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Infinite Dimensional Cauchy Process Associated with the Lévy Laplacian

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1 Introduction

Let $E \equiv \mathcal{S}(\mathbf{R})$ be the Schwartz space consisting of rapidly decreasing functions defined on $\mathbf{R}$, where $\mathbf{R}$ means the set of all real numbers. Fix a finite interval T of $\mathbf{R}$ and take $\eta_T \in E$ satisfing $\eta_T = \frac{1}{|T|}$ on T. Let $\{X_t^m\}_{t\geq 0}, m \in \mathbf{N}$, be an independent sequence of Cauchy processes such that the characteristic function of each X_t^m is given by $E[e^{izX_t^m}] = e^{-a_m t|z|}$ for all $z \in \mathbf{R}$, where $\mathbf{N}$ is the set of natural numbers and $\{a_n\}_{n=1}^{\infty}$ is some sequence in $\mathbf{R} \setminus \mathbf{Q}$, $\mathbf{Q}$ being the set of all rational numbers. Then an infinite dimensional stochastic process $\{(\xi_m + X_t^m \eta_T)_{m\in\mathbf{N}}; t \geq 0\}, \xi_m \in E_{\mathbf{C}}, m = 1, 2, \dots$, has the Lévy Laplacian as the infinitesimal generator on some domain consisting of S-transforms of generalized white noise functionals.

In this paper we give the domain of the Lévy Laplacian and prove that the Laplacian on the domain is the infinitesimal generator of the above stochastic process.

The paper is organized as follows. In Section 2 we summarize some basic definitions and results in white noise theory. In Section 3 we introduce the definition of the Lévy Laplacian acting on vectors of generalized white noise functionals and give an equi-continuous semi-group of class (C_0) generated by the Laplacian. In Section 4 we give an infinite dimensional Cauchy process generated by the Lévy Laplacian.

2 Generalized white noise functionals

In this section we assemble some basic notations of white noise analysis following [6, 9, 15, 18, 23, 27].

We take the space $E^* \equiv \mathcal{S}'(\mathbf{R})$ of tempered distributions with the standard Gaussian measure μ such that

$$\int_{E^*} \exp\{i\langle x, \xi\rangle\}\, d\mu(x) = \exp\left(-\frac{1}{2}|\xi|_0^2\right), \qquad \xi \in E \equiv \mathcal{S}(\mathbf{R}),$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E^* \times E$ and $|\cdot|_0$ is the $L^2(\mathbf{R})$-norm.

Let $A = -(d/du)^2 + u^2 + 1$. Then this is a densely defined self-adjoint operator on $L^2(\mathbf{R})$ and there exists an orthonormal basis $\{e_\nu; \nu \geq 0\} \subset E$ for $L^2(\mathbf{R})$ such that $Ae_\nu = 2(\nu+1)e_\nu$. Define the norm $|\cdot|_p$ by $|f|_p = |A^p f|_0$ for $f \in E$ and $p \in \mathbf{R}$, and let E_p be the completion of E with respect to the norm $|\cdot|_p$. Then E_p is a real separable Hilbert space with the norm $|\cdot|_p$ and the dual space E_p' of E_p is the same as E_{-p} (see [12]). The space E is the projective limit space of $\{E_p; p \geq 0\}$ and the space E^* is the dual space of E. We denote the complexifications of $L^2(\mathbf{R})$, E and E_p by $L^2_{\mathbf{C}}(\mathbf{R})$, $E_{\mathbf{C}}$ and $E_{\mathbf{C},p}$, respectively.

The space $(L^2) = L^2(E^*, \mu)$ of complex-valued square-integrable functionals defined on E^* admits the well-known Wiener-Itô decomposition:

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where H_n is the space of multiple Wiener integrals of order $n \in \mathbf{N}$ and $H_0 = \mathbf{C}$. Let $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$ denote the n-fold symmetric tensor product of $L^2_{\mathbf{C}}(\mathbf{R})$. If $\varphi \in (L^2)$ is represented by $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n),\ f_n \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$, then the (L^2)-norm $\|\varphi\|_0$ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^{\infty} n!|f_n|_0^2\right)^{1/2},$$

where $|\cdot|_0$ means also the norm of $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$.

