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Carleson inequalities in weighted harmonic Bergman spaces, $0 < p \leq 1$

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ABSTRACT. We give a necessary and sufficient condition for positive measures μ and ν on the upper half-space of $\mathbb{R}^n$ to satisfy the inequality

$$\int |D^\alpha u|^p d\mu \leq C \int |D_y^m u|^p d\nu$$

for all u in a subclass of a harmonic Bergman space when $0 < p \leq 1$, $d\nu = \omega dV$, and ω satisfies a certain condition.

1. Introduction

Let H be the upper half-space of the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, that is, $H = \{z = (x, y) \in \mathbb{R}^n\ ;\ y > 0\}$, where we have written a point $z \in \mathbb{R}^n$ as $z = (x, y)$ with $x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. For $0 < p < \infty$, let $b^p = b^p(H, dV)$ be the class of all harmonic functions u on H such that

$$\| u \|_p = \left(\int_H |u|^p dV \right)^{1/p} < \infty$$

where dV denotes the Lebesgue volume measure on H. The class b^p is called the harmonic Bergman space. Properties of functions in the harmonic Bergman space on the upper half-space were studied by Ramey and Yi [13] when $1 \leq p < \infty$, and by the author [15] when $0 < p \leq 1$.

Let μ and ν be σ-finite positive Borel measures on H. We consider conditions on μ and ν for which there exists a constant $C > 0$ such that $\int |u| d\mu \leq C \int |D_y u| d\nu$ for all u in a subclass of b^1, where D_y denotes the differetiation operator with respect to y.(Our consideration is more general.) Such inequalities on the unit disk in the complex plane were studied by Stegenga, and multipliers of the Dirichlet space were characterized [14]. When $d\nu = (1 - |\zeta|)^r dA$ and $r \geq 1$, Stegenga proved that finite positive Borel measures μ and ν on the unit disk satisfy the inequality $\int |f|^2 d\mu \leq C \int |f'|^2 d\nu$ for all holomorphic functions f, $f(0) = 0$ if and only if there is a constant K such that $\mu(S_I) \leq K|I|^r$ for any interval I in the unit circle, where dA denotes the Lebesgue area measure, $|I|$ denotes the normalized arc length of I, and S_I is the corresponding Carleson square over I. It was also proved that when $0 \leq r < 1$ such measures are those satisfying $\mu(\cup S_{I_j}) \leq K\mathrm{Cap}(\cup I_j)$ for all finite disjoint collections of intervals $\{I_j\}$, where Cap is an appropriate Bessel capacity (if $r < 0$ any finite Borel measure satisfies this

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inequality). It is known that these characterizations can be generalized to the case of $p > 1$ (see also [14]). When $0 < p \leq 1$, $d\nu = (1 - |\zeta|)^r dA$, and $-1 < r \leq p - 1$, Ahern and Jevtić [1] proved that there is a constant $C > 0$ such that $\int |f|^p d\mu \leq C \int |f'|^p d\nu$ if and only if $\mu(S_I) \leq K|I|^{2-p+r}$. Using this result, Ahern and Jevtić characterized inner multipliers of the Besov spase in case $0 < p \leq 1$. Such investigations for several variables are in [4]. In these investigations, when $p > 1$ necessary and sufficient conditions were not obtained completely. It was also shown that, in general, the above condition is not necessary. When $0 < p \leq 1$ and $d\nu = y^r dV$, such a inequality on the upper half-space was studied by author [15]. On the unit disk of the complex plane, for more general measures μ and ν, the properties of mesures satisfying a inequality $\int |f|^p d\mu \leq C \int |f|^p d\nu$ were studied in [8], [9], and [12], and partial results were obtained.

If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index of nonnegative integers with order ℓ, then D^α denotes the partial differentiation operator $\partial^\ell / \partial x_1^{\alpha_1} \cdots \partial x_{n-1}^{\alpha_{n-1}} \partial y^{\alpha_n}$. We now state our main result in this paper.

THEOREM 1. *Let* $0 < p \leq 1$ *and* ℓ, m *be nonnegative integers. Suppose that* μ *is a* σ-*finite positive Borel measure on* H, $d\nu = \omega dV$ *and* ω *satisfies the* $(A_q)_\partial$-*condition for some* $1 < q < \infty$. *Then, the following* (1) $\sim$ (3) *are equivalent.*

(1) *There is a constant* $C > 0$ *such that*

$$\int_H |D^\alpha u|^p d\mu \leq C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$ *and multi-indices* α *of order* ℓ,

(2) *There is a constant* $C > 0$ *such that*

$$\int_H |D_y^\ell u|^p d\mu \leq C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$.

