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Quantum White Noise Calculus Based on Nuclear Algebras of Entire Functions

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Introduction

In recent years operator theory over white noise functions has been considerably studied keeping close contacts with infinite dimensional harmonic analysis [5, 7, 13, 17], Cauchy problems in infinite dimension [6], quantum stochastic differential equations [9, 10, 34], and so forth. In particular, for its interesting application to quantum white noises and their nonlinear functions [8, 20, 23, 35, 36], we have started using the term *quantum white noise calculus*, see also the forthcoming survey [24].

In the first comprehensive work [33], adopting the framework of Kubo–Takenaka [30], we developed operator theory on Hida–Kubo–Takenaka space for which the famous characterization theorem for S-transform was first proved [38]. Meanwhile, the framework of white noise distributions has been generalized by many authors in different ways. Among others, generalization keeping the characterization theorem for S-transform valid has been made by Kondratiev–Streit [27], Cochran–Kuo–Sengupta [11] and Asai–Kubo–Kuo [1]. In paralell with these works, an almost equivalent but slightly more general construction has been achieved by Gannoun–Hachaichi–Ouerdiane–Rezgui [14] by means of infinite dimensional holomorphic functions. In the white noise operator theory a key role has been played by the characterization theorem for symbols and, in fact, such characterization theorems have been proved for many variants of white noise function spaces, e.g., see [8]. In order to terminate this routine a unified aspect is proposed by Ji–Obata [22] on the basis of a CKS-space. It turns out, however, that further unification is possible along with the approach proposed by Gannoun–Hachaichi–Ouerdiane–Rezgui [14], since their argument is simply based on a nuclear triplet $N \subset H \subset N^*$ and does not require the famous constant ρ in white noise theory [18, 31, 33].

In this paper we prove some basic results in white noise operator theory within the framework of nuclear algebras of entire functions. Analysis of such nuclear algebras, tracing back to Krée's pioneering works in the early 70's, see e.g., [29], has been developed by Ouerdiane and his collaborators [14, 37] making a close connection with white noise calculus, see also Berezansky–Kondratiev [2], Kondratiev [26] and Lee [32]. There are some advantages of this approach; First the characterization theorem of S-transform follows simply by the combination of Taylor series map $\mathcal{T}$ and the Laplace transform $\mathcal{L}$, both of which admit straightforward extensions to the multi-variable case, in this relation see also [25]. The characterization for operator symbols is obtained from the two-variable extension. Second, the standard construction of white noise functions is based on a choice of defining Hilbertian norms satisfying $|\xi|_p \le \rho |\xi|_{p+1}$ with a constant $0 < \rho < 1$. In fact, this constant plays rather essential roles in convergence of various infinite series appearing in white noise theory. Nevertheless, in our new approach such a constant ρ is not required and is replaced by another constant δ independent of the defining norms. Thanks to this replacement many norm estimates has become more transparent than before. Finally, this new framework is independent of Gaussian analysis and we expect some interesting applications to non-Gaussian analysis. This topic is, however, somehow beyond the scope of this paper and we hope to discuss it elsewhere.

1 Entire Functions with θ-Exponential Growth

1.1 Entire function on a locally convex space

Let $\mathfrak{X}$ be a locally convex space over the complex number field $\mathbf{C}$. A function $f : \mathfrak{X} \to \mathbf{C}$ is called *Gâteaux-entire* if for each $\xi, \eta \in \mathfrak{X}$, the $\mathbf{C}$-valued function of one complex variable $\lambda \mapsto f(\xi + \lambda\eta)$ is holomorphic at every $\lambda \in \mathbf{C}$. A Gâteaux-entire function $f : \mathfrak{X} \to \mathbf{C}$ is called *entire* if it is continuous on $\mathfrak{X}$, or equivalently if it is locally bounded, i.e., every point of $\mathfrak{X}$ is contained in a neighborhood on which f is bounded, see e.g., Dineen [12].

Consider a complex Banach space $(B, |\cdot|)$. We classify entire functions on B by means of their growth rate at the infinity. Let θ be a Young function (see Appendix). An entire function $f : B \to \mathbf{C}$ is said to be *with θ-exponential growth of finite type* $\delta > 0$ if

$$\| f \|_{\theta,\delta} \equiv \sup_{z \in B} |f(z)| e^{-\theta(\delta |z|)} < +\infty.$$

Let $\mathcal{E}_\theta(B, \delta)$ denote the space of all such entire functions, which becomes a Banach space equipped with the norm $\| \cdot \|_{\theta,\delta}$.

1.2 A real nuclear chain and its complexification

We start with a real nuclear Fréchet space E which is continuously and densely imbedded in a real Hilbert space $H_{\mathbf{R}}$. The norm of $H_{\mathbf{R}}$ is denoted by $|\cdot|_0$. It is known that there exists a sequence of Hilbertian norms $\{|\cdot|_p\}$ determining the topology of E such that

$$|\xi|_0 \le |\xi|_1 \le |\xi|_2 \le \cdots, \qquad \xi \in E. \tag{1.1}$$

For each $p \ge 0$ let E_p denote the real Hilbert space obtained by completing E with respect to $|\cdot|_p$. Equipped with the canonical map $\pi_{p,p+1} : E_{p+1} \to E_p$, which is continuous and has a dense image, $\{E_p\}_{p=0}^{\infty}$ forms a projective sequence of Hilbert spaces and it holds that

$$E \cong \operatorname*{proj\,lim}_{p \to \infty} E_p \left(= \bigcap_{p=0}^{\infty} E_p \text{ as sets} \right).$$

Let E^* be the dual space[a)] of E. We recall a standard expression of E^*. For each $p \ge 0$ we denote by E_{-p} the dual space of E_p. By duality the map $\pi^*_{p,p+1} : E_{-p} \to E_{-(p+1)}$ is a

[a)] For a locally convex space $\mathfrak{X}$ the dual space, denoted by $\mathfrak{X}^*$, is by definition the space of all continuous linear functions on $\mathfrak{X}$. The dual space is assumed to carry the strong dual topology unless otherwise stated.

continuous injection with a dense image. Thus, $\{E_{-p}\}_{p=0}^{\infty}$ becomes an inductive sequence of Hilbert spaces and it holds that

$$E^* \cong \operatorname*{ind\,lim}_{p\to\infty} E_{-p} \left(= \bigcup_{p=0}^{\infty} E_{-p} \text{ as sets} \right).$$

In particular, the strong dual topology and the inductive limit topology coincide.

In the above consideration there is a distinguished Hilbert space $H_{\mathbf{R}} = E_0$. Identifying $H_{\mathbf{R}}$ with its dual space $H_{\mathbf{R}}^*$ by the Riesz theorem, we obtain a chain of Hilbert spaces and their limits:

$$E \subset \cdots \subset E_p \subset \cdots \subset E_1 \subset H_{\mathbf{R}} \cong H_{\mathbf{R}}^* \subset E_{-1} \subset \cdots \subset E_{-p} \subset \cdots \subset E^*. \tag{1.2}$$

The canonical bilinear forms on $E^* \times E$ and on $E_{-p} \times E_p$, and the inner product of $H_{\mathbf{R}}$ are denoted by the same symbol $\langle \cdot, \cdot \rangle$ for they are all compatible[b)].

Now we consider the complexification. For each $p \in \mathbf{R}$ we set $N_p = E_p + iE_p$, which becomes a complex Hilbert space in an obvious manner. In particular, for $\xi = \xi_1 + i\xi_2$ and $\eta = \eta_1 + i\eta_2$ the Hermitian inner product is defined by

$$\begin{aligned} \langle \xi, \eta \rangle_{N_p} &= \langle \xi_1 + i\xi_2, \eta_1 + i\eta_2 \rangle_{N_p} \\ &= \langle \xi_1, \eta_1 \rangle_{E_p} + i \langle \xi_1, \eta_2 \rangle_{E_p} - i \langle \xi_2, \eta_1 \rangle_{E_p} + \langle \xi_2, \eta_2 \rangle_{E_p}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{E_p}$ is the inner product of E_p. Then (1.2) is extended to a complex nuclear chain:

$$N \subset \cdots \subset N_p \subset \cdots \subset N_1 \subset H = N_0 \subset N_{-1} \subset \cdots \subset N_{-p} \subset \cdots \subset N^*. \tag{1.3}$$

The canonical $\mathbf{C}$-bilinear forms on $N^* \times N$ and on $N_{-p} \times N_p$, $p \geq 0$, are denoted by the same symbol $\langle \cdot, \cdot \rangle$. It is then noted that $|\xi|_0^2 = \langle \xi, \xi \rangle_H = \langle \bar{\xi}, \xi \rangle$ for $\xi \in H = N_0$.

Lemma 1.1 *Let $p \in \mathbf{R}$ be fixed. There exists uniquely an isometric, anti-linear isomorphism $\xi \mapsto \xi^*$ from N_p onto N_{-p} such that*

$$\langle \xi^*, \eta \rangle = \langle \xi, \eta \rangle_{N_p}, \qquad \xi, \eta \in N_p,$$

where the right hand side is the Hermitian inner product of the Hilbert space N_p.

PROOF. Given $\xi \in N_p$, we consider the map $\eta \mapsto \langle \xi, \eta \rangle_{N_p}$, where $\eta \in N_p$. Since this map is continuous and linear, by definition there exists a unique $\xi^* \in N_{-p}$ such that $\langle \xi, \eta \rangle_{N_p} = \langle \xi^*, \eta \rangle$. It is easy to see that $(\alpha\xi + \beta\eta)^* = \bar{\alpha}\xi^* + \bar{\beta}\eta^*$. Moreover, it is isometric since

$$|\xi^*|_{-p} = \sup_{|\eta|_p \leq 1} |\langle \xi^*, \eta \rangle| = \sup_{|\eta|_p \leq 1} |\langle \xi, \eta \rangle_{N_p}| = |\xi|_p.$$

Finally, the map $\xi \mapsto \xi^*$ is surjective, which can be verified by the Riesz theorem. ∎

b) The right and left arguments of $\langle \cdot, \cdot \rangle$ are sometimes confused when there is no danger.