For $p \in \mathbf{R}$, let $\|\varphi\|_p = \|\Gamma(A)^p \varphi\|_0$, where $\Gamma(A)$ is the second quantization operator of A. If $p \geq 0$, let $(E)_p$ be the domain of $\Gamma(A)^p$. If $p < 0$, let $(E)_p$ be the completion of (L^2) with respect to the norm $\|\cdot\|_p$. Then $(E)_p$, $p \in \mathbf{R}$, is a Hilbert space with the norm $\|\cdot\|_p$. It is easy to see that for $p > 0$, the dual space $(E)_p^*$ of $(E)_p$ is given by $(E)_{-p}$. Moreover, for any $p \in \mathbf{R}$, we have the decomposition

$$(E)_p = \bigoplus_{n=0}^{\infty} H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{\mathbf{I}_n(f); f \in E_{\mathbf{C}}^{\hat{\otimes} n}\}$ with respect to $\|\cdot\|_p$. Here $E_{\mathbf{C}}^{\hat{\otimes} n}$ is the n-fold symmetric tensor product of $E_{\mathbf{C}}$. We also have $H_n^{(p)} = \{\mathbf{I}_n(f); f \in E_{\mathbf{C},p}^{\hat{\otimes} n}\}$ for any $p \in \mathbf{R}$, where $E_{\mathbf{C},p}^{\hat{\otimes} n}$ is also the n-fold symmetric tensor product of $E_{\mathbf{C},p}$. The norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)_p$ is given by

$$\|\varphi\|_p = \left(\sum_{n=0}^{\infty} n!|f_n|_p^2\right)^{1/2}, \qquad f_n \in E_{\mathbf{C},p}^{\hat{\otimes} n},$$

where the norm of $E_{\mathbf{C},p}^{\hat{\otimes} n}$ is denoted also by $|\cdot|_p$.

The projective limit space (E) of spaces $(E)_p$, $p \in \mathbf{R}$ is a nuclear space. The inductive limit space $(E)^*$ of spaces $(E)_p, p \in \mathbf{R}$ is nothing but the dual space of (E). The space $(E)^*$ is called the space of *generalized white noise functionals.* We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical bilinear form on $(E)^* \times (E)$. Then we have

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle$$

for any $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in (E)^*$ and $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (E)$, where the canonical bilinear form on $(E_{\mathbf{C}}^{\hat{\otimes} n})^* \times (E_{\mathbf{C}}^{\hat{\otimes} n})$ is denoted also by $\langle \cdot, \cdot \rangle$.

Since $\varphi_\xi \equiv \exp\{\langle \cdot, \xi \rangle - \frac{1}{2}\langle \xi, \xi \rangle\} \in (E)$, the *S-transform* is defined on $(E)^*$ by

$$S[\Phi](\xi) = \langle\langle \Phi, \varphi_\xi \rangle\rangle, \qquad \xi \in E_{\mathbf{C}}.$$

A complex-valued function F on $E_{\mathbf{C}}$ is called a *U-functional* if for every $\xi, \eta \in E_{\mathbf{C}}$, the function $z \to F(\xi + z\eta)$, $z \in \mathbf{C}$, is an entire function of z and there exist non-negative constants K, a and p such that

$$|F(\xi)| \leq K \exp\{a|\xi|_p^2\}, \quad \xi \in E_{\mathbf{C}}.$$

Theorem 1 (see e.g. [9, 18, 23, 27]) *A complex-valued function F on $E_{\mathbf{C}}$ is the S-transform of an element in $(E)^*$ if and only if F is a U-functional.*

3 The Lévy Laplacian acting on vectors of generalized white noise functionals

Let $F \in S[(E)^*]$. Then, by Theorem 1, we see that for any $\xi, \eta \in E_{\mathbf{C}}$ the function $F(\xi + z\eta)$ is an entire function of $z \in \mathbf{C}$. Hence we have the series expansion:

$$F(\xi + z\eta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} F^{(n)}(\xi)(\eta, \cdots, \eta),$$

where $F^{(n)}(\xi) : E_{\mathbf{C}} \times \cdots \times E_{\mathbf{C}} \to \mathbf{C}$ is a continuous n-linear functional.