(3) *There are constants* $K > 0$ *and* $0 < \varepsilon < 1$ *such that* $\mu(S(w)) \leq K t^{(\ell - m)p} \nu(D_\varepsilon(w))$ *for all* $w = (s, t) \in H$.

In §2, we give some lemmas for investigations of Theorem 1. In §3, the necessity of the condition is shown. In §4, we define the notion of the $(A_p)_\partial$-conditon on the upper half-space, and study some properties of the $(A_p)_\partial$-conditon. The $(A_p)_\partial$-conditon on the unit disk of the complex plane is defined in [12]. In the definition of the $(A_p)_\partial$-conditon on the unit disk, the normarized reproducing kernel in the Bergman space is used. However, on the upper half-space of $\mathbb{R}^n$, we can not use arguments in the complex plane. Therefore, we will extend the notion of the $(A_p)_\partial$-conditon using anothor function. In §5, the sufficiency of the condition is contained.

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

2. Preliminary lemmas

Recall that a point $z \in H$ will be written as $z = (x, y)$ with $x \in \mathbb{R}^{n-1}$ and $y > 0$. We use the absolute value symbol $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$. For $z = (x, y)$, let

$\bar{z} = (x, -y)$. The pseudohyperbolic metric ρ in H is defined by $\rho(z, w) = |w - z|/|\bar{w} - z|$. It is clear that ρ is invariant under horizontal translations. Let $D_\varepsilon(w) = \{z \in H \ ; \ \rho(z, w) < \varepsilon\}$ when $w = (s, t) \in H$ and $0 < \varepsilon < 1$. $D_\varepsilon(w)$ is a Euclidean ball whose center and radius are $(s, \dfrac{1+\varepsilon^2}{1-\varepsilon^2}t)$ and $\dfrac{2\varepsilon t}{1-\varepsilon^2}$ respectively. It follows that there is a constant $C = C_\varepsilon > 0$ such that $C^{-1}t^n \leq V(D_\varepsilon(w)) \leq Ct^n$ for all $w \in H$. The following lemma is stated in [15].

LEMMA 1. *Let $0 < \varepsilon < 1$. Then, the following are true.*

(1) *If z, w, ζ are in H and $\rho(z, w) < \varepsilon$, then $C^{-1}|\bar{\zeta} - z| \leq |\bar{\zeta} - w| \leq C|\bar{\zeta} - z|$ with a positive constant C depending only on ε.*

(2) *If $z = (x, y), w = (s, t)$ are in H and $\rho(z, w) < \varepsilon$, then $C^{-1}y \leq t \leq Cy$ with a positive constant C depending only on ε.*

(3) *If $0 < \varepsilon < 1/2$ then there exist a positive integer N and a sequence $\{\zeta_j\}$ in H satisfying the following conditions :* (a) $H = \cup D_\varepsilon(\zeta_j)$, (b) *any point in H belongs to at most N of the sets $D_{2\varepsilon}(\zeta_j)$.*

For a function u on H and $\delta > 0$, let $\tau_\delta u$ denote the function on H defined by $\tau_\delta u(x, y) = u(x, y+\delta)$, and let $\mathcal{T}^p = \{\tau_\delta u \ ; \ u \in b^p, \delta > 0\}$. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index of nonnegative integers with order ℓ, then D^α denotes the partial differentiation operator $\partial^\ell/\partial x_1^{\alpha_1} \cdots \partial x_{n-1}^{\alpha_{n-1}} \partial y^{\alpha_n}$ The following lemma is stated in [15].