Let $\{e_i\}$ be a complete orthonormal basis of N_p. Then the Fourier expansion of $\xi \in N_p$ is expressed in the form:

$$\xi = \sum_i \langle e_i^*, \xi \rangle e_i, \qquad |\xi|_p^2 = \sum_i |\langle e_i^*, \xi \rangle|^2.$$

Moreover, as is easily verified, $\{e_i^*\}$ becomes a complete orthonormal basis of N_{-p}. The Fourier expansion of $f \in N_{-p}$ is expressed in the form:

$$f = \sum_i \langle f, e_i \rangle e_i^*, \qquad |f|_{-p}^2 = \sum_i |\langle f, e_i \rangle|^2.$$

Note also that $\langle e_i^*, e_j \rangle = \delta_{ij}$.

1.3 Entire functions on nuclear spaces

Let θ be a fixed Young function. We note that $\{\mathcal{E}_\theta(N_{-p}, \delta)\}$ becomes a projective system of Banach spaces as $p \to \infty$ and $\delta \downarrow 0$. We then define

$$\mathcal{F}_\theta(N^*) = \operatorname*{proj\,lim}_{p\to\infty;\delta\downarrow 0} \mathcal{E}_\theta(N_{-p}, \delta),$$

which is called the *space of entire functions on N^* with θ-exponential growth of minimal type*. Similarly, $\{\mathcal{E}_\theta(N_p, \delta)\}$ becomes an inductive system of Banach spaces as $p \to \infty$ and $\delta \to \infty$. We define

$$\mathcal{G}_\theta(N) = \operatorname*{ind\,lim}_{p\to\infty;\delta\to\infty} \mathcal{E}_\theta(N_p, \delta),$$

which is called the *space of entire functions on N with θ-exponential growth of finite type.*

Proposition 1.2 *$\mathcal{F}_\theta(N^*)$ is identified with the space of all functions $f : N^* \to \mathbf{C}$ such that $f|_{N_{-p}} = f \circ \pi_p^*$ is entire on N_{-p} for any $p \geq 0$ and*

$$\| f \|_{\theta,-p,\delta} \equiv \sup_{z\in N_{-p}} |f(z)| e^{-\theta(\delta |z|_{-p})} < +\infty \tag{1.4}$$

for any $\delta > 0$. Moreover, such f is entire on N^*.

PROOF. By definition an element of the projective limit space is a consistent family $(f_{p\delta})$, where $f_{p\delta} \in \mathcal{E}_\theta(N_{-p}, \delta)$. For $z \in N^*$ we choose some $p \geq 0$ such that $z \in N_{-p}$ and define $f(z) = f_{p\delta}(z)$, which is independent of the choice of p, δ. This f satisfies the desired property. This argument is easily converted and we see that the correspondence $(f_{p\delta}) \leftrightarrow f$ is one-to-one. We shall show now that f is entire on N^*. Obviously, f is Gâteaux-entire. Since any (strongly) bounded subset of N^* is contained in N_{-p} for some $p \geq 0$ and is bounded in the norm [16, Chap.1.5.3], the local boundedness of f follows from that of $f|_{N_{-p}}$, which is already shown in (1.4). Hence f is entire on N^*. ∎

Proposition 1.3 *$\mathcal{G}_\theta(N)$ is identified with the space of all functions $g : N \to \mathbf{C}$ for which there exists a pair $p \geq 0$, $\delta > 0$ such that g admits an entire extension $g_p : N_p \to \mathbf{C}$ and*

$$\| g_p \|_{\theta,p,\delta} \equiv \sup_{z\in N_p} |g_p(z)| e^{-\theta(\delta |z|_p)} < +\infty.$$

Moreover, such a function g is entire on N.

PROOF. Similar to the proof of Proposition 1.2. ∎

Both $\mathcal{F}_\theta(N^*)$ and $\mathcal{G}_\theta(N)$ are constructed after choosing a sequence of Hilbertian norms (1.1). We shall show that the construction does not depend on the choice. Let $|\cdot|_\alpha$ be a continuous seminorm on N. Then, in a canonical manner we have a Banach space N_α and a continuous map $\pi_\alpha : N \to N_\alpha$ with a dense image. By duality $\pi_\alpha^* : N_\alpha^* \to N^*$ is a continuous injection. The dual norm is defined by

$$|f|_{-\alpha} = \sup_{|x|_\alpha \le 1} |\langle \pi_\alpha^* f, x\rangle|, \qquad f \in N_\alpha^*.$$

Proposition 1.4 *A function $f : N^* \to \mathbf{C}$ belongs to $\mathcal{F}_\theta(N^*)$ if and only if for any continuous seminorm $|\cdot|_\alpha$ of N, $f \circ \pi_\alpha^*$ is entire on N_α^* and*

$$\sup_{z \in N_\alpha^*} |f(\pi_\alpha^* z)| e^{-\theta(m|z|_{-\alpha})} < +\infty.$$

PROOF. We need only show the "only if" part. Since $|\cdot|_\alpha$ is continuous, there exist $p \ge 0$ and $c \ge 0$ such that

$$|\xi|_\alpha \le c|\xi|_p, \qquad \xi \in N.$$

Then the natural map $\pi_{\alpha p} : N_p \to N_\alpha$ is continuous and has a dense image. By duality we have a continuous injection $\pi_{\alpha p}^* : N_{-\alpha} \to N_{-p}$ and $c^{-1}|\pi_{\alpha p}^* z|_{-p} \le |z|_{-\alpha}$. Note also the following commutative diagrams:

$$\begin{tikzcd} & N \arrow[dl, "\pi_p"'] \arrow[dr, "\pi_\alpha"] & \\ N_p \arrow[rr, "\pi_{\alpha p}"'] & & N_\alpha \end{tikzcd} \qquad \begin{tikzcd} & N^* & \\ N_{-p} \arrow[ur, "\pi_p^*"] & & N_\alpha^* \arrow[ul, "\pi_\alpha^*"'] \arrow[ll, "\pi_{\alpha p}^*"] \end{tikzcd}$$

Now suppose $f \in \mathcal{F}_\theta(N^*)$. Then

$$\begin{aligned} \sup_{z \in N_\alpha^*} |f(\pi_\alpha^* z)| e^{-\theta(|z|_{-\alpha})} &= \sup_{z \in N_\alpha^*} |f(\pi_p^* \circ \pi_{\alpha p}^* z)| e^{-\theta(|z|_{-\alpha})} \\ &\le \sup_{z \in N_\alpha^*} |f(\pi_p^* \circ \pi_{\alpha p}^* z)| e^{-\theta(c^{-1}|\pi_{\alpha p}^* z|_{-p})} \\ &\le \sup_{w \in N_{-p}} |f(\pi_p^* w)| e^{-\theta(c^{-1}|w|_{-p})} = \| f \|_{\theta,-p,c^{-1}}, \end{aligned}$$

where π_p^* is injective so that N_{-p} is regarded as a subspace of N^*. By assumption the last norm is finite and the proof is completed. ∎

Similarly, we have the following

Proposition 1.5 *A function $g : N \to \mathbf{C}$ belongs to $\mathcal{G}_\theta(N)$ if and only if thete exist a continuous seminorm $|\cdot|_\alpha$ of N and an entire function $g_\alpha : N_\alpha \to \mathbf{C}$ such that $g = g_\alpha \circ \pi_\alpha$ and*

$$\sup_{z \in N_\alpha} |g_\alpha(z)| e^{-\theta(|z|_\alpha)} < +\infty.$$

Propositions 1.4 and 1.5 mean that

$$\mathcal{F}_\theta(N^*) = \operatorname*{proj\,lim}_{\alpha;\delta\downarrow 0} \mathcal{E}_\theta(N_{-\alpha}, \delta), \qquad \mathcal{G}_\theta(N) = \operatorname*{ind\,lim}_{\alpha;\delta\to\infty} \mathcal{E}_\theta(N_\alpha, \delta),$$

where α runs over all continuous seminorms of N.

1.4 Equivalent Young functions

We here only mention the following fact, the proof of which is easy, see [14].

Proposition 1.6 *If two Young functions θ_1 and θ_2 are equivalent at infinity, i.e.,*

$$\lim_{x\to+\infty}\frac{\theta_1(x)}{\theta_2(x)}=1,$$

then $\mathcal{F}_{\theta_1}(N^*)=\mathcal{F}_{\theta_2}(N^*)$ and $\mathcal{G}_{\theta_1}(N)=\mathcal{G}_{\theta_2}(N)$.

1.5 Multiplication

Proposition 1.7 *$\mathcal{F}_\theta(N^*)$ is closed under pointwise multiplication. Moreover, the pointwise multiplication yields a continuous bilinear map from $\mathcal{F}_\theta(N^*)\times\mathcal{F}_\theta(N^*)$ into $\mathcal{F}_\theta(N^*)$.*

PROOF. For $f,g\in\mathcal{F}_\theta(N^*)$,

$$|f(z)g(z)|e^{-\theta(\delta|z|_{-p})}\le|f(z)|e^{-\theta(\frac{\delta}{2}|z|_{-p})}|g(z)|e^{-\theta(\frac{\delta}{2}|z|_{-p})},$$

where an obvious inequality $\frac{1}{2}\theta(x)=\frac{1}{2}(\theta(x)+\theta(0))\ge\theta(\frac{1}{2}x)$ was used. Then, taking the supremum over $z\in N_{-p}$, we obtain

$$\|fg\|_{\theta,-p,\delta}\le\|f\|_{\theta,-p,\delta/2}\|g\|_{\theta,-p,\delta/2}\,.$$

This proves the assertion. ∎

Proposition 1.8 *$\mathcal{G}_\theta(N)$ is closed under pointwise multiplication. Moreover, the pointwise multiplication yields a separately continuous bilinear map from $\mathcal{G}_\theta(N)\times\mathcal{G}_\theta(N)$ into $\mathcal{G}_\theta(N)$.*

PROOF. Suppose $f,g\in\mathcal{G}_\theta(N)$. Then, by definition there exist $p\ge0$, $\delta>0$ and an entire function $f_p:N_p\to\mathbf{C}$ which extends f such that

$$\|f\|_{\theta,p,\delta}=\sup_{z\in N_p}|f_p(z)|e^{-\theta(\delta|z|_p)}<\infty.$$