We fix a finite interval T in $\mathbf{R}$. Take an orthonormal basis $\{\zeta_n\}_{n=0}^{\infty} \subset E$ for $L^2(T)$ satisfying the equally dense and uniform boundedness property (see e.g. [18, 19]). Let $\mathcal{D}_L$ denote the set of all $\Phi \in (E)^*$ such that the limit

$$\widetilde{\Delta}_L S[\Phi](\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} S[\Phi]''(\xi)(\zeta_n, \zeta_n)$$

exists for any $\xi \in E_{\mathbf{C}}$ and is in $S[(E)^*]$. The Lévy Laplacian Δ_L is defined by

$$\Delta_L \Phi = S^{-1} \widetilde{\Delta}_L S \Phi$$

for $\Phi \in D_L$. We denote the set of all functionals $\Phi \in \mathcal{D}_L$ such that $S[\Phi](\eta) = 0$ for all $\eta \in E$ with $\mathrm{supp}(\eta) \subset T^c$ by $\mathcal{D}_L^T$.

Take a generalized white noise functional

$$\Phi = \int_{T^n} f(u_1, \cdots, u_n) : e^{iax(u_1)} \cdots e^{iax(u_n)} : d\mathbf{u}, \quad f \in E_{\mathbf{C}}^{\hat{\otimes} n}, a \in \mathbf{R},$$

which its S-transform $S[\Phi]$ is given by

$$S[\Phi](\xi) = \int_{T^n} f(\mathbf{u}) e^{ia\xi(u_1)} \cdots e^{ia\xi(u_n)} d\mathbf{u}.$$

We put

$$\mathbf{D}_n^a = \left\{ \int_{T^n} f(\mathbf{u}) : \prod_{\nu=1}^{n} e^{iax(u_\nu)} : d\mathbf{u}; f \in E_{\mathbf{C}}^{\hat{\otimes} n}, \ \mathrm{supp} f \subset T^n \right\}$$

for each $n \in \mathbf{N} \cup \{0\}$. Then $\mathbf{D}_n^a$ is a linear subspace of $(E)_{-p}$ for any $p > \frac{5}{12}$. (See [20].) We define a space $\overline{\mathbf{D}_n^a}$ by the completion of $\mathbf{D}_n^a$ in $(E)_{-p}$ with respect to $\|\cdot\|_{-p}$. Then for each

$n \in \mathbf{N} \cup \{0\}$ and $a \in \mathbf{R}$, $\overline{\mathbf{D}_n^a}$ becomes a Hilbert space with the inner product of $(E)_{-p}$. Using the simillar method in [33], we get the following:

Theorem 2 (cf. [33], see also [20, 30]) *For each $n \in \mathbf{N} \cup \{0\}$ and $a \in \mathbf{R}$, the operator Δ_L becomes a continuous linear operator from $\overline{\mathbf{D}_n^a}$ into itself such that*

$$\Delta_L \Phi = -\frac{na^2}{|T|}\Phi \text{ for any } \Phi \in \overline{\mathbf{D}_n^a}.$$

Proposition 3 (cf. [33]) *Let $\Phi = \sum_{n=0}^{\infty} \Phi_n$ and $\Psi = \sum_{n=0}^{\infty} \Psi_n$ be generalized white noise functionals such that Φ_n and Ψ_n are in $\overline{\mathbf{D}_n^a}$ for each $a \in \mathbf{R}$ and $n \in \mathbf{N} \cup \{0\}$. If $\Phi = \Psi$ in $(E)^*$, then $\Phi_n = \Psi_n$ for all $n \in \mathbf{N} \cup \{0\}$.*

The operator Δ_L is a self-adjoint operator on $\overline{\mathbf{D}_n^a}$ for each $n \in \mathbf{N} \cup \{0\}$ and $a \in \mathbf{R}$.

Put $\alpha_N^a(n) = \sum_{\ell=0}^{N} \left(\frac{na^2}{|T|}\right)^{2\ell}$ and define a space $\mathbf{E}_{-p,N}^a$ by

$$\mathbf{E}_{-p,N}^a = \left\{ \sum_{n=1}^{\infty} \Phi_n \in (E)^*; \sum_{n=1}^{\infty} \alpha_N^a(n) \|\Phi_n\|_{-p}^2 < \infty, \Phi_n \in \overline{\mathbf{D}_n^a}, n = 0, 1, 2, \ldots \right\}$$

with the norm $|||\cdot|||_{-p,N,a}$ given by

$$|||\Phi|||_{-p,N,a} = \left(\sum_{n=1}^{\infty} \alpha_N^a(n) \|\Phi_n\|_{-p}^2 \right)^{1/2}, \quad \Phi = \sum_{n=1}^{\infty} \Phi_n \in \mathbf{E}_{-p,N}^a$$

for each $N \in \mathbf{N}, p > \frac{5}{12}$ and $a \in \mathbf{R}$. Then for any $N \in \mathbf{N}, p > \frac{5}{12}$ and $a \in \mathbf{R}$, $\mathbf{E}_{-p,N}^a$ is in $(E)_{-p}$ and is a Hilbert space with respect to the norm $|||\cdot|||_{-p,N,a}$.

