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CERTAIN CLASSES AND INEQUALITIES INVOLVING FRACTIONAL CALCULUS AND MULTIVALENT FUNCTIONS

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Abstract

In this paper we intriduce two novel subclasses $\mathcal{V}_\delta(p;\mu)$ and $\mathcal{W}_\delta(p;\mu)$ of analytic and p-valent functions which is defined by using the fractional calculus (fractional derivatives). We obtain a sufficient condition for a function to belong to each of these subclasses and investigate the charactristics of functions in these subclasses. Geometric properties of multivalent functions (p-valently close-to-convex, p-valently starlike and p-valently convex functions) are also considered.

Key words: Open unit disk, analytic, multivalent, starlike, convex, close-to-convex functions, fractional calculus and Jack's Lemma.

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1. Introduction and Definitions

Let $p \in \mathcal{N} = \{1, 2, 3, ...\}$ and $\mathcal{T}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1.1}$$

being analytic and p-valent in the open unit disk

$$\mathcal{U} = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}.$$

A function $f(z) \in \mathcal{T}(p)$ is said to be *p-valently starlike* in $\mathcal{U}$, if it satisfies the inequality:

$$\Re e \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.2}$$

A function $f(z) \in \mathcal{T}(p)$ is said to be *p-valently convex* in $\mathcal{U}$, if it satisfies the inequality:

$$\Re e \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.3}$$

Further, a function $f(z) \in \mathcal{T}(p)$ is said to *p-valently close-to-convex* in $\mathcal{U}$, if it satisfies the inequality:

$$\Re e \left\{ \frac{f'(z)}{z^{p-1}} \right\} > 0 \qquad (z \in \mathcal{U}). \tag{1.4}$$

(See, for details, [3], [5], and [13] for the above definitions.)

The following definitions of fractional calculus will be required in our present investigation:

Definition 1. (cf., [10] and [12]; see also [2]) Let a function $f(z)$ be analytic in a simply-connected region of the z-plane containing the origin. The fractional integral of order μ $(\mu > 0)$ is defined by

$$D_z^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)} \int_0^z f(\xi)(z - \xi)^{\mu-1} d\xi, \tag{1.5}$$

and the fractional derivative of order μ $(0 \leq \mu < 1)$ is defined by

$$D_z^{\mu}\{f(z)\} = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z f(\xi)(z - \xi)^{-\mu} d\xi, \tag{1.6}$$

where the multiplicity of $(z-\xi)^{\mu-1}$ involved in (1.5) and that of $(z-\xi)^{-\mu}$ in (1.6) are removed by requiring $\log(z-\xi)$ to be real when $z-\xi>0$.

Definition 2. (cf., [10] and [12]; see also [2]) Under the hypotheses of Definition 1, the *fractional derivative of order* $m+\mu$ $(m \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}; 0 \le \mu < 1)$ is defined by

$$D_z^{m+\mu}\{f(z)\} = \frac{d^m}{dz^m} D_z^{\mu}\{f(z)\}. \tag{1.7}$$

Now, by making use of the fractional derivative operator $D_z^{m+\mu}$, we define two important families $\mathcal{V}_\delta(p;\mu)$ and $\mathcal{W}_\delta(p;\mu)$ in $\mathcal{T}(p)$, where $\delta \in \mathcal{R} \setminus \{0\}$, $p \in \mathcal{N}$ and $0 \le \mu < 1$.

Definition 3. Let $\delta \in \mathcal{R}\backslash\{0\}, p \in \mathcal{N}$ and $0 \le \mu < 1$. Then a function $f(z) \in \mathcal{T}(p)$ is said to belong to $\mathcal{V}_\delta(p;\mu)$, if it satisfies the inequality:

$$\left| \left(\frac{z D_z^{1+\mu} f(z)}{D_z^{\mu} f(z)} \right)^{\delta} - (p-\mu)^{\delta} \right| < (p-\mu)^{\delta} \quad (z \in \mathcal{U}), \tag{1.8}$$

where the value of $(z D_z^{1+\mu} f(z)/D_z^{\mu} f(z))^{\delta}$ is taken its principal value.

Definition 4. Let $\delta \in \mathcal{R}\backslash\{0\}, p \in \mathcal{N}$ and $0 \le \mu < 1$. Then a function $f(z) \in \mathcal{T}(p)$ is said to belong to $\mathcal{W}_\delta(p;\mu)$, if

$$\left| \left(z^{\mu-p} D_z^{\mu} f(z) \right)^{\delta} - \left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} \right)^{\delta} \right| < \left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} \right)^{\delta} \quad (z \in \mathcal{U}), \tag{1.9}$$

by taking the principal value for $(z^{\mu-p} D_z^{\mu} f(z))^{\delta}$.

Note that functions in $\mathcal{V}_1(p;0)$ are p-valently starlike in $\mathcal{U}$ (e.g. [9]). See, for examples, the papers involving the fractional calculus and/or certain inequalities, [1], [4], [6], [7], and [11]. In [6, 7], Irmak and Çetin studied starlikeness and convexity for multivalent functions involving inequalities. In this paper we investigate various interesting properties for $\mathcal{V}_\delta(p;\mu)$ and $\mathcal{W}_\delta(p;\mu)$ associated with fractional calculus and also extend the results of Irmak and Çetin ([6, 7]).