Similarly, for g we have $q\ge0$, $\delta'>0$ and an entire function $g_q:N_q\to\mathbf{C}$ which extends g such that

$$\|g\|_{\theta,q,\delta'}=\sup_{z\in N_q}|g_q(z)|e^{-\theta(\delta'|z|_q)}<\infty.$$

We may assume that $p\le q$. Then we have $N\subset N_q\subset N_p$. Set $f_q=f_p\big|_{N_q}$. Then, it is obvious that f_q is Gâteaux-entire on N_q. Moreover, since $|z|_p\le|z|_q$, we have

$$|f_q(z)|=|f_p(z)|\le\|f\|_{\theta,p,\delta}\,e^{\theta(\delta|z|_p)}\le\|f\|_{\theta,p,\delta}\,e^{\theta(\delta|z|_q)},\qquad z\in N_q.$$

Hence f_q is locally bounded on N_q, and hence f_q is entire on N_q. Now, f_qg_q extends fg and is entire on N_q. Moreover,

$$|f_q(z)g_q(z)|\le\|f\|_{\theta,p,\delta}\,e^{\theta(\delta|z|_q)}\|g\|_{\theta,q,\delta}\,e^{\theta(\delta'|z|_q)}\le\|f\|_{\theta,p,\delta}\|g\|_{\theta,q,\delta}\,e^{\theta(2(\delta+\delta')|z|_q)}.$$

Consequently,

$$\| fg \|_{\theta, p\vee q, 2(\delta+\delta')} \le \| f \|_{\theta,p,\delta} \| g \|_{\theta,q,\delta'} ,$$

which ompletes the proof. ∎

Remark 1.9 It is plausible that the above separately continuous bilinear map from $\mathcal{G}_\theta(N) \times \mathcal{G}_\theta(N)$ into $\mathcal{G}_\theta(N)$ is, in fact, continuous.

2 Taylor Series Map

2.1 Symmetric tensor powers and Taylor expansion

For two locally convex spaces $\mathfrak{X}, \mathfrak{Y}$ we denote by $\mathfrak{X} \otimes_{\text{alg}} \mathfrak{Y}$ the algebraic tensor product. The completion of $\mathfrak{X} \otimes_{\text{alg}} \mathfrak{Y}$ with respect to the π-topology is called the π-tensor product and is denoted by $\mathfrak{X} \otimes \mathfrak{Y}$ for simplicity. If both $\mathfrak{X} = H$, $\mathfrak{Y} = K$ are Hilbert spaces, $H \otimes K$ stands for the Hilbert space tensor product though it is different from the π-tensor product.

For a locally convex space $\mathfrak{X}$ the n-fold symmetric tensor power $\mathfrak{X}^{\widehat{\otimes} n}$ is the closed subspace of $\mathfrak{X}^{\otimes n}$ spanned by the elements of the form $\xi^{\otimes n}$, where $\xi \in \mathfrak{X}$. Similar definition is adopted for the n-fold symmetric tensor power of a Hilbert space.

Lemma 2.1 *For a nuclear Fréchet space N we have $(N^*)^{\widehat{\otimes} n} \cong (N^{\widehat{\otimes} n})^*$.*

By Propositions 1.2 and 1.3 $f \in \mathcal{F}_\theta(N^*)$ and $g \in \mathcal{G}_\theta(N)$ admit the Taylor expansions:

$$f(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, f_n \rangle, \qquad z \in N^*, \quad f_n \in N^{\widehat{\otimes} n}, \tag{2.1}$$

$$g(\xi) = \sum_{n=0}^{\infty} \langle g_n, \xi^{\otimes n} \rangle, \qquad \xi \in N, \quad g_n \in (N^{\widehat{\otimes} n})^*,$$

where we used the common symbol $\langle \cdot, \cdot \rangle$ for the canonical bilinear form on $(N^{\widehat{\otimes} n})^* \times N^{\widehat{\otimes} n}$ for all n.

Here is a just notation. A sequence $\Phi = (F_n)$, where $F_n \in (N^{\widehat{\otimes} n})^*$, is called a *formal power series* on N. With a formal power series $\Phi = (F_n)$ we associate a function F_Φ on N defined by

$$F_\Phi(\xi) = \sum_{n=0}^{\infty} \left\langle F_n, \xi^{\otimes n} \right\rangle, \qquad \xi \in N,$$

though the convergence is not taken into consideration here.

2.2 Nuclear spaces of power series

We shall characterize $\mathcal{F}_\theta(N^*)$ and $\mathcal{G}_\theta(N)$ in terms of the Taylor expansions. First we define a sequence $\{\theta_n\}$ by

$$\theta_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n}, \qquad n = 0, 1, 2, \cdots .$$

Suppose a pair $p \geq 0$, $\delta > 0$ is given. Then, for $\phi = (f_n)_{n=0}^{\infty}$ with $f_n \in N_p^{\widehat{\otimes} n}$ we put

$$\| \phi \|_{+;p,\delta}^2 = \sum_{n=0}^{\infty} \theta_n^{-2} \delta^{-n} \mid f_n \mid_p^2 ,$$

and for $\Phi = (F_n)_{n=0}^{\infty}$ with $F_n \in N_{-p}^{\widehat{\otimes} n}$,

$$\| \Phi \|_{-;p,\delta}^2 = \sum_{n=0}^{\infty} (n! \theta_n)^2 \delta^n \mid F_n \mid_{-p}^2 .$$

Accordingly, we put

$$F_\theta(N_p, \delta) = \left\{ \phi = (f_n) \, ; \, f_n \in N_p^{\widehat{\otimes} n}, \, \| \phi \|_{+;p,\delta} < \infty \right\} ,$$
$$G_\theta(N_{-p}, \delta) = \left\{ \Phi = (F_n) \, ; \, F_n \in N_{-p}^{\widehat{\otimes} n}, \| \Phi \|_{-;p,\delta} < \infty \right\} .$$

These are sometimes referred to as weighted Fock spaces too. Finally, we define

$$F_\theta(N) = \operatorname*{proj\,lim}_{p \to \infty; \delta \downarrow 0} F_\theta(N_p, \delta), \qquad G_\theta(N^*) = \operatorname*{ind\,lim}_{p \to \infty; \delta \to \infty} G_\theta(N_{-p}, \delta). \tag{2.2}$$

It is easily verified that $F_\theta(N)$ becomes a nuclear Fréchet space. By definition, $F_\theta(N)$ and $G_\theta(N^*)$ are dual each other, namely, the strong dual of $F_\theta(N)$ is identified with $G_\theta(N^*)$ through the canonical bilinear form:

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle , \tag{2.3}$$

and we have the Schwartz inequality:

$$| \langle\langle \Phi, \phi \rangle\rangle | \leq \| \Phi \|_{-;p,\delta} \| \phi \|_{+;p,\delta} , \qquad \Phi \in G_\theta(N^*), \quad \phi \in F_\theta(N).$$

2.3 Taylor series map

With each entire function the sequence of Taylor coefficients is associated by the *Taylor series map* $\mathcal{T}$ (at zero). For example, if the Taylor expansion of $f \in \mathcal{F}_\theta(N^*)$ is given as in (2.1), the Taylor series map is defined by $\mathcal{T} f = \vec{f} = (f_n)$. Then we come to the following fundamental result due to Gannoun–Hachaichi–Ouerdiane–Rezgui [14].

Theorem 2.2 *The Taylor series map $\mathcal{T}$ yields topological isomorphisms $\mathcal{F}_\theta(N^*) \cong F_\theta(N)$ and $\mathcal{G}_{\theta^*}(N) \cong G_\theta(N^*)$, where θ^* is the polar function of θ.*

2.4 Exponential vector and exponential function

For $\xi \in N$ we define

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots, \frac{\xi^{\otimes n}}{n!}, \cdots \right) .$$

Then $\phi_\xi \in F_\theta(N)$. In fact,

$$\| \phi_\xi \|^2_{+;p,\delta} = \sum_{n=0}^{\infty} \theta_n^{-2} \delta^{-n} \frac{| \xi |_p^{2n}}{(n!)^2} = \sum_{n=0}^{\infty} \frac{1}{(\theta_n n!)^2} (\delta^{-1} | \xi |_p^2)^n < \infty,$$

for all $p \geq 0$ and $\delta > 0$, see Lemma A.10 in Appendix. On the other hand, we define

$$e^\xi(z) = e^{\langle z, \xi \rangle}, \qquad z \in N^*. \tag{2.4}$$

Obviously, $e^\xi \in \mathcal{F}_\theta(N^*)$. We see from the obvious relations:

$$(\mathcal{T}^{-1}\phi_\xi)(z) = \sum_{n=0}^{\infty} \left\langle z^{\otimes n}, \frac{\xi^{\otimes n}}{n!} \right\rangle = e^{\langle z, \xi \rangle} = e^\xi(z), \qquad z \in N^*,$$

that $\mathcal{T} e^\xi = \phi_\xi$. Both ϕ_ξ and e^ξ are called an *exponential function* or an *exponential vector* or a *coherent vector.*

Lemma 2.3 *The set of exponential vectors $\{\phi_\xi \,;\, \xi \in N\}$ is linearly independent and spans a dense subspace of $F_\theta(N)$. Hence so is $\{e^\xi \,;\, \xi \in N\}$ in $\mathcal{F}_\theta(N^*)$.*

2.5 Laplace transform

Let $\mathcal{F}_\theta(N^*)^*$ denote the dual space of $\mathcal{F}_\theta(N^*)$. Noting that $e^\xi \in \mathcal{F}_\theta(N^*)$ for all $\xi \in N$, we define the *Laplace transform* of $\Phi \in \mathcal{F}_\theta(N^*)^*$ by

$$\mathcal{L}\Phi(\xi) = \langle\!\langle \Phi, e^\xi \rangle\!\rangle, \qquad \xi \in N.$$

The following result, due to Gannoun–Hachaichi–Ouerdiane–Rezgui [14], is now immediate from Theorem 2.2 and the fact that $F_\theta(N)^*$ and $G_\theta(N^*)$ are identified through the bilinear form (2.3).