Put $\mathbf{E}_{-p,\infty}^a = \bigcap_{N=1}^{\infty} \mathbf{E}_{-p,N}^a$ with projective limit topology. Then we have the following inclusion relations:

$$\mathbf{E}_{-p,\infty}^a \subset \cdots \subset \mathbf{E}_{-p,N+1}^a \subset \mathbf{E}_{-p,N}^a \subset \cdots \subset \mathbf{E}_{-p,1}^a \subset (E)_{-p}.$$

The space $\mathbf{E}_{-p,\infty}^a$ includes $\overline{\mathbf{D}_n^a}$ for any $n \in \mathbf{N} \cup \{0\}$ and $m \in \mathbf{N}$. The operator Δ_L becomes a continuous linear operator defined on $\mathbf{E}_{-p,2}^a$ into $\mathbf{E}_{-p,1}^a$ satisfying $|||\Delta_L\Phi|||_{-p,N,a} \leq |||\Phi|||_{-p,N+1,a}$, $N = 1, 2, 3, \ldots$, $\Phi \in \mathbf{E}_{-p,\infty}^a$. With these properties, we have the following:

Theorem 4 ([20, 33]) *The operator Δ_L is a self-adjoint operator densely defined on $\mathbf{E}_{-p,N}^a$ for each $N \in \mathbf{N}$ and $p > \frac{5}{12}$ and $a \in \mathbf{R}$.*

For each $t \geq 0$ and $a \in \mathbf{R}$ we consider an operator G_t^a on $\mathbf{E}_{-p,\infty}^a$ defined by

$$G_t^a \Phi = \sum_{n=1}^{\infty} e^{-t\frac{na^2}{|T|}} \Phi_n$$

for $\Phi = \sum_{n=1}^{\infty} \Phi_n \in \mathbf{E}_{-p,\infty}^a$. Then we have the following:

Theorem 5 (cf. [20, 31]) *For each $a \in \mathbf{R}$ the family $\{G_t^a; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by Δ_L as a continuous linear operator defined on $\mathbf{E}^a_{-p,\infty}$.*

Proposition 6 *Let $\{a_n\}_{n=1}^\infty$ be a sequence in $\mathbf{R} \setminus \mathbf{Q}$ such that $a_\ell \neq a_m$ holds for any $\ell, m \in \mathbf{N}, \ell \neq m$. Then $\mathbf{E}^{a_\ell}_{-p,N} \cap \mathbf{E}^{a_m}_{-p,N} = \{0\}$ holds for each $N \in \mathbf{N}$.*

Proof Let $\ell, m \in \mathbf{N}, \ell \neq m$. Take an element Φ in $\mathbf{E}^{a_\ell}_{-p,N} \cap \mathbf{E}^{a_m}_{-p,N}$. Then the functional Φ has two expressions:

$$\Phi = \sum_{n=1}^\infty \Phi_n = \sum_{n=1}^\infty \Psi_n, \quad \Phi_n \in \overline{\mathbf{D}_n^{a_\ell}}, \Psi_n \in \overline{\mathbf{D}_n^{a_m}}.$$