LEMMA 2. *Let $0 < p \leq 1$. Then, the following are true.*

(1) *For any $u \in b^p$, there is a constant $C > 0$ such that $|D^\alpha u(s, t)| \leq C/t^{n/p+|\alpha|}$ for all $(s, t) \in H$.*

(2) *For any $u \in b^p$, there is a constant $C > 0$ such that $|(D^\alpha \tau_\delta u)(s, t)| \leq C/(t + \delta)^{n/p+|\alpha|}$ for all $(s, t) \in H$.*

Let $w = (s, t) \in H$. The Poisson kernel P_w is the function on $\mathbb{R}^{n-1}$ given by $P_w(x) = P(s - x, t) = \gamma_n t/(|s - x|^2 + t^2)^{n/2}$ (γ_n is the positive constant $\gamma_n = 2/(nV(\mathbb{B}_n))$, where $\mathbb{B}_n$ denotes the unit ball in $\mathbb{R}^n$). The harmonic extension of this function to H is $P(s - x, t + y)$. If $z = (x, y) \in H$, then we may write $P_w(z)$. We note that $P_w(z) = \gamma_n(t + y)/|\bar{w} - z|^n$, $|D_z^\alpha P_w(z)| \leq C/|\bar{w} - z|^{n+|\alpha|-1}$, and $D_z^\alpha P_w(z) = (-1)^{\alpha_1+\cdots+\alpha_{n-1}} D_w^\alpha P_w(z)$. The following lemma is useful and stated in [13, Lemma 3.1]

LEMMA 3. *Let $0 < c < 1$. Then, there is a constant $C > 0$ depending on c and n such that*

$$\int_H \frac{y^{-c}}{|w - \bar{z}|^n} dV(z) = Ct^{-c}$$

for all $w = (s, t) \in H$.

Let m be a nonnegative integer and let $c_m = (-2)^m/m!$. The following Lemma 4 is given in [15].

LEMMA 4. *Let $0 < p \leq 1$. If $u \in \mathcal{T}^p$, then*

$$u(w) = -2c_{m+k} \int_H y^{m+k} (D_y^m u)(z) D_y^{k+1} P_w(z) dV(z)$$

for all $m, k \geq 0$ *and* $w \in H$.

We show that Lemma 4 is also valid for $u \in b^p$ when the integer k is sufficiently large.

LEMMA 5. *Let* $0 < p \leq 1$ *and* k *be a nonnegative integer such that* $k > n/p$. *If* $u \in b^p$, *then*

$$u(w) = -2c_{m+k} \int_H y^{m+k} (D_y^m u)(z) D_y^{k+1} P_w(z) dV(z)$$

for all $m \geq 0$ *and* $w \in H$.

3. (μ, ν)–Carleson inequality

We give a sufficient condition for measures μ and ν which satisfy the (μ, ν)–Carleson inequality with derivatives.

PROPOSITION 2. *Let* $0 < p \leq 1$, $1 < q < \infty$, *and* $k > n/p$. *Suppose that* ℓ, m *be nonnegative integers. Assume that* μ *is a* σ*-finite positive Borel measure on* H *and* $d\nu = \omega dV$ *such that* $\omega \in L^1_{loc}(H, dV)$. *If there are constants* $K > 0$ *and* $0 < \varepsilon < 1$ *such that*

$$\int_H \left(\int_{D_\varepsilon(w)} \omega^{\frac{-1}{q-1}} dV \right)^{q-1} \frac{t^{p(n+m+k)-nq}}{|w - \bar{z}|^{p(n+\ell+k)}} d\mu(z) \leq K,$$

for all $w = (s, t) \in H$, *then there is a constant* $C > 0$ *such that*

$$\int_H |D^\alpha u|^p d\mu \leq C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$ *and multi-indices* α *of order* ℓ.

We will also give a necessary condition for the (μ, ν)–Carleson inequality. We need the following lemma, and Lemma 6 is stated in [15].

LEMMA 6. *Let* k *be a nonnegative integer. Then, there exist constants* $0 < \sigma \leq 1$ *and* $C > 0$ *such that* $|D_y^k P_w(z)| \geq C/t^{n+k-1}$ *for all* $w = (s, t) \in H$ *and* $z \in S(s, \sigma t)$.

PROPOSITION 3. *Let* $0 < p \leq 1$. *Suppose that* ℓ, m *be nonnegative integers. Assume that* μ *and* ν *are* σ*-finite positive Borel measures on* H. *If there is a constant* $C > 0$ *such that*

$$\int_H |D_y^\ell u|^p d\mu \leq C \int_H |D_y^m u|^p d\nu$$

for all $u \in b^p$, *then there are constants* $0 < \sigma \leq 1$ *and* $K = K_\sigma > 0$ *such that*

$$\mu(S(s, \sigma t)) \leq K t^{p(\ell+n+k-1)} \int_H \frac{1}{|\overline{w} - z|^{p(m+k+n-1)}} d\nu$$

for all $w = (s, t) \in H$.