Theorem 2.4 *The Laplace transform induces a topological isomorphism $\mathcal{L} : \mathcal{F}_\theta(N^*)^* \to \mathcal{G}_{\theta^*}(N)$.*

By Theorem 2.2 we have

$$\mathcal{L}\Phi(\xi) = \langle\!\langle \Phi, e^\xi \rangle\!\rangle = \langle\!\langle \mathcal{T}\Phi, \mathcal{T}e^\xi \rangle\!\rangle = \langle\!\langle \mathcal{T}\Phi, \phi_\xi \rangle\!\rangle . \tag{2.5}$$

In the context of white noise theory, for $\Psi \in F_\theta(N)^*$

$$S\Psi(\xi) = \langle\!\langle \Psi, \phi_\xi \rangle\!\rangle , \qquad \xi \in N,$$

is called the *S-transform.* Hence (2.5) implies that $\mathcal{L}\Phi(\xi) = S\mathcal{T}\Phi(\xi)$, that is,

$$\mathcal{L} = S \circ \mathcal{T}.$$

Since $\mathcal{T}$ is an isomorphism, the images of $\mathcal{L}$ and S coinside, and we obtain the famous characterization theorem of S-transform[c)].

Theorem 2.5 *The S-transform S is a topological isomorphism from $F_\theta(N)^*$ onto $\mathcal{G}_{\theta^*}(N)$.*

[c)] The statement in the present theorem is more general than that in the usual context of white noise theory. Because we do not assume that $F_\theta(N)$ is a subspace of $\Gamma(H) \cong L^2(E^*, \mu)$, see also §4.

3 Operator Theory

We are interested in a continuous operator from $F_\theta(N)$ into $F_\theta(N)^*$. The space of such operators is denoted by $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$ and is assumed to carry the bounded convergence topology.

3.1 Symbols and kernels

There is an isomorphism: $\mathcal{L}(F_\theta(N), F_\theta(N)^*) \cong (F_\theta(N) \otimes F_\theta(N))^*$ which follows from the famous kernel theorem. If $\Xi \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ and $\Xi^K \in (F_\theta(N) \otimes F_\theta(N))^*$ are related under this isomorphism, we have

$$\langle\langle \Xi\phi, \psi \rangle\rangle = \langle\langle \Xi^K, \phi \otimes \psi \rangle\rangle, \qquad \phi, \psi \in F_\theta(N).$$

We call Ξ^K the *kernel* of Ξ. The *symbol* of $\Xi \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle = \langle\langle \Xi^K, \phi_\xi \otimes \phi_\eta \rangle\rangle, \qquad \xi, \eta \in N. \tag{3.1}$$

Since the exponential vectors $\{\phi_\xi \,;\, \xi \in N\}$ span a dense subspace of $F_\theta(N)$, an operator $\Xi \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ is uniquely specified by the symbol. We shall discuss an analytic characterization of the symbols in terms of the Laplace transform.

3.2 Holomorphic functions in two-variables

Let M and N be two nuclear Fréchet spaces with defining Hilbertian norms $\{|\cdot|_{M,p}\}$ and $\{|\cdot|_{N,p}\}$, respectively. Let $M_p \oplus N_p$ be the Hilbert space direct sum[d)]. Then the direct sum $M \oplus N$ is by definition

$$M \oplus N = \operatorname*{proj\,lim}_{p\to\infty} M_p \oplus N_p,$$

Similarly,

$$(M \oplus N)^* = M^* \oplus N^* = \operatorname*{ind\,lim}_{p\to\infty} M_{-p} \oplus N_{-p}.$$

Consider a function $f : M \times N \to \mathbf{C}$ such that (i) $z \mapsto f(z, w)$ is entire for any fixed $w \in N$; and (ii) $w \mapsto f(z, w)$ is entire for any fixed $z \in M$. Such a function is called an entire function in two variables. On the other hand, a function $f : M \times N \to \mathbf{C}$ is in one-to-one correspondence to a function $\tilde{f} : M \oplus N \to \mathbf{C}$ in an obvious manner. Since $M \oplus N$ is another nuclear space, we can consider an entire function on it. It is known (e.g., [12]), however, that f is entire in two-variables if and only if $\tilde{f}$ is entire on $M \oplus N$. Therefore, we do not need make distinction.

Proposition 3.1 *A function $f : (M \oplus N)^* \to \mathbf{C}$ belongs to $\mathcal{F}_\theta((M \oplus N)^*)$ if and only if*

$$\sup_{w\in M, z\in N} |f(w \oplus z)| e^{-\theta(m|w|_{M,-p}) - \theta(m|z|_{N,-p})} < \infty \tag{3.2}$$

for any pair $p \geq 0$ and $m > 0$.

---

d) In general, for two Hilbert spaces H, K we denote by $H \oplus K$ the Hilbert space direct sum. The norm is defined by $|\xi \oplus \eta|^2_{H\oplus K} = |\xi|^2_H + |\eta|^2_K$ for $\xi \in H$ and $\eta \in K$.

PROOF. First note that

$$\frac{1}{\sqrt{2}}(|w|_{M,-p}+|z|_{N,-p}) \le |w\oplus z|_{M\oplus N,-p}$$
$$= (|w|_{M,-p}^2+|z|_{N,-p}^2)^{1/2} \le |w|_{M,-p}+|z|_{N,-p}.$$

Since θ is an increasing function,

$$\theta\Big(\frac{m}{\sqrt{2}}(|w|_{M,-p}+|z|_{N,-p})\Big) \le \theta(m|w\oplus z|_{M\oplus N,-p})$$
$$\le \theta(m|w|_{M,-p}+m|z|_{N,-p}). \tag{3.3}$$

Note that any Young function satisfies the following inequality:

$$\theta\Big(\frac{s}{2}\Big)+\theta\Big(\frac{t}{2}\Big) \le \theta(s+t) \le \frac{\theta(2s)+\theta(2t)}{2} \le \theta(2s)+\theta(2t), \qquad s,t\ge 0.$$

Then (3.3) becomes

$$\theta\left(\frac{m}{2\sqrt{2}}|w|_{M,-p}\right)+\theta\left(\frac{m}{2\sqrt{2}}|z|_{N,-p}\right) \le \theta(m|w\oplus z|_{M\oplus N,-p})$$
$$\le \theta(2m|w|_{M,-p})+\theta(2m|z|_{N,-p}).$$

This shows that (3.2) is equivalent to that $f\in\mathcal{F}_\theta((M\oplus N)^*)$. ∎

Similarly we have

Proposition 3.2 *A function $f: M\oplus N\to \mathbf{C}$ belongs to $\mathcal{G}_\theta(M\oplus N)$ if and only if there exist a pair $p\ge 0$ and $m>0$ such that*

$$\sup_{w\in M, z\in N} |f(w\oplus z)|e^{-\theta(m|w|_{M,p})-\theta(m|z|_{N,p})} < \infty.$$

Proposition 3.3 *There is a unique topological isomorphism $\mathcal{F}_\theta((M\oplus N)^*)\cong\mathcal{F}_\theta(M^*)\otimes\mathcal{F}_\theta(N^*)$ which extends the correspondence $e^{\xi\oplus\eta}\leftrightarrow e^\xi\otimes e^\eta$.*

PROOF. For $f_1\in\mathcal{F}_\theta(M^*)$ and $f_2\in\mathcal{F}_\theta(N^*)$ we define $f_1\otimes f_2$ as usual:

$$f_1\otimes f_2(w\oplus z) = f_1(w)f_2(z).$$

Then $(f_1,f_2)\mapsto f_1\otimes f_2$ is a bilinear map from $\mathcal{F}_\theta(M^*)\times\mathcal{F}_\theta(N^*)$ into $\mathcal{F}_\theta((M\oplus N)^*)$, and hence we have $h:\mathcal{F}_\theta(M^*)\otimes_{\mathrm{alg}}\mathcal{F}_\theta(N^*)\to\mathcal{F}_\theta((M\oplus N)^*)$. It follows from Proposition 3.1 that h is continuous so that h is extended to a continuous map $\mathcal{F}_\theta(M^*)\otimes\mathcal{F}_\theta(N^*)\to\mathcal{F}_\theta((M\oplus N)^*)$. Moreover, we see from

$$e^{\xi\oplus\eta}(w\oplus z) = e^{\langle w\oplus z,\xi\oplus\eta\rangle} = e^{\langle w,\xi\rangle+\langle z,\eta\rangle} = e^\xi(w)e^\eta(z)$$

that $h(e^\xi\otimes e^\eta) = e^{\xi\oplus\eta}$. Recall that $\{e^{\xi\oplus\eta}\}$ spans a dense subspace of $\mathcal{F}_\theta((M\oplus N)^*)$, see Lemma 2.3. By a standard argument with the Taylor expansion we conclude that h is extended to an isomorphism from $\mathcal{F}_\theta(M^*)\otimes\mathcal{F}_\theta(N^*)$ onto $\mathcal{F}_\theta((M\oplus N)^*)$. ∎

Corollary 3.4 *There is a unique topological isomorphism $F_\theta(N \oplus M) \cong F_\theta(N) \otimes F_\theta(M)$ which extends the correspondence $\phi_{\xi\oplus\eta} \leftrightarrow \phi_\xi^N \otimes \phi_\eta^M$, where the exponential vectors in $F_\theta(N)$ and $F_\theta(M)$ are denoted by ϕ_ξ^N and ϕ_η^M, respectively.*

We now come to the characterization of operator symbols.

Theorem 3.5 *A function $\Theta : N \times N \to \mathbf{C}$ is the symbol of some $\Xi \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ if and only if $\Theta \in \mathcal{G}_{\theta^*}(N \oplus N)$.*

PROOF. In view of (3.1) we have

$$\widehat{\Xi}(\xi, \eta) = \langle\!\langle \Xi^K, \phi_\xi \otimes \phi_\eta \rangle\!\rangle = \langle\!\langle \Xi^K, \phi_{\xi\oplus\eta} \rangle\!\rangle = \mathcal{L}\Xi^K(\xi \oplus \eta).$$

By Theorem 2.4 we see that $\mathcal{L}\Xi^K \in \mathcal{G}_{\theta^*}(N \oplus N)$, which proves the assertion. ∎

3.3 Chaotic expansion of operators

Given $\Xi \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$, we consider the Taylor expansion of the symbol $\widehat{\Xi} \in \mathcal{G}_{\theta^*}(N \oplus N)$:

$$\widehat{\Xi}(\xi, \eta) = \sum_{l,m=0}^{\infty} \left\langle \lambda_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle, \qquad \lambda_{l,m} \in (N^{\otimes(l+m)})^*.$$

It is obvious by Theorem 3.5 that there exists $\Xi_{l,m} \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ such that

$$\widehat{\Xi}_{l,m}(\xi, \eta) = \left\langle \lambda_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle.$$

Thus we come to

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m},$$

which is called the *chaotic expansion* of $\Xi \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$.