For each $n \in \mathbf{N}$, Φ_n and Ψ_n are expressed in the forms

$$\Phi_n = \lim_{N\to\infty} \int_{T^n} f_n^{[N]}(u_1, \dots, u_n) : \prod_{\nu=1}^n e^{ia_\ell x(u_\nu)} : d\mathbf{u}$$

and

$$\Psi_n = \lim_{N\to\infty} \int_{T^n} g_n^{[N]}(u_1, \dots, u_n) : \prod_{\nu=1}^n e^{ia_m x(u_\nu)} : d\mathbf{u},$$

respectively, where $f_n^{[N]} \in E_{\mathbf{C}}^{\hat{\otimes} n}$ and $g_n^{[N]} \in E_{\mathbf{C}}^{\hat{\otimes} n}$. Take $\xi_T \in E_{\mathbf{C}}$ with $\xi_T = 1$ on T. Put $\xi = \alpha\xi_T + \eta$ for arbitrarily fixed $\alpha \in \mathbf{C}$. Then, by taking the S-transforms, we have

$$\begin{aligned}&\sum_{n=1}^\infty e^{ina_\ell\alpha} \lim_{N\to\infty} \int_{T^n} f_n^{[N]}(u_1, \dots, u_n) \prod_{\nu=1}^n e^{ia_\ell \eta(u_\nu)} d\mathbf{u} \\ &= \sum_{n=1}^\infty e^{ina_m\alpha} \lim_{N\to\infty} \int_{T^n} g_n^{[N]}(u_1, \dots, u_n) \prod_{\nu=1}^n e^{ia_m \eta(u_\nu)} d\mathbf{u}.\end{aligned}$$

This implies $\Phi_n = \Psi_n = 0$ for all $n \in \mathbf{N}$, i.e., $\Phi = 0$. □

By Proposition 6, for $N \in \mathbf{N}$ we can define a space $\mathcal{E}_{-p,N}$ by

$$\mathcal{E}_{-p,N} = \left\{ (\Phi^m)_{m\in\mathbf{N}}; \Phi^m = \sum_{n=1}^\infty \Phi_n^m \in \mathbf{E}^{a_m}_{-p,N}, \forall m, \sum_{m=1}^\infty |||\Phi^m|||^2_{-p,N,m} < \infty \right\}$$

with norm given by

$$|||\Phi|||_{\mathcal{E}_{-p,N}} = \left(\sum_{m=1}^\infty |||\Phi^m|||^2_{-p,N,m} \right)^{1/2}$$

for $\Phi \in \mathcal{E}_{-p,N}$. Then for each $N \in \mathbf{N}$, we have $\mathcal{E}_{-p,N+1} \subset \mathcal{E}_{-p,N}$. Put $\mathcal{E}_{-p,\infty} = \bigcap_{N=1}^\infty \mathcal{E}_{-p,N}$ with projective limit topology.

The Lévy Laplacian Δ_L is defined on the space $\mathcal{E}_{-p,\infty}$ by

$$\Delta_L \Phi = (\Delta_L \Phi^m)_{m\in\mathbf{N}} \quad \text{for each } \Phi = (\Phi^m)_{m\in\mathbf{N}} \in \mathcal{E}_{-p,\infty}.$$

The operator Δ_L becomes a continuous linear operator from $\mathcal{E}_{-p,\infty}$ into itself. For any $t \geq 0$, we introduce an operator G_t on $\mathcal{E}_{-p,\infty}$ by

$$G_t \Phi = (G_t^{a_m} \Phi^m)_{m\in\mathbf{N}} \quad \text{for each} \quad \Phi = (\Phi^m)_{m\in\mathbf{N}} \in \mathcal{E}_{-p,\infty}.$$

Then we have the following:

Theorem 7 *The family $\{G_t; t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by Δ_L as a continuous linear operator defined on $\mathcal{E}_{-p,\infty}$.*

Proof Let $p > \frac{5}{12}$ be a number arbitrarily fixed. For any $t \geq 0$ and $N \in \mathbf{N}$, the norm $|||G_t\Phi|||_{\mathcal{E}_{-p,N}}$ for $\Phi = (\Phi^m)_{m\in\mathbf{N}} \in \mathcal{E}_{-p,N}$ can be estimated as follows:

$$|||G_t\Phi|||^2_{\mathcal{E}_{-p,N}} = \sum_{m=1}^{\infty} |||G_t^{a_m}\Phi^m|||^2_{-p,N,m} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_N^{a_m}(n)||e^{-t\frac{na_m^2}{|T|}}\Phi_n^m||^2_{-p}$$
$$\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_N^{a_m}(n)||\Phi_n^m||^2_{-p} = \sum_{m=1}^{\infty} |||\Phi^m|||^2_{-p,N,m} = |||\Phi|||^2_{\mathcal{E}_{-p,N}}.$$

