数理解析研究所講究録 1276 巻 2002 年 48-53

On a sufficient condition for starlikeness

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In [1], Miller, Mocanu and Reade proved the following theorem by applying a geometrical properties.

Theorem A. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disc $E = \{ z : |z| < 1 \}$, with $f(z) f'(z) / z \neq 0$ there, and let α be a real number. Then $f(z)$ is said to be α-convex in E if and only if

$$\mathrm{Re}\left[(1-\alpha) \frac{z f'(z)}{f(z)} + \alpha \left(1 + \frac{z f''(z)}{f'(z)} \right) \right] > 0 \quad \text{in } E.$$

Then, if $f(z)$ is α-convex in E, then $f(z)$ is starlike in E and moreover, if $1 \leq \alpha$, then $f(z)$ is convex in E.

In this paper, we improve Theorem A by applying the following lemma [2].

Lemma. Let $p(z)$ be analytic in E, $p(0) = 1$ and suppose that there exists a point $z_0 \in E$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for} \quad |z| < |z_0|$$

$$\operatorname{Re} p(z_0) = 0 \quad \text{and} \quad p(z_0) \neq 0 .$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and $1 \leq |k|$.

Theorem 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E and suppose that

$$(1) \qquad (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \neq il \quad \text{in } E$$

where α is a positive real number and l is also real number and $\sqrt{\alpha(2+\alpha)} \leq |l|$.

Then $f(z)$ is starlike in E.

Proof. If we put

$$p(z) = \frac{zf'(z)}{f(z)}, \qquad (p(0) = 1)$$

then we obtain

$$(2) \qquad (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

$$= p(z) + \alpha\frac{zp'(z)}{p(z)}$$

From (2) and assumption (1), we obtain $p(z) \neq 0$ in E, because if $p(z)$ has zero of order m at $z = \beta$, $|\beta| < 1$ then we can write

$$p(z) = (z-\beta)^m p_1(z)$$

where $p_1(z)$ is analytic in E, $p_1(\beta) \neq 0$ and m is a positive integer.

Then it follows that

$$(3) \qquad p(z) + \alpha \frac{z p'(z)}{p(z)}$$

$$= (z-\beta)^m p_1(z) + \frac{\alpha m z}{z-\beta} + \frac{z p_1'(z)}{p_1(z)} .$$

In a sufficiently small neighborhood of the point $z = \beta$ the right hand side of (3) can take a pure imaginary complex value whose modulus is sufficiently large. This contradicts the hypothesis of the theorem and therefore $p(z) \neq 0$ in E.

Suppose that there exists a point $z_0 \in E$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for} \quad |z| < |z_0|$$

$$\operatorname{Re} p(z_0) = 0, \quad p(z_0) = ia \quad \text{and} \quad p(z_0) \neq 0,$$

then, from [2], we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is real and

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}$$

and

$$k \leq -\frac{1}{2}\left(|a| + \frac{1}{|a|}\right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}$$

For the case $\arg p(z_0) = \frac{\pi}{2}$, $(0 < a)$, we have

$$p(z_0) + \alpha \frac{z_0 p'(z_0)}{p(z_0)} = i\,(a + \alpha k)$$

and

$$a + \alpha k \geq \frac{1}{2}\left\{(2+\alpha)a + \frac{\alpha}{a}\right\}$$
$$\geq \sqrt{\alpha(2+\alpha)}.$$

This contradicts assumption of the theorem and for the case $\arg p(z_0) = -\frac{\pi}{2}$, we also have a contradicti by applying the same method as the above.

This completes the proof.

By applying Theorem 1, we have the following very interesting result.

Theorem 2. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E and suppose that

$$1 + \frac{zf''(z)}{f'(z)} \neq il \qquad \text{in } E$$

where l is a real number and $\sqrt{3} \leq |l|$.

Then $f(z)$ is starlike in E.

Remark. By applying the same method as the proof of Theorem 1, Theorem 1 holds to be true for the case $\alpha \leq -2$.

References

1. S. S. Miller, P. T. Mocanu and M. O. Reade, All α-convex functions are univalent and starlike, Proc. Amer. Math. Soc., 37(2) (1973), 553-554.

2. M. Nunokawa, On properties of non-Carathéodory functions, Proc. Japan Acad., 68(6) (1993), 152-153.

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