4. $(A_p)_\partial$-condition

Let $1 < p < \infty$, and ω be a non-negative L^1_{loc} function on H. We say that the function ω satisfies the $(A_p)_{\partial}$-conditon if there exists a constant $\gamma > 0$ such that for every $w = (s,t) \in H$,

$$\int_H \frac{t^n}{|\overline{w} - z|^{2n}} \omega dV(z) \left(\int_H \frac{t^n}{|\overline{w} - z|^{2n}} \omega^{\frac{-1}{p-1}} dV(z) \right)^{p-1} \leq \gamma.$$

The $(A_p)_{\partial}$-conditon on the unit disk Δ of the complex plane is defined in [12]. In the definition of the $(A_p)_{\partial}$-conditon on the unit disk, the normarized reproducing kernel in the Bergman space is used. The B_p-condition is defined in [3] for characterizing the boundedness of a projection from $L^p(\omega)$ onto $L^p_a(\omega)$. And the C_p-condition is defined in [10]. For $z, w \in \Delta$, let $k_w(z) = \dfrac{1 - |w|^2}{(1 - \overline{w}z)^2}$ and $\phi_w(z) = \dfrac{w - z}{1 - \overline{w}z}$. The functions $k_w(z)$ and $\phi_w(z)$ are called the normalized reproducing kernel of the Bergman space on Δ and the Möbius mapping of Δ, respectively. Let $S_w = \{z \in \Delta; 1 - |w| < |z| < 1, |\arg z - \arg w| < 1 - |w|\}$ and $\Delta_w = \Delta_{w,\varepsilon} = \{z \in \Delta; |\phi_z(w)| < \varepsilon\}$. The $(A_p)_{\partial}$, B_p, and C_p-conditions on the unit disk Δ are the following.

The $(A_p)_{\partial}$-conditon: there exists a constant $\gamma > 0$ such that for every $w \in \Delta$,

$$\int_\Delta |k_w(z)|^2 \omega dA(z) \left(\int_\Delta |k_w(z)|^2 \omega^{\frac{-1}{p-1}} dA(z) \right)^{p-1} \leq \gamma.$$

The B_p-conditon: there exists a constant $\gamma > 0$ such that for every $w \in \Delta$,

$$\frac{1}{A(S_w)} \int_{S_w} \omega dA(z) \left(\frac{1}{A(S_w)} \int_{S_w} \omega^{\frac{-1}{p-1}} dA(z) \right)^{p-1} \leq \gamma.$$

The C_p-conditon: there exists a constant $\gamma > 0$ such that for every $w \in \Delta$,

$$\frac{1}{A(\Delta_w)} \int_{\Delta_w} \omega dA(z) \left(\frac{1}{A(\Delta_w)} \int_{\Delta_w} \omega^{\frac{-1}{p-1}} dA(z) \right)^{p-1} \leq \gamma.$$

In general, it is easy to see that

$$\frac{1}{A(\Delta_w)} \int_{\Delta_w} \omega dA(z) \leq C \frac{1}{A(S_w)} \int_{S_w} \omega dA(z) \leq C' \int_\Delta |k_w(z)|^2 \omega dA(z).$$

On the upper half-space H, it is also easy to see that there is a constant $C > 0$ suth that $\dfrac{1}{V(D_\varepsilon(w))} \displaystyle\int_{D_\varepsilon(w)} \omega dV(z) \leq C \dfrac{1}{V(S(w))} \displaystyle\int_{S(w)} \omega dA(z)$. However, we do not know that the second inequality is satisfied or not. For $z = (x,y), w = (s,t) \in H$, let

$$R_w(z) = \frac{4}{nV(B)} \frac{n(y+t)^2 - |\overline{w} - z|^2}{|\overline{w} - z|^{n+2}}$$

and

$$r_w(z) = \frac{(2t)^{\frac{n}{2}}}{\sqrt{n-1}} \frac{n(y+t)^2 - |\overline{w} - z|^2}{|\overline{w} - z|^{n+2}}.$$

The functions $R_w(z)$ and $r_w(z)$ are called the reproducing kernel and the normalized reproducing kernel of the harmonic Bergman space, respectively.