Hence the family $\{G_t; t \geq 0\}$ is equi- continuous in t. By Theorem 5, for any $t, s \geq 0$, and $\Phi = (\Phi^m)_{m\in\mathbf{N}} \in \mathcal{E}_{-p,N}$, we get

$$G_tG_s\Phi = (G_t^{a_m}G_s^{a_m}\Phi^m)_{m\in\mathbf{N}} = (G_{t+s}^{a_m}\Phi^m)_{m\in\mathbf{N}} = G_{t+s}\Phi$$

and

$$G_0\Phi = (G_0^{a_m}\Phi^m)_{m\in\mathbf{N}} = (\Phi^m)_{m\in\mathbf{N}} = \Phi.$$

Since

$$|||(G_t^{a_m} - G_s^{a_m})\Phi^m|||^2_{-p,N,m} \leq 4|||\Phi^m|||^2_{-p,N,m}, \quad m = 1, 2, \dots,$$

and

$$\sum_{m=1}^{\infty} |||\Phi^m|||^2_{-p,N,m} < \infty,$$

by Theorem 5 we have

$$|||G_t\Phi - G_{t_0}\Phi|||^2_{\mathcal{E}_{-p,N}} = \sum_{m=1}^{\infty} |||(G_t^{a_m} - G_s^{a_m})\Phi^m|||^2 \to 0,$$

as $t \to t_0$ for each $t, t_0 \geq 0, N \geq 1$ and $\Phi = (\Phi^m)_{m\in\mathbf{N}} \in \mathcal{E}_{-p,\infty}$. Thus the family $\{G_t; t \geq 0\}$ is an equi-continuous semigroup of class (C_0). We next prove that the infinitesimal generator of the semigroup is given by Δ_L. For $N \geq 1$ we see that

$$\left|\left|\left|\frac{G_t\Phi - \Phi}{t} - \Delta_L\Phi\right|\right|\right|^2_{\mathcal{E}_{-p,N}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_N^{a_m}(n)\left|\left|\left(\frac{e^{-t\frac{na_m^2}{|T|}} - 1}{t} + \frac{na_m^2}{|T|}\right)\Phi_n^m\right|\right|^2_{-p}. \tag{3.1}$$

Since $\Phi = (\Phi^m)_{m\in\mathbf{N}} \in \mathcal{E}_{-p,\infty}$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_{N+1}^{a_m}(n)||\Phi_n^m||^2_{-p} < \infty. \tag{3.2}$$

By the mean value theorem, for any $t > 0$ there exists a constant $\theta \in (0,1)$ such that

$$\left|\frac{e^{-t\frac{na_m^2}{|T|}} - 1}{t}\right| = \frac{na_m^2}{|T|}e^{-t\theta\frac{na_m^2}{|T|}} < \frac{na_m^2}{|T|}.$$

Therefore we can estimate each term in (3.1) as follows.

$$\begin{aligned}\alpha_N^{a_m}(n)\left\|\left(\frac{e^{-t\frac{na_m^2}{|T|}} - 1}{t} + \frac{na_m^2}{|T|}\right)\Phi_n^m\right\|_{-p}^2 &= \alpha_N^{a_m}(n)\left|\frac{e^{-t\frac{na_m^2}{|T|}} - 1}{t} + \frac{na_m^2}{|T|}\right|^2 \|\Phi_n^m\|_{-p}^2 \\ &\leq 4\alpha_{N+1}^{a_m}(n)\|\Phi_n^m\|_{-p}^2.\end{aligned}$$

By (3.2),

$$\lim_{t\to 0}\left|\frac{e^{-t\frac{na_m^2}{|T|}} - 1}{t} + \frac{na_m^2}{|T|}\right| = 0$$

and the Lebesgue convergence theorem, we obtain

$$\lim_{t\to 0}\left|\left|\left|\frac{G_t\Phi - \Phi}{t} - \Delta_L\Phi\right|\right|\right|_{\mathcal{E}_{-p,N}}^2 = 0.$$

Thus the proof is completed. □

4 An infinite dimensional stochastic process associated with the Lévy Laplacian

For any $p \in \mathbf{R}$ we define a space $E_{\mathbf{C},p}^{\infty}$ by

$$E_{\mathbf{C},p}^{\infty} = \{(\xi_m)_{m\in\mathbf{N}}; \xi_m \in E_{\mathbf{C},p}, \forall m\}.$$

