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A remark on convex and starlike functions

By

Mamoru Nunokawa

Let A denote the set of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $E = \{ z : |z| < 1 \}$.

A function $f(z) \in A$ is said to be starlike of order α $(0 \leq \alpha < 1)$ if and only if

$$\operatorname{Re} \frac{z f'(z)}{f(z)} > \alpha \quad \text{in } E,$$

and it is said to be convex of order α $(0 \leq \alpha < 1)$ if and only if

$$1 + \operatorname{Re} \frac{z f''(z)}{f'(z)} > \alpha \quad \text{in } E.$$

Marx [1] and Strohhäcker [4] have shown respectively that if a function $f(z) \in A$ is convex of order 0, then $f(z)$ is starlike of order at least $1/2$.

On the other hand, a function $f(z) \in A$ is said to be strongly starlike of order α $(0 < \alpha \leq 1)$ if and only if

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}\alpha \quad \text{in } E,$$

and it is said to be strongly convex of order α $(0 < \alpha \leq 1)$ if and only if

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2}\alpha \quad \text{in } E.$$

Mocanu [2] obtained the following result.

If a function $f(z) \in A$ satisfies

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2}\gamma \quad \text{in } E,$$

then

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}\beta \quad \text{in } E$$

where

$$\operatorname{Tan} \frac{\pi\gamma}{2} = \operatorname{Tan} \frac{\pi\beta}{2} + \frac{\beta}{(1-\beta)\cos\frac{\pi\beta}{2}} \left(\frac{1-\beta}{1+\beta} \right)^{\frac{1+\beta}{2}}$$

and $0 < \beta < 1$.

After that, Nunokawa [3] also obtained the following result by applying another method of Mocanu's proof.

If a function $f(z) \in A$ satisfies

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} \right) \right| < \frac{\pi}{2} \alpha(\beta) \quad \text{in } E,$$

where

$$\alpha(\beta) = \beta + \frac{2}{\pi} \mathrm{Tan}^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2}(1-\beta)}{p(\beta) + \beta q(\beta) \cos \frac{\pi}{2}(1-\beta)},$$

$$p(\beta) = (1+\beta)^{\frac{1+\beta}{2}}, \quad q(\beta) = (1-\beta)^{\frac{\beta-1}{2}}$$

$$\text{and} \quad 0 < \beta \le 1,$$

then we have

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2}\beta \quad \text{in } E.$$

To obtain the last result, Nunokawa [3] proved the following lemma.

Lemma. Let $p(z)$ be analytic in E, $p(0) = 1$, $p(z) \neq 0$ in E and suppose that there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

where $0 < \alpha$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha$$

where

$$p(z_0)^{1/\alpha} = \pm ia \quad \text{and} \quad 0 < a.$$

Applying the above lemma, we obtain the following result.

Theorem. Let $f(z) \in A$ satisfy the condition

$$\left| \arg\left(1 + \frac{z f''(z)}{f'(z)}\right) \right| < \frac{\pi}{2}\delta(\beta) \quad \text{in } E$$

where $0 < \beta \leq 1$,

$$\delta(\beta) = \frac{2}{\pi}\operatorname{Tan}^{-1}\frac{\psi(a(\beta))}{\phi(a(\beta))},$$

$$\phi(a) = \frac{1}{2} + \frac{1}{2}a^{\beta}\cos\frac{\pi}{2}\beta - \frac{\beta(a + a^{-1})a^{\beta}\sin\frac{\pi}{2}\beta}{2(1 + 2a^{\beta}\cos\frac{\pi}{2}\beta + a^{2\beta})},$$

$$\psi(a) = \frac{1}{2}a^{\beta}\sin\frac{\pi}{2}\beta + \frac{\beta(a^{1+\beta} + a^{\beta-1})(a^{\beta} + \cos\frac{\pi}{2}\beta)}{2(1 + 2a^{\beta}\cos\frac{\pi}{2}\beta + a^{2\beta})}$$

$$\operatorname*{Min}_{0<a<\infty} \operatorname{Tan}^{-1} \frac{\psi(a)}{\phi(a)} = \operatorname{Tan}^{-1} \frac{\psi(a(\beta))}{\phi(a(\beta))}$$

where we must take

$$0 \leq \operatorname{Tan}^{-1} \Theta \leq \frac{\pi}{2} \qquad \text{when} \quad 0 \leq \Theta$$

and

$$\frac{\pi}{2} < \operatorname{Tan}^{-1} \Theta \leq \pi \qquad \text{when} \quad \Theta < 0 .$$