THEOREM 2. *Let ω be a non-negative L^1_{loc} function on H. Then, the following* (1) *and* (2)

(1) *There are constans* $0 < \sigma \leq 1$ *and* $C > 0$ *such that for every* $w = (s,t) \in H$,

$$\frac{1}{V(S(s,\sigma t))}\int_{S(s,\sigma t)} \omega dV(z) \leq C\int_H |r_w(z)|^2 \omega dV(z).$$

(2) *There is a constan* $C > 0$ *such that for every* $w \in H$,

$$\frac{1}{V(S(w))}\int_{S(w)} \omega dV(z) \leq C\int_H \frac{t^n}{|\overline{w}-z|^{2n}} \omega dV(z).$$

By Theorem 2, we obtain the following result.

THEOREM 3. *Let* $1 < p < \infty$ *and* ω *be a non-negative* L^1_{loc} *function on* H. *Then, the following* (1) *and* (2) *are hold.*

(1) *If* ω *satisfies the* $(A_p)_{\partial}$*-conditon on* H, *then there is a constant* $C > 0$ *such that for every* $w \in H$,

$$C^{-1}\int_H |r_w(z)|^2 \omega dV(z) \leq \int_H \frac{t^n}{|\overline{w}-z|^{2n}} \omega dV(z) \leq C\int_H |r_w(z)|^2 \omega dV(z).$$

(2) *If* ω *satisfies the* $(A_p)_{\partial}$*-conditon on* H, *then* ω *satisfies the* B_p*-conditon on* H, *and hence* ω *satisfies the* C_p*-conditon on* H.

5. Proof of Theorem 1

In this senction, we give a proof of Theorem 1.

PROOF OF THEOREM 1. (1) $\Rightarrow$ (2) *is trivial. We show that* (2) $\Rightarrow$ (3). *We suppose that (2) is hold. Then, Proposition 3 implies that there are constants* $0 < \sigma \leq 1$ *and* $K = K_\sigma > 0$ *such that* $\mu(S(s,\sigma t)) \leq Kt^{p(\ell+n+k-1)}\int_H 1/|\overline{w}-z|^{p(m+k+n-1)}d\nu$ *for all* $w = (s,t) \in H$. *Since* $|\overline{w}-z| \geq t$, *We have* $\mu(S(s,\sigma t)) \leq Kt^{p(\ell-m)+n}\int_H t^n/|\overline{w}-z|^{2n}d\nu$. *Moreover, since* ω *satisfies the* $(A_q)_{\partial}$*-conditon, we obtain* $\mu(S(s,\sigma t)) \leq Kt^{p(\ell-m)}\nu(D_\varepsilon(s,\sigma t))$. *Since* s *and* t *are arbitrary, we can replace* t *by* t/σ. *This implies that* $\mu(S(w)) \leq Ct^{p(\ell-m)}\nu(D_\varepsilon(w))$. *We will show* (3) $\Rightarrow$ (1). *Let* $c = p(\ell - m)$ *and suppose that* $\mu(S(\zeta)) \leq K\eta^c\nu(D_\varepsilon(\zeta))$ *for all* $\zeta = (\xi,\eta) \in H$. *Since* ω *satisfies the* $(A_q)_{\partial}$*-conditon, the sufficient condition in Proposition 2 is equivalent to the condition* $\int_H t^{p(n+m+k)}/|\overline{w}-z|^{p(n+\ell+k)}d\mu(z) \leq K\nu(D_\varepsilon(w))$. *Therefore, it is enough to prove that* $\int_H 1/|\overline{w}-z|^\gamma d\mu(z) \leq Ct^{c-\gamma}\nu(D_\varepsilon(w))$ *for all* $w = (s,t) \in H$, *where* $\gamma = p(n+\ell+k)$ *and* k *is sufficiently large. Let* $w \in H$. *Clearly, if* $z \notin S(s, 2^{j-1}t)$, *then* $|w-\overline{z}| \geq 2^{j-1}t$ $(j \geq 1)$. *Therefore, the hypothesis implies that*