This space is a linear space and a complete metric space with metric d_p given by

$$d_p(\xi,\eta) = \sum_{m=1}^{\infty}\frac{1}{2^m}\frac{|\xi_m - \eta_m|_p}{1 + |\xi_m - \eta_m|_p}$$

for $\xi = (\xi_m)_{m\in\mathbf{N}}, \eta = (\eta_m)_{m\in\mathbf{N}} \in E_{\mathbf{C},p}^{\infty}$. We also define a space $\mathbf{C}^{\infty}$ by

$$\mathbf{C}^{\infty} = \{(z_m)_{m\in\mathbf{N}}; z_m \in \mathbf{C}, \forall m\}$$

with metric

$$\rho(\mathbf{z},\mathbf{w}) = \sum_{m=1}^{\infty}\frac{1}{2^m}\frac{|z_m - w_m|}{1 + |z_m - w_m|}$$

for $\mathbf{z} = (z_m)_{m\in\mathbf{N}}, \mathbf{w} = (w_m)_{m\in\mathbf{N}} \in \mathbf{C}^{\infty}$. This space is also a linear space and a complete metric space with metric ρ.

Since $d_p \leq d_q$ if $p \geq q$, we introduce the projective limit topology to $E_{\mathbf{C}}^{\infty} \equiv \bigcap_p E_{\mathbf{C},p}^{\infty}$. The S-transform can be extended to a continuous linear operator on $\mathcal{E}_{-p,\infty}$ by

$$S\Phi(\xi) = (S\Phi^m(\xi_m))_{m\in\mathbf{N}}, \quad \xi = (\xi_m)_{m\in\mathbf{N}} \in E_{\mathbf{C}}^{\infty},$$

for any $\Phi = (\Phi^m)_{m\in\mathbf{N}} \in \mathcal{E}_{-p,\infty}$. The space $S[\mathcal{E}_{-p,\infty}]$ is endowed with the topology induced from $\mathcal{E}_{-p,\infty}$ by the S-transform. Then the S-transform becomes a homeomorphism from $\mathcal{E}_{-p,\infty}$ onto $S[\mathcal{E}_{-p,\infty}]$. The transform $S\Phi$ of $\Phi \in \mathcal{E}_{-p,\infty}$ is a continuous operator from $E_{\mathbf{C}}^{\infty}$ into $\mathbf{C}^{\infty}$. We denote the operator by the same notation $S\Phi$.

Let $\widetilde{G_t}$ be an operator defined on $S[\mathcal{E}_{-p,\infty}]$ by

$$\widetilde{G_t} = SG_tS^{-1}$$

for $t \geq 0$. Then by Theorem 7, $\{\widetilde{G_t};\, t \geq 0\}$ is an equi-continuous semigroup of class (C_0) generated by the operator $\widetilde{\Delta_L}$.

Let $\{X_t^m\}_{t\geq 0}$ be an independent sequence of the Cauchy processes such that the characteristic function of each X_t^m is given by $E[e^{izX_t^m}] = e^{-a_m t|z|}$ for all $z \in \mathbf{R}$. Take a smooth function $\eta_T \in E$ with $\eta_T = \frac{1}{|T|}$ on T. Define an infinite dimensional stochastic process $\{\mathbf{X}_t; t \geq 0\}$ starting at $\xi = (\xi_m)_{m\in\mathbf{N}} \in E_{\mathbf{C}}^{\infty}$ by

$$\mathbf{X}_t = (\xi_m + X_t^m \eta_T)_{m\in\mathbf{N}}, \quad t \geq 0.$$

Then this is an $E_{\mathbf{C}}^{\infty}$-valued stochastic process and we have the following:

Theorem 8 *Let F be the S-transform in $S[\mathcal{E}_{-p,\infty}]$. Then the equality*

$$\widetilde{G_t}F(\xi) = E[F(\mathbf{X}_t)|\mathbf{X}_0 = \xi]$$

holds for $t \geq 0$.