2. Main Results

Now, we mention the following result which is used in the sequal.

Lemma (cf., Jack [8]; see also Miller and Mocanu [9]). *Let $w(z)$ be an analytic function in the unit disk $\mathcal{U}$ with $w(0) = 0$ and let $0 < r < 1$. If $|w(z)|$ attains at z_0 its maximum value on the circle $|z| = r$, then*

$$z_0 w'(z_0) = c w(z_0) \qquad (c \geq 1). \tag{2.1}$$

Making use of this lemma, we first give the following theorem:

Theorem 1. *Let $\delta \in \mathcal{R} \setminus \{0\}$, $p \in \mathcal{N}$ and $0 \leq \mu < 1$. If a funtion $f(z) \in \mathcal{T}(p)$ satisfies the inequality:*

$$\Re e \left\{ 1 + z \left(\frac{D_z^{2+\mu} f(z)}{D_z^{1+\mu} f(z)} - \frac{D_z^{1+\mu} f(z)}{D_z^{\mu} f(z)} \right) \right\} \left\{ \begin{array}{ll} < 1/(2\delta) & when\ \delta > 0 \\ > 1/(2\delta) & when\ \delta < 0 \end{array} \right\} \quad (z \in \mathcal{U}), \tag{2.2}$$

then $f(z) \in \mathcal{V}_\delta(p; \mu)$.

Proof. First of all, Definition 1 readily provides us the following fractional derivative formula for a power function :

$$D_z^{\mu}\{z^{\kappa}\} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\mu+1)} z^{\kappa-\mu} \qquad (\kappa > -1; 0 \leq \mu < 1). \tag{2.3}$$

Define the function $w(z)$ by

$$\left(\frac{z D_z^{1+\mu} f(z)}{D_z^{\mu} f(z)} \right)^{\delta} = (p-\mu)^{\delta}[1 + w(z)] \qquad (z \in \mathcal{U}). \tag{2.4}$$

Then it follows from (2.3) that $w(z)$ is an analytic function in $\mathcal{U}$ and $w(0) = 0$. The logarithmically differentiation of (2.4) implies that

$$\mathcal{G}(z) := \left\{ 1 + z \left(\frac{D_z^{2+\mu} f(z)}{D_z^{1+\mu} f(z)} - \frac{D_z^{1+\mu} f(z)}{D_z^{\mu} f(z)} \right) \right\} = \frac{1}{\delta} \cdot \frac{z w'(z)}{1 + w(z)}. \tag{2.5}$$

Now, suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \qquad (w(z_0) \neq -1).$$

Then, applying Jack's Lemma, we can write

$$z_0 w'(z_0) = c w(z_0) \qquad (c \geq 1)$$

and $w(z_0) = e^{i\theta}$ $(\theta \neq \pi)$. Thus, from (2.5) we obtain

$$\Re e\{\mathcal{G}(z_0)\} = \frac{1}{\delta} \, \Re e \left(\frac{z_0 w'(z_0)}{1 + w(z_0)} \right)$$

$$= \frac{c}{\delta} \Re e \left(\frac{e^{i\theta}}{1+e^{i\theta}} \right)$$

$$= \frac{c}{2\delta} \left\{ \begin{array}{ll} \geq \frac{1}{2\delta} & when\ \delta > 0 \\ \leq \frac{1}{2\delta} & when\ \delta < 0 \end{array} \right\}, \tag{2.6}$$

where $\theta \neq \pi$ and $c \geq 1$. Therefore, (2.6) contradict our condition (2.2), and we conclude from the definition (2.4) that

$$\left| \left(\frac{zD_z^{1+\mu} f(z)}{D_z^{\mu} f(z)} \right)^{\delta} - (p-\mu)^{\delta} \right| = (p-\mu)^{\delta} |w(z)| < (p-\mu)^{\delta},$$

which completes the proof of Theorem 1.

Theorem 2. *Let* $\delta \in \mathcal{R} \setminus \{0\}$, $p \in \mathcal{N}$ *and* $0 \leq \mu < 1$. *If a funtion* $f(z) \in \mathcal{T}(p)$ *satisfies the inequality:*

$$\Re e \left(\frac{zD_z^{1+\mu} f(z)}{D_z^{\mu} f(z)} \right) \left\{ \begin{array}{ll} < p-\mu+1/(2\delta) & when\ \delta > 0 \\ > p-\mu+1/(2\delta) & when\ \delta < 0 \end{array} \right\} \qquad (z \in \mathcal{U}), \tag{2.7}$$

then $f(z) \in \mathcal{W}_{\delta}(p;\mu)$.

Proof. Put

$$\left(z^{\mu-p} D_z^{\mu} f(z) \right)^{\delta} = \left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} \right)^{\delta} [1+w(z)] \qquad (z \in \mathcal{U}), \tag{2.8}$$

then, using the same technique as in the proof of Theorem 1, we get the desired result.