$$\begin{aligned}
\int_H \frac{1}{|w-\overline{z}|^\gamma}d\mu(z) &\leq t^{-\gamma}\int_{S(s,t)} d\mu + t^{-\gamma}\sum_{j=1}^{\infty}\frac{1}{2^{\gamma(j-1)}}\int_{S(s,2^jt)\setminus S(s,2^{j-1}t)} d\mu \\
&\leq t^{-\gamma}\mu(S(s,t)) + t^{-\gamma}\sum_{j=1}^{\infty}\frac{1}{2^{\gamma(j-1)}}\mu(S(s,2^jt)) \\
&\leq Kt^{c-\gamma}\nu(D_\varepsilon(s,t)) + Kt^{-\gamma}\sum_{j=1}^{\infty}\frac{1}{2^{\gamma(j-1)}}(2^jt)^c\nu(D_\varepsilon(s,2^jt)) \\
&= Kt^{c-\gamma}\left(\nu(D_\varepsilon(s,t)) + 2^\gamma\sum_{j=1}^{\infty}\frac{1}{2^{(\gamma-c)j}}\nu(D_\varepsilon(s,2^jt))\right)
\end{aligned}$$

Since ω satisfies the $(A_q)_\partial$–condition, $\nu = \omega dV$ satisfies the doubling condition. Therefore, there is a constant $\lambda > 0$ such that $\nu(D_\varepsilon(s, 2t)) \leq 2^\lambda \nu(D_\varepsilon(s,t))$. Hence, we have

$$\begin{aligned}\int_H \frac{1}{|w-\bar{z}|^\gamma} d\mu(z) &\leq Kt^{c-\gamma}\left(\nu(D_\varepsilon(w)) + 2^\gamma \sum_{j=1}^{\infty} \frac{1}{2^{(\gamma-c)j}} 2^{\lambda j}\nu(D_\varepsilon(w))\right) \\ &= Kt^{c-\gamma}\left(1 + 2^\gamma \sum_{j=1}^{\infty} \frac{1}{2^{(\gamma-c-\lambda)j}}\right)\nu(D_\varepsilon(w)).\end{aligned}$$

If we choose an integer k such that $\gamma - c - \lambda = p(n+m+k) - \lambda > 0$, then we obtain $\int_H 1/|\overline{w} - z|^\gamma d\mu(z) \leq Ct^{c-\gamma}\nu(D_\varepsilon(w))$.

References

[1] *P.Ahern and M.Jevtić, Inner multipliers of the Besov space, $0 < p \leq 1$, Rocky Mountain J. Math.* **20**(1990), 753–764.

[2] *S.Axler, P.Bourdon and W.Ramey, Harmonic Function Theory, Springer-Verlag, New York, 1992.*

[3] *D.Békollé, Inégalités à poids pour le projecteur de Bergman dans la boule unité de $\mathbb{C}^n$, Studia Math.* **71**(1982), 305–323.

[4] *C.Cascante and J.Ortega, Carleson measures on spaces of Hardy-Sobolev type, Can. J. Math.* **47**(1995), 1177–1200.

[5] *C.Fefferman and E.Stein, H^p–Spaces of several variables, Acta Math.* **129**(1972), 137–193.

[6] *J.Garnett, Bounded Analytic Functions, Academic Press, New York, 1981.*

[7] *R.Hunt, B.Muckenhoupt, and R.Wheeden, Weighted norm ineqalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc.* **176**(1973), 227–251.

[8] *D.Luecking, Inequalities on Bergman spaces, Illinois. J. Math.* **25**(1981), 1–11.

[9] *D.Luecking, Forword and reverse Carleson inequalities for functions in Bergman spaces and their derivatives, Amer. J. Math.* **107**(1985), 85–111.

[10] *D.Luecking, Representation and duality in weighted spaces of analytic functions, Indiana Univ. Math. J.* **34**(1985), 319–336.

[11] *D.Luecking, Embedding derivatives of Hardy spaces into Lebesgue spaces, Proc. London Math. Soc.* **63**(1991), 595–619.

[12] *T.Nakazi and M.Yamada, (A_2)-conditions and Carleson inequalities in Bergman spaces, Pacific J. Math.* **173**(1996), 151–171.

[13] *W.Ramey and H.Yi, Harmonic Bergman functions on half-spaces, Trans. Amer. Math. Soc.* **348**(1996), 633–660.

[14] D.Stegenga, *Multipliers of the Dirichlet space, Ill. J. Math.* **24**(1980), 113–139.

[15] M.Yamada, *Carleson inequalities in classes of derivatives of harmonic Bergman functions with* $0 < p \leq 1$, *Hiroshima Math. J.* **29**(1999), 161-174.

[16] K.Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990.

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