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MAJORIZATION OF SUBORDINATE HARMONIC FUNCTIONS

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ABSTRACT. It is well known that if $f(z)$ and $F(z)$ with $f(0)=F(0)$ are analytic in $|z|<1$ and if $f(z)$ is subordinate to $F(z)$, then for $0<p$ and $0<r<1$,

$$\int_0^{2\pi}|f(re^{i\theta})|^p d\theta \le \int_0^{2\pi}|F(re^{i\theta})|^p d\theta.$$

In this paper, we research the relationship of large and small of the

$$\int_0^{2\pi}|\mathrm{Re}f(re^{i\theta})|^p d\theta \quad \text{and} \quad \int_0^{2\pi}|\mathrm{Re}F(re^{i\theta})|^p d\theta.$$

Suppose that a function $f(z)$ is analytic in the unit disc $E=\{z:|z|<1\}$ and that a function $F(z)$ is analytic and univalent in E. Suppose that $f(0)=F(0)$. If the image of the disc E under the mapping $w=f(z)$ is contained in the image of that disc under the mapping $w=F(z)$, we say that the function $f(z)$ is subordinate to $F(z)$ in the disc E and that the function $F(z)$ is a univalent majorant of $f(z)$. We denote this by writting $f(z)\prec F(z)$. This is equivalent to regularity of the function $F^{-1}(f(z))=\varphi(z)$ in E, where $\varphi(0)=0$ and $|\varphi(z)|\le 1$ in E.

It follows that the set of all functions $f(z)$ that are subordinate in the disc $|z|<r$ to a given univalent majorant $F(z)$ is defined by the formula

$$f(z)=F(\varphi(z)),$$

where $\varphi(z)$ is an arbitrary function satisfying the conditions of the Schwarz lemma, that is, it is analytic in E, $\varphi(0)=0$, $|\varphi(z)|<1$ in E. Then, Rogosinski [3] proved the following theorem.

Theorem A. *If $f(z)$ and $F(z)$ with $f(0)=F(0)$ are analytic in E and if $f(z)\prec F(z)$, then for $0<p$ and $0<r<1$,*

$$\int_0^{2\pi}|f(re^{i\theta})|^p d\theta \le \int_0^{2\pi}|F(re^{i\theta})|^p d\theta.$$

On the other hand, Avhadiev and Aksent'ev [1] obtained the following result.

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Theorem B. *If $f(z)$ and $F(z)$ satisfy the conditions of Theorem A, then it follows that*

$$\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})| d\theta \leq \int_0^{2\pi} |\mathrm{Re} F(re^{i\theta})| d\theta$$

for $0 < r < 1$.

After the Theorem B was obtained, Nunokawa, Fukui and Saitoh [2] proved the following theorem.

Theorem C. *If $f(z)$ and $F(z)$ satisfy the conditions of Theorem A, then we have*

$$\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})|^2 d\theta \leq \int_0^{2\pi} |\mathrm{Re} F(re^{i\theta})|^2 d\theta$$

for $0 < r < 1$.

It is the purpose of the present paper to generalize Theorem B.
In this paper, we need Hölder's theorem.

Lemma 1. (Hölder) *Let $f(x)$ and $g(x)$ are continuous on $a \leq x \leq b$, $f(x) \geq 0$ and $g(x) \geq 0$ on $a \leq x \leq b$. Then we have*

$$\int_a^b f(x)g(x)\,dx \leq \left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \left(\int_a^b g(x)^q dx\right)^{\frac{1}{q}} \tag{1}$$

where $1/p + 1/q = 1$, $1 < p$ *and* $0 < q$ *and we have*

$$\int_a^b f(x)g(x)\,dx \geq \left(\int_a^b f(x)^p dx\right)^{\frac{1}{p}} \left(\int_a^b g(x)^q dx\right)^{\frac{1}{q}} \tag{2}$$

where $1/p + 1/q = 1$, $0 < p < 1$ *and* $q < 0$.

Proof. (1) is very popular and applying the same method as the proof of (1), we can obtain (2). □

Theorem 1. *If $f(z)$ and $F(z)$ with $f(0) = F(0)$ are analytic in E and if $f(z) \prec F(z)$, then we have for $0 < r < 1$ and for the case $1 \leq p$,*

$$\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} |\mathrm{Re} F(re^{i\theta})|^p d\theta. \tag{3}$$

and (3) *does not hold for the case* $0 < p < 1$ *or* (3) *is not always true for the case* $0 < p < 1$.

Proof. For the case $1 \leq p$, from the hypothesis of Theorem 1, we have

$$f(z) = F(\varphi(z))$$

where $\varphi(z)$ is analytic in E, $\varphi(0) = 0$ and $|\varphi(z)| < 1$ in E. Applying the same reason as the proof of Theorem B [1, p.934] and by using Lemma 1, we have for $0 < r < \rho < 1$,

$$
\begin{aligned}
&\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})|^p \, d\theta \\
\leq &\int_0^{2\pi} \left(\frac{1}{2\pi} \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})| \,\mathrm{Re}\, \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} \, d\nu \right)^p d\theta \\
= &\int_0^{2\pi} \left(\frac{1}{2\pi} \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})| \left(\mathrm{Re}\, \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} \right)^{\frac{1}{p} + \frac{p-1}{p}} d\nu \right)^p d\theta \\
\leq &\int_0^{2\pi} \left\{ \left(\frac{1}{2\pi} \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p \,\mathrm{Re}\, \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} \, d\nu \right)^{\frac{1}{p}} \left(\frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\, \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} \, d\nu \right) \right\}^p d\theta \\
= &\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p \,\mathrm{Re}\, \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} \, d\nu \, d\theta \\
= &\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p \,\mathrm{Re}\, \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} \, d\theta \, d\nu \\
= &\frac{1}{2\pi} \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})|^p \, d\nu.
\end{aligned}
$$

Putting $r \to \rho$, we obtain (3).

For the case $0 < p < 1$, let us take a function $F(z)$ whose real part is positive in E, then we have (e.g. [4, p.227])

$$
\begin{aligned}
&\mathrm{Re} f(re^{i\theta}) = |\mathrm{Re} f(re^{i\theta})| = \mathrm{Re} F(\varphi(re^{i\theta})) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re} F(\rho e^{i\nu}) \,\mathrm{Re}\, \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} \, d\nu \\
&= \frac{1}{2\pi} \int_0^{2\pi} |\mathrm{Re} F(\rho e^{i\nu})| \,\mathrm{Re}\, \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})} \, d\nu. \qquad (4)
\end{aligned}
$$

Applying the same method as the proof of (3), Lemma 1 and equation (4), we have

$$\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})|^p d\theta \geq \int_0^{2\pi} |\mathrm{Re} F(re^{i\theta})|^p d\theta$$

where $\mathrm{Re} F(z) > 0$ in E and so $\mathrm{Re} f(z) > 0$ in E. This completes the proof of Theorem 1. □

Remark.

$$0 < \mathrm{Re}\, \frac{\rho e^{i\nu} + \varphi(re^{i\theta})}{\rho e^{i\nu} - \varphi(re^{i\theta})}$$

for $0 < r < \rho < 1$.

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