4 Quantum White Noise Calculus

4.1 Gaussian space

In white noise analysis the Gaussian space (E^*, μ) plays a central role, where μ is the standard Gaussian measure uniquely specified by

$$e^{-|\xi|_0^2/2} = \int_{E^*} e^{i\langle x, \xi\rangle} \mu(dx), \qquad \xi \in E.$$

The famous Wiener–Itô–Segal theorem says that there is a unique unitary isomorphism between $L^2(E^*, \mu)$ and the Boson Fock space $\Gamma(H)$ which is uniquely specified by the correspondence between the exponential vectors:

$$e^\xi(x) = e^{\langle x, \xi\rangle} \quad \leftrightarrow \quad \phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right),$$

where ξ runs over $N = E + iE$. Recall that the Boson Fock space[e)] is defined by

$$\Gamma(H) = \left\{ \phi = (f_n)\,;\; f_n \in H^{\widehat{\otimes} n}, \quad \| \phi \|_0^2 \equiv \sum_{n=0}^{\infty} n! \, | f_n |_0^2 \right\}.$$

From now on we assume an additional condition on the Young function θ:

(G) $\displaystyle\limsup_{x\to\infty} \frac{\theta(x)}{x^2} < \infty.$

An equivalent condition is mentioned in Proposition A.4.

Lemma 4.1 *If a Young function θ satisfies condition (G), there exist constant numbers $a > 0$ and $b > 0$ such that*

$$\theta_n \le a \left(\frac{2be}{n} \right)^{n/2}.$$

PROOF. By condition (G) there exist constant numbers $a > 0$ and $b > 0$ such that

$$e^{\theta(r)} \le a e^{br^2}, \qquad r \ge 0.$$

Then, by an elementary calculus we obtain

$$\theta_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n} \le \inf_{r>0} \frac{a e^{br^2}}{r^n} = a \left(\frac{2be}{n} \right)^{n/2},$$

as desired. ∎

Proposition 4.2 *If the Young function θ satisfies condition (G), then $F_\theta(N) \subset \Gamma(H)$, where the inclusion is continuous and has a dense image.*

PROOF. For $\phi = (f_n)$ we have

$$\| \phi \|_0^2 = \sum_{n=0}^{\infty} n! \, \| f_n \|_0^2 = \sum_{n=0}^{\infty} \theta_n^2 \delta^n n! \times \theta_n^{-2} \delta^{-n} \, \| f_n \|_0^2, \tag{4.1}$$

where we have by Lemma 4.1

$$\theta_n^2 \delta^n n! \le a^2 \left(\frac{2be}{n} \right)^n \delta^n n! = a^2 (2b\delta)^n \sqrt{n} \, \frac{e^n n!}{n^n \sqrt{n}}.$$

With the help of the Stirling formula, the last fraction tends to $\sqrt{2\pi}$ as $n \to \infty$. Therefore, for $\delta < (2b)^{-1}$ we have

$$M^2 \equiv \sup_{n\ge 0} \theta_n^2 \delta^n n! < \infty.$$

[e)] In the definition of norm $\| \phi \|_0$ we put the factor $n!$ due to white noise convention.

Thus (4.1) becomes

$$\|\phi\|_0^2 \le M^2 \sum_{n=0}^{\infty} \theta_n^{-2} \delta^{-n} \|f_n\|_0^2 = M^2 \|\phi\|_{+;0,\delta}^2,$$

which means that $F_\theta(N) \subset \Gamma(H)$ and the inclusion is continuous. It is obvious that $F_\theta(N) \subset \Gamma(H)$ is a dense subspace. ∎

In that case we have a nuclear triple:

$$F_\theta(N) \subset \Gamma(H) \subset F_\theta(N)^*. \tag{4.2}$$

Moreover, $\mathcal{L}(F_\theta(N), F_\theta(N))$ and $\mathcal{L}(F_\theta(N)^*, F_\theta(N)^*)$ are subspaces of $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$. The bounded operators on $\Gamma(H)$ form also a subspace of $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$. A member of $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$ is called a *white noise operator.*

4.2 Contraction of tensor product

Consider a 4-linear map $L : (N^*)^{\otimes l} \times (N^*)^{\otimes m} \times N^{\otimes n} \times N^{\otimes m} \to (N^*)^{\otimes l} \otimes N^{\otimes n}$ defined by

$$L(\kappa_l, \kappa_m, f_n, f_m) = \langle \kappa_m, f_m \rangle \kappa_l \otimes f_n.$$

Since L is continuous, there is a continuous bilinear map $\tilde{L} : (N^*)^{\otimes(l+m)} \times \times N^{\otimes(n+m)} \to (N^*)^{\otimes l} \otimes N^{\otimes n}$ such that

$$\tilde{L}(\kappa_l \otimes \kappa_m, f_n \otimes f_m) = L(\kappa_l, \kappa_m, f_n, f_m) = \langle \kappa_m, f_m \rangle \kappa_l \otimes f_n.$$

For $\kappa_{l+m} \in (N^*)^{\otimes(l+m)}$ and $f_{n+m} \in N^{\otimes(l+m)}$ we put

$$\kappa_{l+m} \otimes_m f_{n+m} = \tilde{L}(\kappa_{l+m}, f_{n+m}),$$

which is called the *right contraction* of degree m.

Lemma 4.3 *For $\kappa_{l+m} \in N_{-p}^{\otimes(l+m)}$ and $f_{n+m} \in N_p^{\otimes(n+m)}$,*

$$|\kappa_{l+m} \otimes_m f_{n+m}|_{-p} \le |\kappa_{l+m}|_{-p} |f_{n+m}|_p. \tag{4.3}$$

For $\kappa_{l+m} \in N_p^{\otimes(l+m)}$ and $f_{n+m} \in N_p^{\otimes(n+m)}$,

$$|\kappa_{l+m} \otimes_m f_{n+m}|_p \le |\kappa_{l+m}|_p |f_{n+m}|_p. \tag{4.4}$$

PROOF. Let $\{e_i\}$ be a complete orthonormal basis of N_p. Recall that $\{e_i^*\}$ be a complete orthonormal basis of N_{-p}. Moreover,

$$e_{\vec{i}} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$$

form a complete orthonormal basis of $N_p^{\otimes n}$. Now, consider the Fourier expansions:

$$\kappa_{l+m} = \sum_{\vec{i},\vec{j}} \left\langle \kappa_{l+m}, e_{\vec{i}} \otimes e_{\vec{j}} \right\rangle e_{\vec{i}}^* \otimes e_{\vec{j}}^*, \qquad f_{n+m} = \sum_{\vec{k},\vec{j}} \left\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*, f_{n+m} \right\rangle e_{\vec{k}} \otimes e_{\vec{j}}.$$

Then the right m-contraction is given by

$$\kappa_{l+m} \otimes_m f_{n+m} = \sum_{\vec{i},\vec{j},\vec{k}} \left\langle \kappa_{l+m},\, e_{\vec{i}} \otimes e_{\vec{j}} \right\rangle \left\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \right\rangle e_{\vec{i}}^* \otimes e_{\vec{k}}.$$

Hence

$$\begin{aligned}
|\, \kappa_{l+m} \otimes_m f_{n+m} \,|_{-p}^2 &= \sum_{\vec{p},\vec{q}} \left| \sum_{\vec{i},\vec{j},\vec{k}} \left\langle \kappa_{l+m},\, e_{\vec{i}} \otimes e_{\vec{j}} \right\rangle \left\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \right\rangle \left\langle e_{\vec{i}}^* \otimes e_{\vec{k}},\, e_{\vec{p}} \otimes e_{\vec{q}} \right\rangle \right|^2 \\
&= \sum_{\vec{p},\vec{q}} \left| \sum_{\vec{j},\vec{k}} \left\langle \kappa_{l+m},\, e_{\vec{p}} \otimes e_{\vec{j}} \right\rangle \left\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \right\rangle \left\langle e_{\vec{k}},\, e_{\vec{q}} \right\rangle \right|^2 .
\end{aligned}$$

Fixing $\vec{p}$, we continue computation:

$$\begin{aligned}
&\sum_{\vec{q}} \left| \sum_{\vec{j},\vec{k}} \left\langle \kappa_{l+m},\, e_{\vec{p}} \otimes e_{\vec{j}} \right\rangle \left\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \right\rangle \left\langle e_{\vec{k}},\, e_{\vec{q}} \right\rangle \right|^2 \\
&\quad = \sum_{\vec{q}} \left| \sum_{\vec{j},\vec{k}} \left\langle \left\langle \kappa_{l+m},\, e_{\vec{p}} \otimes e_{\vec{j}} \right\rangle \left\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \right\rangle e_{\vec{k}},\, e_{\vec{q}} \right\rangle \right|^2 \\
&\quad = \left| \sum_{\vec{j},\vec{k}} \left\langle \kappa_{l+m},\, e_{\vec{p}} \otimes e_{\vec{j}} \right\rangle \left\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \right\rangle e_{\vec{k}} \right|_{-p}^2 \\
&\quad \le \left| \sum_{\vec{j},\vec{k}} \left\langle \kappa_{l+m},\, e_{\vec{p}} \otimes e_{\vec{j}} \right\rangle \left\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \right\rangle e_{\vec{k}} \right|_{p}^2 \\
&\quad = \sum_{\vec{j},\vec{k}} \left| \left\langle \kappa_{l+m},\, e_{\vec{p}} \otimes e_{\vec{j}} \right\rangle \left\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \right\rangle \right|^2 \\
&\quad \le \sum_{\vec{k}} \sum_{\vec{j}} |\langle \kappa_{l+m},\, e_{\vec{p}} \otimes e_{\vec{j}} \rangle|^2 \sum_{\vec{j}} |\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \rangle|^2 .
\end{aligned}$$