Then we have

$$\left| \arg \left(\frac{zf'(z)}{f(z)} - \frac{1}{2} \right) \right| < \frac{\pi}{2}\beta \quad \text{in } E$$

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)} , \qquad (p(0) = 1)$$

and

$$q(z) = 2\left(p(z) - \frac{1}{2} \right) .$$

If there exists a point $z_0 \in E$ such that

$$| \arg q(z) | < \frac{\pi}{2}\beta \quad \text{for } |z| < |z_0|$$

and

$$| \arg q(z_0) | = \frac{\pi}{2}\beta$$

where $0 < \beta \leq 1$, then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = i\beta k$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \quad \text{when } \arg q(z_0) = \frac{\pi}{2}\beta$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \quad \text{when } \arg q(z_0) = -\frac{\pi}{2}\beta$$

$$q(z_0)^{1/\beta} = \pm i a \quad \text{and} \quad 0 < a .$$

For the case $\arg q(z_0) = \frac{\pi}{2}\beta$, it follows that

$$1 + \frac{z_0 f''(z_0)}{f'(z_0)}$$

$$= p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} = \frac{1}{2}\left(q(z_0) + 1\right) + \frac{z_0 q'(z_0)}{q(z_0) + 1}$$

$$= \frac{1}{2}\left(1 + a^{\beta} e^{i\frac{\pi}{2}\beta}\right) + i\beta k \frac{q(z_0)}{q(z_0) + 1}$$

$$= \frac{1}{2} + \frac{1}{2} a^{\beta} \cos\frac{\pi}{2}\beta + i\frac{1}{2} a^{\beta} \sin\frac{\pi}{2}\beta$$

$$+ i\beta k \frac{a^{2\beta} + a^{\beta}\cos\frac{\pi}{2}\beta + i a^{\beta}\sin\frac{\pi}{2}\beta}{1 + 2a^{\beta}\cos\frac{\pi}{2}\beta + a^{2\beta}}$$

Then from Lemma, we have

$$\arg\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right)$$

$$\geq \operatorname{Tan}^{-1}\frac{\psi(a)}{\phi(a)} \geq \operatorname{Tan}^{-1}\frac{\psi(a(\beta))}{\phi(a(\beta))}$$

where

$$\phi(a) = \frac{1}{2} + \frac{1}{2}a^{\beta}\cos\frac{\pi}{2}\beta - \frac{\beta(a+a^{-1})a^{\beta}\sin\frac{\pi}{2}\beta}{2(1+2a^{\beta}\cos\frac{\pi}{2}\beta + a^{2\beta})}$$

and

$$\psi(a) = \frac{1}{2}a^{\beta}\sin\frac{\pi}{2}\beta + \frac{\beta(a^{1+\beta}+a^{\beta-1})(a^{\beta}+\cos\frac{\pi}{2}\beta)}{2(1+2a^{\beta}\cos\frac{\pi}{2}\beta + a^{2\beta})}$$

This contradicts the hypothesis of the Theorem and for the case $\arg g(z) = -\frac{\pi}{2}\beta$, applying the same method as the above, we can obtain

$$\arg\left(1+\frac{z_1 f''(z_1)}{f'(z_1)}\right)$$

$$\leq -\operatorname{Tan}^{-1}\frac{\psi(a)}{\phi(a)} \leq -\operatorname{Tan}^{-1}\frac{\psi(a(\beta))}{\phi(a(\beta))}.$$

This also contradicts the assumption of the Theorem and so, it completes the proof.

Putting $\beta = 1$ in the Theorem, we can have the following theorem.

Corollary (Marx - Strohhäcker's theorem). If $f(z) \in A$ is convex of order 0, then $f(z)$ is starlike of order at least $\frac{1}{2}$.

References

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Department of Mathematics
University of Gunma
Aramaki, Maebashi 371-8510
Japan

e-mail address: nunokawa@edu.gunma-u.ac.jp