Proof We first consider the case when $F \in S[\mathcal{E}_{-p,\infty}]$ is given by

$$F(\xi) = (F^m(\xi_m))_{m\in\mathbf{N}}, \quad F^m(\xi_m) = \int_{T^n} f(\mathbf{u}) e^{ia_m\xi_m(u_1)} \cdots e^{ia_m\xi_m(u_n)} d\mathbf{u}$$

with $f \in E_{\mathbf{C}}^{\hat{\otimes} n}$. Then we have

$$\begin{aligned} E[F(\mathbf{X}_t|\mathbf{X}_0 = \xi)] &= (E[F^m(\xi_m + X_t^m \eta_T)])_{m\in\mathbf{N}} \\ &= \left(\int_{T^n} f(\mathbf{u}) e^{ia_m\xi_m(u_1)} \cdots e^{ia_m\xi_m(u_n)} E[e^{ina_m X_t^m}] d\mathbf{u}\right)_{m\in\mathbf{N}} \\ &= \left(e^{-t\frac{na_m^2}{|T|}} F^m(\xi_m)\right)_{m\in\mathbf{N}} \\ &= (\widetilde{G_t^{a_m}} F^m(\xi_m))_{m\in\mathbf{N}} \\ &= \widetilde{G_t}F(\xi). \end{aligned}$$

Next let $F = (\sum_{n=1}^{\infty} F_n^m)_{m\in\mathbf{N}} \in S[\mathcal{E}_{-p,\infty}]$. Then for any $m, n \in \mathbf{N}$, F_n^m is expressed in the following form:

$$F_n^m(\xi_m) = \lim_{N\to\infty} \int_{T^n} f_n^{[N]}(\mathbf{u}) e^{ia_m\xi_m(u_1)} \cdots e^{ia_m\xi_m(u_n)} d\mathbf{u},$$

where $(f_n^{[N]})_N$ is a sequence of functions in $E_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$. Hence, for each $m \in \mathbf{N}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} E[|F_n^m(\xi_m + X_t^m \eta_T)|] &= \sum_{n=1}^{\infty} E\left[\lim_{N\to\infty}\left|\int_{T^n} f_n^{[N]}(\mathbf{u})\prod_{\nu=1}^{n} e^{ia_m \xi_m(u_\nu)} e^{ia_m X_t^m \eta_T(u_\nu)} d\mathbf{u}\right|\right] \\ &= \sum_{n=1}^{\infty} \lim_{N\to\infty}\left|\int_{T^n} f_n^{[N]}(\mathbf{u})\prod_{\nu=1}^{n} e^{ia_m \xi_m(u_\nu)} d\mathbf{u}\right| \\ &= \sum_{n=1}^{\infty} |F_n^m(\xi_m)|.\end{aligned}$$

Since $F_n^m \in S[\mathbf{E}_{-p,\infty}]$, there exists some $\Phi_n^m \in \mathbf{E}_{-p,\infty}$ such that $F_n^m = S[\Phi_n^m]$ for any m and n. By the Schwarz inequality, we see that

$$\begin{aligned}\sum_{n=1}^{\infty} |F_n^m(\xi_m)| &\leq \sum_{n=1}^{\infty} \|\Phi_n^m\|_{-p} \|\varphi_{\xi_m}\|_p \\ &\leq \left\{\sum_{n=1}^{\infty} \alpha_N^{a_m}(n)^{-1}\right\}^{1/2} \left\{\sum_{n=1}^{\infty} \alpha_N^{a_m}(n) \|\Phi_n^m\|_{-p}^2\right\}^{1/2} \|\varphi_{\xi_m}\|_p < \infty,\end{aligned}$$

for each $m \in \mathbf{N}$, $\xi_m \in E_{\mathbf{C}}$ and $N \in \mathbf{N}$. Therefore by the continuity of $\widetilde{G_t^{a_m}}, m \in \mathbf{N}$, we get

$$\begin{aligned}E[F(\mathbf{X}_t|\mathbf{X}_0 = \xi)] &= (E[F^m(\xi_m + X_t^m \eta_T)])_{m\in\mathbf{N}} \\ &= \left(\sum_{n=1}^{\infty} E[F_n^m(\xi_m + X_t^m \eta_T)]\right)_{m\in\mathbf{N}} \\ &= \left(\sum_{n=1}^{\infty} \widetilde{G_t^{a_m}} F_n^m(\xi_m)\right)_{m\in\mathbf{N}} \\ &= \left(\widetilde{G_t^{a_m}} \sum_{n=1}^{\infty} F_n^m(\xi_m)\right)_{m\in\mathbf{N}} \\ &= \widetilde{G_t} F(\xi).\end{aligned}$$

Thus we obtain the assertion. □

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