Summing up over $\vec{p}$, we come to

$$\begin{aligned}
|\, \kappa_{l+m} \otimes_m f_{n+m} \,|_{-p}^2 &\le \sum_{\vec{p},\vec{j}} |\langle \kappa_{l+m},\, e_{\vec{p}} \otimes e_{\vec{j}} \rangle|^2 \sum_{\vec{k},\vec{j}} |\langle e_{\vec{k}}^* \otimes e_{\vec{j}}^*,\, f_{n+m} \rangle|^2 \\
&= |\, \kappa_{l+m} \,|_{-p}^2 \, |\, f_{n+m} \,|_{p}^2 .
\end{aligned}$$

This completes the proof of (4.3). In a similar way (4.4) is proved. ∎

4.3 Integral kernel operators

Lemma 4.4 *For each $\kappa \in (N^{\otimes(l+m)})^*$ there exists an operator $\Xi_{l,m}(\kappa) \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ whose symbol is given by*

$$\widehat{\Xi_{l,m}(\kappa)}(\xi, \eta) = \langle\!\langle \Xi_{l,m}(\kappa)\phi_\xi,\, \phi_\eta \rangle\!\rangle = \langle \kappa,\, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle\, e^{\langle \xi, \eta \rangle}, \qquad \xi, \eta \in N. \tag{4.5}$$

PROOF. We write $\Theta(\xi,\eta)$ for the righthand side of (4.5). It is sufficient to show that $\Theta \in \mathcal{G}_{\theta^*}(N \oplus N)$ by Theorem 3.5. Since $\langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle$ is of polynomial growth, it belongs to $\mathcal{G}_{\theta^*}(N \oplus N)$. From the nuclear triplet (4.2) we see that the identity operator I on $F_\theta(N)$ is a member of $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$, and hence $\widehat{I} \in \mathcal{G}_{\theta^*}(N \oplus N)$. Note that

$$\widehat{I}(\xi,\eta) = \langle\langle \phi_\xi, \phi_\eta \rangle\rangle = e^{\langle \xi, \eta\rangle}.$$

Since $\mathcal{G}_{\theta^*}(N \oplus N)$ is closed under pointwise multiplication (Proposition 1.8), we conclude that $\Theta \in \mathcal{G}_{\theta^*}(N \oplus N)$. ∎

It is noteworthy that the above proof is much simpler than the original proof, see e.g., [33], where the norm of $\Xi_{l,m}(\kappa)\phi$ is estimated directly. Below we record the norm estimate.

Let $\phi = (f_n) \in F_\theta(N)$. Consider a formal power series $\Phi = (g_n)$, where

$$g_n = 0, \quad 0 \le n < l; \qquad g_n = \frac{(n-l+m)!}{(n-l)!}\,\kappa \otimes_m f_{n-l+m}, \quad n \ge l.$$

Let us calculate the norm:

$$\begin{aligned}
\| \Phi \|^2_{-;p,\delta} &= \sum_{n=0}^{\infty} (n!\theta_n)^2 \delta^n \,| g_n |^2_{-p} \\
&= \sum_{n\ge l} (n!\theta_n)^2 \delta^n \left(\frac{(n-l+m)!}{(n-l)!}\right)^2 | \kappa \otimes_m f_{n-l+m} |^2_{-p} \\
&= \sum_{n=0}^{\infty} ((n+l)!\theta_{n+l})^2 \delta^{n+l} \left(\frac{(n+m)!}{n!}\right)^2 | \kappa \otimes_m f_{n+m} |^2_{-p}
\end{aligned} \tag{4.6}$$

Since $| \kappa \otimes_m f_{n+m} |_{-p} \le | \kappa |_{-p} \, | f_{n+m} |_p$ by Lemma 4.3, (4.6) becomes

$$\begin{aligned}
\| \Phi \|^2_{-;p,\delta} &\le \sum_{n=0}^{\infty} ((n+l)!\theta_{n+l})^2 \delta^{n+l} \left(\frac{(n+m)!}{n!}\right)^2 | \kappa |^2_{-p} \, | f_{n+m} |^2_p \\
&= \delta^{l+m} | \kappa |^2_{-p} \sum_{n=0}^{\infty} ((n+l)!\theta_{n+l}\theta_{n+m})^2 \delta^{2n} \left(\frac{(n+m)!}{n!}\right)^2 \\
&\qquad \times \theta_{n+m}^{-2} \delta^{-(n+m)} \, | f_{n+m} |^2_p .
\end{aligned} \tag{4.7}$$

Suppose that

$$M = M_{l,m}(\delta) \equiv \sup_{n \ge 0} \, (n+l)!\theta_{n+l}\theta_{n+m}\delta^n \frac{(n+m)!}{n!} < \infty.$$

Then (4.7) becomes

$$\| \Phi \|^2_{-;p,\delta} \le \delta^{l+m} | \kappa |^2_{-p} \, M_{l,m}(\delta)^2 \sum_{n=0}^{\infty} \theta_{n+m}^{-2} \delta^{-(n+m)} \, | f_{n+m} |^2_p ,$$

that is,

$$\| \Phi \|_{-;p,\delta} \le \delta^{(l+m)/2} | \kappa |_{-p} \, M_{l,m}(\delta) \, \| \phi \|_{+;p,\delta} .$$

This imples that $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$.

We shall prove that $M_{l,m}(\delta) < \infty$ for small $\delta > 0$. Take constant numbers $a > 0$ and $b > 0$ as in Lemma 4.1. Then,

$$\theta_{n+l} \le a\left(\frac{2be}{n+l}\right)^{(n+l)/2}, \qquad \theta_{n+m} \le a\left(\frac{2be}{n+m}\right)^{(n+m)/2},$$

and

$$\begin{aligned}(n+l)!\theta_{n+l}\theta_{n+m}\delta^n \frac{(n+m)!}{n!} \\ \le a^2(2be)^{(l+m)/2}(2be\delta)^n \frac{(n+l)!}{(n+l)^{(n+l)/2}} \frac{(n+m)!}{(n+m)^{(n+m)/2}} \frac{1}{n!}.\end{aligned} \tag{4.8}$$

Using the Stirling formula, we see that the right hand side of (4.8) is equivalent[f] to

$$\sqrt{2\pi}\, a^2(2b)^{(l+m)/2}(2b\delta)^n\{(n+l)(n+m)\}^{1/4}\left\{\frac{(n+l)!(n+m)!}{n!n!}\right\}^{1/2},$$

which goes to 0 as $n \to \infty$ whenever $0 < \delta < (2b)^{-1}$. Consequently, $M_{l,m}(\delta) < \infty$ for all $0 < \delta < (2b)^{-1}$.

4.4 Fock expansion

Theorem 4.5 *For each operator $\Xi \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ there exist an integral kernel operators $\Xi_{l,m}(\kappa_{l,m})$ with $\kappa_{l,m} \in (N^{\widehat{\otimes}(l+m)})^*$ such that*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}),$$

which converges in $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$.

Proof. Set $\Psi(\xi,\eta) = \widehat{\Xi}(\xi,\eta)e^{-\langle\xi,\eta\rangle}$. Since $\widehat{I}(\xi,\eta) = e^{\langle\xi,\eta\rangle}$ belongs to $\mathcal{G}_{\theta^*}(N \oplus N)$, so does $e^{-\langle\xi,\eta\rangle}$. Hence $\Psi \in \mathcal{G}_{\theta^*}(N \oplus N)$ and admits the Taylor expansion:

$$\Psi(\xi,\eta) = \sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle.$$

With these coefficients we define integral kernel operators $\Xi_{l,m}(\kappa_{l,m})$. These are what we looked for. The rest of the proof is just a routine. ∎

[f] Two positive sequences $\{a_n\}$ and $\{b_n\}$ are called equivalent if $\lim_{n\to\infty} a_n/b_n = 1$.

5 Applications

5.1 D. M. Chung's new products of white noise functions

Let $B : F_\theta(N) \times F_\theta(N) \to F_\theta(N)$ be a bilinear map assume that for any pair $\xi, \eta \in N$ there exists $\Theta(\xi, \eta) \in \mathbf{C}$ such that

$$B(\phi_\xi, \phi_\eta) = \Theta(\xi, \eta)\phi_{\xi+\eta}, \qquad \xi, \eta \in N,$$

The following result[g] is due to Chung [3].

Theorem 5.1 (1) *B is continuous if and only if $\Theta \in \mathcal{G}_{\theta^*}(N \oplus N)$.*

(2) *B is associative if and only if*

$$\Theta(\xi, \eta)\Theta(\xi + \eta, \zeta) = \Theta(\xi, \eta + \zeta)\Theta(\eta, \zeta), \qquad \xi, \eta, \zeta \in N. \tag{5.1}$$

We shall give examples of such $\Theta(\xi, \eta)$.

Lemma 5.2 *For a polynomial $H(u)$ define*

$$g(x, y) = \int_0^y (H(x + u) - H(u))du. \tag{5.2}$$

Then $g(x, y)$ is a polynomial satisfying

$$g(x, y) + g(x + y, z) = g(x, y + z) + g(y, z). \tag{5.3}$$

PROOF. It is sufficient to prove the assertion for $H(u) = u^n$. In that case we have

$$g(x, y) = \frac{1}{n+1}\left\{(x + y)^{n+1} - x^{n+1} - y^{n+1}\right\}.$$

Then, by a direct computaion we see that both sides of (5.3) become

$$\frac{1}{n+1}\left\{(x + y + z)^{n+1} - x^{n+1} - y^{n+1} - z^{n+1}\right\}.$$

This completes the proof. ∎

Remark 5.3 The above $g(x, y)$ is symmetric. In fact, suppose $x < y$. Then

$$\begin{aligned} g(x, y) &= \int_0^y (H(x + u) - H(u))du = \int_x^{x+y} H(u)du - \int_0^y H(u)du \\ &= \int_y^{x+y} H(u)du - \int_0^x H(u)du = \int_0^x H(y + u)du - \int_0^x H(u)du \\ &= \int_0^x (H(y + u) - H(u))\, du = g(y, x). \end{aligned}$$

Similar argumant is valid also for $x > y$.

