alpha < 1, \\ \dfrac{\Gamma\left(\frac{1}{2} + \frac{1}{\alpha}\right)}{\sqrt{\pi}\,\Gamma\left(1 + \frac{1}{\alpha}\right)}, & 1 \leq \alpha, \end{cases}$$

and this result is sharp.

Mocanu [5] obtained the following result.

Theorem C. *If $f(z) \in \mathcal{A}$ satisfies the condition,*

$$\left|\arg\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}\right| < \frac{\pi}{2}\gamma \quad z \in \mathbb{U},$$

where

$$\tan\frac{\pi}{2}\gamma = \tan\frac{\pi}{2}\beta + \frac{\alpha\beta}{(1-\beta)\cos\frac{\pi}{2}\beta}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}$$

and $0 < \beta < 1$, then $f(z) \in \mathcal{SS}^*(\beta)$.

In this paper, we investigate conditions on α, β and γ for which

$$\left|(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - \gamma\right| < \gamma \quad z \in \mathbb{U}$$

implies $f(z) \in \mathcal{SS}^*(\beta)$ holds.

We make use of the following lemma due to Nunokawa [6].

Lemma. *Let $p(z)$ be analytic, $p(z) \neq 0$ in $\mathbb{U}$ and $p(0) = 1$. Suppose that there exists a point $z_0 \in E$ such that*

$$|\arg p(z)| < \frac{\pi\alpha}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\alpha}{2}$$

where $\alpha > 0$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \text{when} \quad \arg p(z_0) = \frac{\pi\alpha}{2}$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi\alpha}{2}$$

where

$$p(z_0)^{1/\alpha} = \pm ia \quad \text{and} \quad a > 0.$$

2. Main result.

Theorem. *If $f(z) \in \mathcal{A}$ satisfies the condition,*

$$\left|(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - \gamma\right| < \gamma \quad z \in \mathbb{U}, \tag{5}$$

where

$$\gamma = \frac{\alpha\beta(1 + \sin\frac{\pi}{2}\beta)}{\cos\frac{\pi}{2}\beta},$$

$\alpha > 0$ and $0 < \beta < 1$, then $f(z) \in \mathcal{SS}^*(\beta)$.

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)}. \tag{6}$$

From the condition (5), we have $f(z) \in \mathcal{S}^*$, so $p(z) \neq 0$ in $\mathbb{U}$. By logarithmic differentiation of (6), we have

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{zp'(z)}{p(z)}$$

or

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z) + \alpha\frac{zp'(z)}{p(z)}.$$

If there exist a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\beta,$$

then from Lemma, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\beta k$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \text{when} \quad \arg p(z_0) = \frac{\pi\beta}{2}$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi\beta}{2}$$

where

$$p(z_0)^{1/\beta} = \pm ia \quad \text{and} \quad a > 0.$$

At first, let us suppose $p(z_0)^{1/\beta} = ia$, then we have

$$p(z_0) + \alpha\frac{z_0 p'(z_0)}{p(z_0)} = a^\beta e^{i\frac{\pi}{2}\beta} + i\alpha\beta k$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1$$

From this, we have

$$\begin{aligned}
\mathrm{Re}\left(p(z_0) + \alpha\frac{z_0 p'(z_0)}{p(z_0)}\right)^{-1} &= \frac{a^\beta \cos\frac{\pi}{2}\beta}{a^{2\beta}\cos^2\frac{\pi}{2}\beta + (a^\beta \sin\frac{\pi}{2}\beta + \alpha\beta k)^2} \\
&\leq \frac{a^\beta \cos\frac{\pi}{2}\beta}{a^{2\beta}\cos^2\frac{\pi}{2}\beta + (a^\beta \sin\frac{\pi}{2}\beta + \alpha\beta)^2} \\
&= \frac{a^\beta \cos\frac{\pi}{2}\beta}{a^{2\beta} + 2a^\beta\alpha\beta\sin\frac{\pi}{2}\beta + \alpha^2\beta^2}.
\end{aligned}$$

Let us put

$$g(t) = \frac{t\cos\frac{\pi}{2}\beta}{t^2 + 2t\alpha\beta\sin\frac{\pi}{2}\beta + \alpha^2\beta^2}$$

where $t > 0$. Then by easy calculation, we have

$$g'(t) = \frac{(\alpha^2\beta^2 - t^2)\cos\frac{\pi}{2}\beta}{(t^2 + 2t\alpha\beta\sin\frac{\pi}{2}\beta + \alpha^2\beta^2)^2}$$

and we see that $g(t)$ takes the maximum value at $t = \alpha\beta$. From this, we have

$$\operatorname{Re}\left(p(z_0) + \alpha\frac{z_0 p'(z_0)}{p(z_0)}\right)^{-1} \leq \frac{\alpha\beta\cos\frac{\pi}{2}\beta}{\alpha^2\beta^2 + 2\alpha^2\beta^2\sin\frac{\pi}{2}\beta + \alpha^2\beta^2} = \frac{\cos\frac{\pi}{2}\beta}{2\alpha\beta(1+\sin\frac{\pi}{2}\beta)}. \tag{7}$$

Since $|w - h| < h \Leftrightarrow \operatorname{Re}(1/w) > 1/2h$, this contradicts the assumption of this theorem.

For the case $p(z_0)^{1/\beta} = -ia$, applying the same method as the above, we have the condition (7). Therefore we complete the proof. □

REFERENCES

[1] A. W. Goodman, "Univalent Functions," Vol. I, Mariner Publishing Company, Tampa, Florida, 1983.

[2] S. S. Miller, P. T. Mocanu and M. O. Reade, *All α-convex functions are univalent and starlike*, Proc. Amer. Math. Soc., **37** **(2)** (1973), 553-554.

[3] S. S. Miller, P. T. Mocanu and M. O. Reade, *The order of starlikeness of alpha-convex functions*, Mathematica (Cluj) **20** **(43)** (1978), 25-30.

[4] P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme*, Mathematica (Cluj) **11** **(34)** (1969), 127-133.

[5] P. T. Mocanu, *Alpha-convex integral operator and strongly starlike functions*, Studia Univ. Babes-Bolyai Mathematica, **34**, 2, (1989), 18-24.

[6] M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan. Acad., **69**, Ser. A (1993), 234-237.

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