Many interesting results involving analytic and multivalent functions can be obtained by the use of Theorem 1 and Theorem 2 together with definitions (1.8) and (1.9) (respectivelly) and by choosing suitable values of δ, μ and p. Now, we are giving some of the important results for the analytic and geometric function theory (cf., [13]):

Letting $\delta = 1$ in Theorem 1, we have

Corollary 1. *Let* $p \in \mathcal{N}$ *and* $0 \leq \mu < 1$. *If a funtion* $f(z) \in \mathcal{T}(p)$ *satisfies the inequality:*

$$\Re e \left\{ 1 + z \left(\frac{D_z^{2+\mu} f(z)}{D_z^{1+\mu} f(z)} - \frac{D_z^{1+\mu} f(z)}{D_z^{\mu} f(z)} \right) \right\} < \frac{1}{2} \qquad (z \in \mathcal{U}), \tag{2.9}$$

then $f(z) \in \mathcal{V}_1(p;\mu)$.

Making use of Theorem 2 and [2, Corollary 1], we obtain

Corollary 2. *Let* $p \in \mathcal{N}$ *and* $0 \leq \mu < 1$. *If a funtion* $f(z) \in \mathcal{T}(p)$ *satisfies the inequality:*

$$\Re e\left(\frac{zD_z^{1+\mu}f(z)}{D_z^{\mu}f(z)}\right) < p - \mu + \frac{1}{2} \qquad (z \in \mathcal{U}), \tag{2.10}$$

then $f(z) \in \mathcal{W}_1(p;\mu)$ *and*

$$\Re e\left\{\frac{D_z^{\mu-1}f(z)}{z^{p-\mu+1}}\right\} > \frac{\Gamma(p+1)}{\Gamma(p-\mu+2)}\left(1+\sum_{k=1}^{\infty}\frac{2(p-\mu+1)(-1)^k}{p-\mu+k+1}\right) \qquad (z \in \mathcal{U}). \tag{2.11}$$

The estimate (2.11) *is sharp in general.*

Proof. If we take $\delta = 1$ in Theorem 2, then the condition (2.10) implies $f(z) \in \mathcal{W}_1(p;\mu)$. Further, from (1.9) it is easily shown that

$$\Re e\left\{\frac{D_z^{\mu}f(z)}{z^{p-\mu}}\right\} > 0.$$

Therefore, by virtue of [2, Corollary 1], we obtain the result.

Letting $\mu = 0$ in Corollaries 1 and 2 (or, $\delta - 1 = \mu = 0$ in Theorems 1 and 2), we get already known results as indicated.

Corollary 3. (cf., [6, p. 457, Corollary 2] ; see also [7, p. 74, Eq. (2.15), 2.2. Corollary]) *Let* $p \in \mathcal{N}$. *If a funtion* $f(z) \in \mathcal{T}(p)$ *satisfies the inequality:*

$$\Re e\left\{1 + z\left(\frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)}\right)\right\} < \frac{1}{2} \qquad (z \in \mathcal{U}), \tag{2.12}$$

then $f(z)$ *is p-valently starlike in* $\mathcal{U}$.

Corollary 4. (cf., [6, p. 457, Corollary 1]) *Let* $p \in \mathcal{N}$. *If a funtion* $f(z) \in \mathcal{T}(p)$ *satisfies the inequality:*

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} < p + \frac{1}{2} \qquad (z \in \mathcal{U}), \tag{2.13}$$

then

$$\Re e\left\{\frac{f(z)}{z^p}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{2.14}$$

Letting $\mu \to 1-$ in Corollaries 1 and 2 (or, $\mu \to 1-$ and $\delta = 1$ in Theorems 1 and 2), we have

Corollary 5. (cf., [6, p. 458, Corollary 4] ; see also [7, p. 75, Eq. (2.17), 2.3. Corollary]) *If a funtion* $f(z) \in \mathcal{T}(p)$ *satisfies the inequality:*

$$\Re e\left\{1+z\left(\frac{f'''(z)}{f''(z)}-\frac{f''(z)}{f'(z)}\right)\right\}<\frac{1}{2} \qquad (z \in \mathcal{U}; p \in \mathcal{N}\setminus\{1\}), \tag{2.15}$$

then $f(z)$ *is p-valently convex in* $\mathcal{U}$.

Corollary 6. (cf., [2, Corollary 1] and [6, p. 458, Corollary 3]) *Let* $p \in \mathcal{N}$. *If a funtion* $f(z) \in \mathcal{T}(p)$ *satisfies the inequality:*

$$\Re e\left\{\frac{zf''(z)}{f'(z)}\right\}<p-\frac{1}{2} \qquad (z \in \mathcal{U}), \tag{2.16}$$

then $f(z)$ *is p-valently close-to-convex in* $\mathcal{U}$ *and*

$$\Re e\left\{\frac{f(z)}{z^p}\right\}>1+2p\sum_{k=1}^{\infty}\frac{(-1)^k}{p+k} \qquad (z \in \mathcal{U}). \tag{2.17}$$

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