[g] In fact, Chung presented the result within the standard framework of white noise calculus. Adaptation to our framework is straightforward.

For a polynomial $g(x,y) = \sum_{j,k} c_{jk} x^j y^k$ we shall define $\hat{g}(\xi,\eta)$ for $\xi,\eta \in N$. We first decompose g into a sum of homogeneous polynomials:

$$g(x,y) = \sum_{m=0}^{n} g_m(x,y), \qquad g_m(x,y) = \sum_{j+k=m} c_{jk} x^j y^k.$$

We then define

$$\hat{g}_m(\xi,\eta) = \sum_{j+k=m} c_{j,k}\, \xi^{\otimes j} \widehat{\otimes} \eta^{\otimes k}$$

which is a member of $N^{\widehat{\otimes} m}$. Finally, we set $\hat{g}(\xi,\eta) = \sum \hat{g}_m(\xi,\eta)$. Then for a formal power series $\Phi = (F_m)$ on N, i.e., $F_m \in (N^{\widehat{\otimes} m})^*$, $m = 0, 1, 2, \ldots$, we have

$$\langle \Phi, \hat{g} \rangle = \sum_{m=0}^{n} \langle F_m, \hat{g}_m \rangle .$$

Proposition 5.4 *Let $\Phi = (F_n)$ be a formal power series on N and g be a polynomial given as in* (5.2). *Then,*

$$\Theta(\xi,\eta) = e^{\langle \Phi, \hat{g}(\xi,\eta) \rangle}, \qquad \xi,\eta \in N, \tag{5.4}$$

satisfies (5.1).

Chung and Chung [4] introduced the γ-product of white noise functions, which is uniquely determined by

$$\phi_\xi \diamond_\gamma \phi_\eta = e^{\gamma\langle \xi,\eta \rangle} \phi_{\xi+\eta}, \qquad \xi,\eta \in N, \tag{5.5}$$

where $\gamma \in \mathbf{C}$. The product $\diamond_\gamma$ is reduced to the pointwise multiplication for $\gamma = 1$ and the Wick product for $\gamma = 0$. In this case, the function $\Theta(\xi,\eta)$ in (5.4) is given as

$$\Theta(\xi,\eta) = e^{\gamma\langle \xi,\eta \rangle} = e^{\langle \gamma\tau, \xi \widehat{\otimes} \eta \rangle},$$

hence $\Phi = \gamma\tau$ and $g(x,y) = xy$, where $\tau \in (N \otimes N)^*$ is the trace. The γ-product is related with the so-called $G_{\alpha,\beta}$-transform (generalized Gauss transform) and plays an interesting role in Cauchy problems for white noise functions.

Remark 5.5 The converse to Lemma 5.2 is valid in the following sense. Let $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a C^1-function satisfying (5.3). Then there exist a continuous function $H : \mathbf{R} \to \mathbf{R}$ and a constact $c \in \mathbf{R}$ such that

$$g(x,y) = c + \int_0^y (H(x+u) - H(u))du.$$

In fact, we first see from (5.3) that

$$g(x,0) = g(0,y) = c$$

is a constant for all $x, y \in \mathbf{R}$. Then, we have

$$\begin{aligned} g(x, y+z) - g(x, y) &= g(x+y, z) - g(y, z) \\ &= (g(x+y, z) - g(x+y, 0)) - (g(y, z) - g(y, 0)). \end{aligned} \tag{5.6}$$

Put

$$h(x, y) = \frac{\partial g}{\partial y}(x, y).$$

Then in (5.6), dividing by z and letting z tend to zero, we come to

$$h(x, y) = h(x+y, 0) - h(y, 0).$$

Define $H(u) = h(u, 0)$. Then $h(x, y) = H(x+y) - H(y)$ and

$$g(x, y) = g(x, 0) + \int_0^y h(x, u)du = c + \int_0^y (H(x+u) - H(u))du.$$

5.2 Wick product of white noise operators

By Theorem 3.5 we easily obtain the following

Lemma 5.6 *For two white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ there exists a unique operator $\Xi \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ such that*

$$\widehat{\Xi}(\xi, \eta) = \widehat{\Xi}_1(\xi, \eta)\widehat{\Xi}_2(\xi, \eta)e^{-\langle \xi, \eta \rangle}, \qquad \xi, \eta \in N. \tag{5.7}$$

The operator Ξ defined in (5.7) is called the *Wick product* or *normal-ordered product* of Ξ_1 and Ξ_2, and is denoted by $\Xi = \Xi_1 \diamond \Xi_2$. We note some simple properties:

$$\begin{aligned} I \diamond \Xi = \Xi \diamond I = \Xi, &\qquad (\Xi_1 \diamond \Xi_2) \diamond \Xi_3 = \Xi_1 \diamond (\Xi_2 \diamond \Xi_3), \\ (\Xi_1 \diamond \Xi_2)^* = \Xi_2^* \diamond \Xi_1^*, &\qquad \Xi_1 \diamond \Xi_2 = \Xi_2 \diamond \Xi_1. \end{aligned}$$

Thus, equipped with the Wick product, $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$ becomes a commutative $*$-algebra.

5.3 Normal-ordered white noise differential equations

A continuous map $t \mapsto L_t \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ defined on a time interval is called a *quantum stochastic process* in the sense of white noise theory. Given a quantum stochastic process $\{L_t\}$ defined on an interval containing 0, a linear equation for unknown quantum stochastic process $\{\Xi_t\}$ is formulated as follows:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \qquad \Xi(0) = I. \tag{5.8}$$

The above equation is generally called a *normal-ordered white noise differential equation.* Since the equation (5.8) is linear and $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$ is a commutative algebra with the Wick product, the formal solution to (5.8) is obtained by the Wick exponential:

$$\Xi_t = \text{wexp}\left(\int_0^t L_s\, ds\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_0^t L_s\, ds\right)^{\diamond n}. \tag{5.9}$$

A serious question is to show convergence of the above infinite series with respect to a certain topology and has been answered to some extent, see e.g., [9, 10, 34].

Now nonlinear extension is of great interest. We end this paper with a very simple example.

Lemma 5.7 *Let $\mathcal{L}_1$ be the set of Wick invertible elements. Then $\mathcal{L}_1$ is an open subset of $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$.*

PROOF. We first show that

$$\mathcal{L}_1 = \{\Xi \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)\,;\, \widehat{\Xi} \text{ has no zero}\}. \tag{5.10}$$

We note a straightforward equivalence:

$$\Xi_1 \diamond \Xi_2 = I \iff \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta) = e^{2\langle \xi,\eta\rangle}.$$

If $\widehat{\Xi}_1$ has no zero, $e^{\langle \xi,\eta\rangle}/\widehat{\Xi}_1(\xi,\eta)$ is entire. By using the division theorem due to Gannoun–Hachaichi–Krée–Ouerdiane [15], we see that

$$\frac{e^{\langle \xi,\eta\rangle}}{\widehat{\Xi}_1(\xi,\eta)} \tag{5.11}$$

belongs to $\mathcal{G}_{\theta^*}(N \oplus N)$. Hence there exists $\Xi_2 \in \mathcal{L}(F_\theta(N), F_\theta(N)^*)$ whose symbol is (5.11). This Ξ_2 is the Wick inverse of Ξ_1 and hence $\Xi_1 \in \mathcal{L}_1$. The converse is readily clear and (5.10) is shown. Since $\widehat{\mathcal{L}}_1 \subset \mathcal{G}_{\theta^*}(N \oplus N)$ is open, so is $\mathcal{L}_1$ in $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$. ∎

Let $\{L_t\}$ be a quantum stochastic process in $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$. Then,

$$\frac{d\Xi}{dt} = L_t \diamond \Xi \diamond \Xi, \qquad \Xi\big|_{t=0} = \Xi_0 \in \mathcal{L}_1,$$

has a unique solution in $\mathcal{L}(F_\theta(N), F_\theta(N)^*)$. In fact,

$$\Xi_t = \left(\Xi_0^{\diamond(-1)} - \int_0^t L_s\, ds\right)^{\diamond(-1)},$$

which is defined in a neighborhood of t. Note that $\Xi_t \in \mathcal{L}_1$.

Appendix: Young Function

For the sake of the readers' convenience we assemble some basic properties of a Young function. For more details see e.g., Krasnosel'skii–Rutickii [28].

A.1 Definition and integral representation

A function $\theta : [0,\infty) \to [0,\infty)$ is called a *Young function* if the following five conditions are satisfied:

(i) continuous;

(ii) convex, i.e., $\theta(tx_1 + (1-t)x_2) \le t\theta(x_1) + (1-t)\theta(x_2)$ for $0 \le t \le 1$, $x_1 \ge 0$, $x_2 \ge 0$;

(iii) increasing, i.e., $\theta(x_1) \le \theta(x_2)$ for $0 \le x_1 \le x_2$;

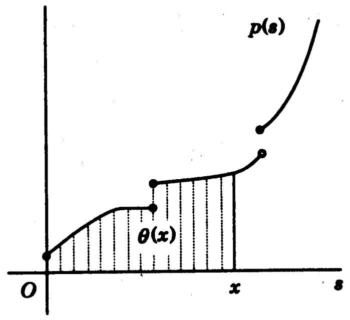
(iv) $\theta(0) = 0$;

(v) $\displaystyle\lim_{x\to\infty} \frac{\theta(x)}{x} = \infty$.

Theorem A.1 *A function $\theta : [0,\infty) \to [0,\infty)$ is a Young function if and only if it admits an expression*

$$\theta(x) = \int_0^x p(s)\,ds, \qquad x \ge 0, \tag{A.1}$$

where $p : [0,\infty) \to [0,\infty)$ satisfies



(i) *right continuous;*

(ii) *increasing;*

(iii) $p(0) \ge 0$;

(iv) $\displaystyle\lim_{s\to\infty} p(s) = \infty$.

In that case p is uniquely determined.

The proof is a slight modification of the argument in [28].

A.2 Polar function

For a Young function θ the *polar function* is defined by

$$\theta^*(x) = \sup\{t \ge 0\,;\, xt - \theta(t)\}.$$

It is shown that θ^* is again a Young function and $(\theta^*)^* = \theta$ holds. In fact, if θ is given as in (A.1), then

$$\theta^*(x) = \int_0^x q(s)\,ds, \qquad x \ge 0,$$

where

$$q(s) = \sup\{t \ge 0\,;\, p(t) \le s\}, \quad s \ge p(0); \qquad q(s) = 0, \quad 0 \le s < p(0).$$

This $q(s)$ is called a generalized inverse function of $p(s)$.

Theorem A.2 *It holds that*

$$st \le \theta^*(s) + \theta(t), \qquad s, t \ge 0. \tag{A.2}$$

The equality holds only when $t = q(s)$.

The proof is obvious from graphical consideration. (A.2) is referred to as the *Young inequality.*

Theorem A.3 *Let θ_1, θ_2 be two Young functions. If there exists $u_0 \geq 0$ such that $\theta_1(u) \leq \theta_2(u)$ for all $u \geq u_0$, there exists $v_0 \geq 0$ such that $\theta_1^*(v) \geq \theta_2^*(v)$ for all $v \geq v_0$.*

PROOF. Let

$$\theta_2(u) = \int_0^u p_2(t)dt, \qquad \theta_2^*(v) = \int_0^v q_2(s)ds$$

be the integral representations of θ_2 and of its polar function θ_2^*, respectively. We set $v_0 = p_2(u_0)$ and let $v \geq v_0$. Then, $q_2(v) \geq u_0$ and

$$q_2(v)v = \theta_2(q_2(v)) + \theta_2^*(v). \tag{A.3}$$

Moreover, by assumption we have $\theta_1(q_2(v)) \leq \theta_2(q_2(v))$. Hence (A.3) becomes

$$q_2(v)v \geq \theta_1(q_2(v)) + \theta_2^*(v). \tag{A.4}$$

On the other hand, by Young's inequality we have

$$q_2(v)v \leq \theta_1(q_2(v)) + \theta_1^*(v). \tag{A.5}$$

The assertion follows immediately by combining (A.4) and (A.5). ∎

Proposition A.4 *Let θ be a Young function. Then*

$$\limsup_{x\to\infty} \frac{\theta(x)}{x^2} < \infty \iff \liminf_{x\to\infty} \frac{\theta^*(x)}{x^2} > 0.$$

PROOF. Assume that $\liminf_{x\to\infty} \theta^*(x)/x^2 > 0$. Then there exist $x_0 > 0$ and $\epsilon > 0$ such that $\epsilon x^2 \leq \theta^*(x)$ for $x \geq x_0$. Note that $\theta_1(x) = \epsilon x^2$ is a Young function and its polar function is given by $\theta_1^*(x) = x^2/4\epsilon$. Then, by Theorem A.3 there exists $y_0 \geq 0$ such that $y^2/4\epsilon \geq \theta(y)$ for $y \geq y_0$. Hence

$$\limsup_{y\to\infty} \frac{\theta(y)}{y^2} \leq \frac{1}{4\epsilon} < \infty.$$

The converse is proved in a similar manner. ∎

A.3 Some properties of Young function

Let θ be a Young function.

Lemma A.5 (1) *$\alpha\theta(x) \geq \theta(\alpha x)$ for $0 \leq \alpha \leq 1$ and $x \geq 0$.*

(2) *$\beta\theta(x) \leq \theta(\beta x)$ for $\beta \geq 1$ and $x \geq 0$.*

PROOF. (1) Since θ is convex,

$$\theta(\alpha x + (1-\alpha)0) \leq \alpha\theta(x) + (1-\alpha)\theta(0) = \alpha\theta(x).$$

Hence $\alpha\theta(x) \geq \theta(\alpha x)$. (2) is immediate from (1) by variable change. ∎

Lemma A.6 $\theta\left(\frac{s}{2}\right) + \theta\left(\frac{t}{2}\right) \le \theta(s+t)$ *for* $s, t \ge 0$.

PROOF. For any $s, t \ge 0$ we have $s \le s+t$ and $\theta(s) \le \theta(s+t)$. Hence

$$\theta(s) + \theta(t) \le 2\theta(s+t) \le \theta(2s+2t),$$

where Lemma 5.10 is taken into account. ∎

Lemma A.7 *For* $0 < x \le y$ *we have*

$$\frac{\theta(x)}{x} \le \frac{\theta(y)}{y}. \tag{A.6}$$

PROOF. Consider the integral representation:

$$\theta(x) = \int_0^x p(u)du.$$

Since $p(u)$ is increasing, for $0 \le u \le x$ we have $p(u) \le p(x)$. Hence

$$\frac{1}{x}\int_0^x p(u)du \le p(x)$$

and for $x \le v$ we have

$$\frac{\theta(x)}{x} \le p(x) \le p(v).$$

Then, integrating by v over an interval $[x, y]$ we have

$$(y-x)\frac{\theta(x)}{x} \le \int_x^y p(v)dv = \int_0^y p(v)dv - \int_0^x p(v)dv = \theta(y) - \theta(x),$$

from which (A.6) follows. ∎

A.4 Properties of $\{\theta_n\}$

For a Young function θ we define a positive sequence $\{\theta_n\}$ by

$$\theta_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n}, \qquad n = 0, 1, 2, \cdots.$$

Lemma A.8 $\limsup_{n\to\infty} \theta_n^{1/n} = 0$.

PROOF. Since by definition $\theta_n \le e^{\theta(r)}/r^n$ for $r > 0$, we have

$$\limsup_{n\to\infty} \theta_n^{1/n} \le \limsup_{n\to\infty} \left(\frac{e^{\theta(r)}}{r^n}\right)^{1/n} = \limsup_{n\to\infty} \frac{e^{\theta(r)/n}}{r} = \frac{1}{r}.$$

Since $r > 0$ is arbitrary, we have the desired assertion. ∎

Lemma A.9 $\theta_n\theta_n^* = \left(\frac{e}{n}\right)^n$ *for* $n \geq 1$.

PROOF. By the Young inequality we have

$$\frac{e^{st}}{s^n t^n} \leq \frac{e^{\theta(s)}}{s^n}\frac{e^{\theta^*(t)}}{t^n}, \qquad s,t>0.$$

The minimum of the left hand side, where (s,t) runs over the region $\{s>0,t>0\}$, is easily obtained and is $(e/n)^n$. Hence

$$\left(\frac{e}{n}\right)^n \leq \inf_{s>0,t>0}\frac{e^{\theta(s)}}{s^n}\frac{e^{\theta^*(t)}}{t^n} = \theta_n\theta_n^*. \tag{A.7}$$

On the other hand, by definition,

$$\theta_n\theta_n^* \leq \frac{e^{\theta(s)}}{s^n}\frac{e^{\theta^*(t)}}{t^n} = \frac{e^{\theta(s)+\theta^*(t)}}{(st)^n}.$$

Consider a pair s,t satisfying $\theta(s)+\theta^*(t)=st$. This occurs only when $s=q(t)$, where $q(t)$ is an intgrand in the integral expression of θ^*. Then,

$$\theta_n\theta_n^* \leq \frac{e^{q(t)t}}{(q(t)t)^n}, \qquad t>0.$$

Since $q(t)t \to \infty$ as $t\to\infty$, we see that

$$\theta_n\theta_n^* \leq \inf_{r>0}\frac{e^r}{r^n} = \left(\frac{e}{n}\right)^n. \tag{A.8}$$

The assertion follows from (A.7) and (A.8). ∎

A.5 Generating function

Lemma A.10 *The power series*

$$\gamma_\theta(x) = \sum_{n=0}^{\infty}\frac{x^n}{(\theta_n n!)^2}$$

has an infinite radius of convergence.

PROOF. In view of Lemma A.9 we have

$$\left(\frac{1}{(\theta_n n!)^2}\right)^{1/n} = \left\{\left(\frac{n}{e}\right)^{2n}\frac{\theta_n^*}{(n!)^2}\right\}^{1/n} = \left(\frac{n^n}{e^n n!}\right)^{2/n}\theta_n^{*2/n}.$$

Using the Stirling formula: $n!e^n \sim \sqrt{2\pi n}n^n$, we see that

$$\limsup_{n\to\infty}\left(\frac{1}{(\theta_n n!)^2}\right)^{1/n} = \limsup_{n\to\infty}\left(\frac{1}{2\pi n}\right)^{1/n}\theta_n^{*2/n} = \limsup_{n\to\infty}\theta_n^{*2/n}.$$

The assertion follows by Lemma A.8. ∎

Lemma A.11 (1) $\theta_m\theta_n \le 2^{m+n}\theta_{m+n}$.

(2) $\theta_{m+n} \le 2^{m+n}\theta_m\theta_n$.

PROOF. (1) By definition

$$\theta_m \le \frac{e^{\theta(r)}}{r^m}, \qquad \theta_n \le \frac{e^{\theta(r)}}{r^n}, \qquad r > 0.$$

Hence

$$\theta_m\theta_n \le \frac{e^{2\theta(r)}}{r^{m+n}} \le \frac{e^{\theta(2r)}}{r^{m+n}} = 2^{m+n}\frac{e^{\theta(2r)}}{(2r)^{m+n}},$$

from which the assertion follows.

(2) Applying the above result (1) to the polar function, we come to

$$\theta^*_m\theta^*_n \le 2^{m+n}\theta^*_{m+n},$$

which is by Lemma A.9 equivalent to

$$\frac{e^{m+n}}{n^n m^m}\theta_m^{-1}\theta_n^{-1} \le 2^{m+n}\frac{e^{m+n}}{(m+n)^{m+n}}\theta_{m+n}^{-1}.$$

That is,

$$\theta_{m+n} \le 2^{m+n}\frac{m^m n^n}{(m+n)^{m+n}}\theta_m\theta_n \le 2^{m+n}\theta_m\theta_n.$$

This completes the proof. ∎

Proposition A.12 $\gamma_\theta\left(\frac{x}{8}\right)\gamma_\theta\left(\frac{y}{8}\right) \le \gamma_\theta(x+y) \le \gamma_\theta(4x)\gamma_\theta(4y)$ *for* $x, y \ge 0$.

PROOF. This is immediate from Lemma A.11 and $\binom{n}{k} \le 2^{n-1}$. ∎

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