数理解析研究所講究録 1278 巻 2002 年 34-47

On Quantum White Noises and Related Transformations

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1 Introduction

Let a_t and a_t^* be respectively annihilation and creation operators at a point $t \in \mathbb{R}$. In the theory of white noise operators [23][24], it is known that a_t is a continuous linear operator from $\Gamma_u(E_{\mathbb{C}})$, the (Fock) space of test functions over a complexified nuclear space $E_{\mathbb{C}}$, into itself, and a_t^* is a continuous linear operator from $\Gamma_u(E_{\mathbb{C}})^*$, the dual space of $\Gamma_u(E_{\mathbb{C}})$, into itself. In paticular, $a_t+a_t^*$ and $a_t+a_t^*+a_t^*a_t+I$ are called the quantum Gaussian white noise and the quantum Poisson white noise, respectively (cf. [11][24]). These are known to be elemental generalized qunatum stochastic processes. The main purpose of this work is to examine relationships between classical and quantum white noises from the point of the classical white noise theory of Gaussian [19][20] and Poisson types [13]. Then it will be clarified that quantum Gaussian and Poisson white noises are the Fock space realizations of classical Gaussian and Poisson white noises, respectively. We shall discuss them in Section 4. As a matter of fact, one can see that such facts can be proved not only through J-transform given by (2.4) and the holomorphy, but also through S_X-transform given by (3.3) depending on the exponential function ϕ_ξ^X. The function ϕ_ξ^X determines a unitary isomorphism between Boson Fock space and $L^2(E^*, \mu_X)$, $X = G,\ P$, where μ_G and μ_P are a Gaussian measure and a Poisson measure on E^*, respectively. As is pointed out in Remark 4.3, the choice of ϕ_ξ^X (kernel function of S_X-transform) is the essential part to represent the above different quantum noises acting on the same space $\Gamma_u(E_{\mathbb{C}})^*$. In Appendix, some connections between [6][7] and [10] will be noted.

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2 Gel'fand Triplet in Terms of Boson Fock Space

Consider $H \equiv L^2(\mathbb{R}, dt)$ with norm $|\cdot|_0$. Let A be an operator in H such that there exists an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ satisfying the conditions:

1. $Ae_j = \lambda_j e_j$.
2. $1 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$.
3. $\sum_{n=0}^{\infty} \lambda_j^{-2\alpha} < \infty$ for some positive constant α.

For each $p \geq 0$, define the norm $|\xi|_p = |A^p \xi|_0$ and let

$$E_p = \{\xi \in H; |\xi|_p < \infty\}.$$

It can be shown that $E_p \subset E_q$ for any $p \geq q \geq 0$ and the inclusion map $i_{p+\alpha,p} : E_{p+\alpha} \hookrightarrow E_p$ is a Hilbert-Schmidt operator for any $p \geq 0$. Let $E = \text{proj}\lim_{p\to\infty} E_p$ and E^* be the dual space of E. Then E is a nuclear space and we obtain a Gel'fand triplet $E \subset H \subset E^*$ with the following continuous inclusions:

$$E \subset E_p \subset H \equiv E_0 \subset E_p^* \subset E^*, \qquad p \geq 0$$

where the norm on E_p^* is given by $|f|_{-p} = |A^{-p} f|_0$. $\langle \cdot, \cdot \rangle$ denotes the bilinear pairing between E^* and E.

In order to discuss operators such as annihilation, creation, and number operators later in the framework of white noise theory, we need to assume the following three conditions:

(H1) each function $\xi \in E$ has a continuous version $\widetilde{\xi}$,

(H2) $\delta_t \in E^*$ for all $t \in \mathbb{R}$ so that $\langle \delta_t, \xi \rangle = \widetilde{\xi}(t)$, where δ_t is the Dirac delta function,

(H3) the mapping $t \mapsto \delta_t$ is continuous with the strong topology for E^*,

(H4) E is algebra.

Thus functions in E will be regarded to be continuous and $\widetilde{\xi}$ be simply denoted by ξ and $(H1) \sim (H4)$ conditions always be assumed throghout this paper.

Let $C_{+,1/2}$ denote the collection of all positive continuous functions u on $[0, \infty)$ satisfying

$$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty. \tag{2.1}$$

For $u \in C_{+,1/2}$, the dual function u^* of u is given by

$$u^*(r) = \sup_{s>0} \frac{e^{\sqrt{rs}}}{u(s)}, \quad r \in [0,\infty). \tag{2.2}$$

For later use, we introduce the following additional conditions on u.

(G1) $\inf_{r\geq 0} u(r) = 1$,

(G2) $\limsup_{r\to\infty} \dfrac{\log u(r)}{r} < \infty$,

(G3) $\log u(x^2)$ is a convex function for $x \in [0,\infty)$.

We denote the complexification of H by $H_{\mathbf{C}}$. It is well-known that the Boson Fock space over $H_{\mathbf{C}}$, denoted by $\Gamma(H_{\mathbf{C}})$, is a Hilbert space consisting of sequences $(f_n)_{n=0}^{\infty}$, where $f_n \in H_{\mathbf{C}}^{\widehat{\otimes} n}$ and $\sum_{n=0}^{\infty} n! |f_n|_0^2 < \infty$. For $(f_n) \in \Gamma(H_{\mathbf{C}})$, $p \geq 0$ and a given function $u \in C_{+,1/2}$ satisfying the conditions $(G1)(G2)(G3)$, define

$$\|(f_n)\|_{\Gamma_u(E_{p,\mathbf{C}})} := \left(\sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^2 \right)^{\frac{1}{2}}$$

where ℓ_u is the Legendre transform of u given by

$$\ell_u(t) = \inf_{r>0} \frac{u(r)}{r^t}, \quad t \in [0,\infty). \tag{2.3}$$

Technical details of Equations (2.2)(2.3) and $(G1)(G2)(G3)$ can be found in [6]. See also Appendix A.2. Let

$$\Gamma_u(E_{p,\mathbf{C}}) = \left\{ (f_n)_{n=0}^{\infty} \in \Gamma(H_{\mathbf{C}}) \ ; \ \|(f_n)\|_{\Gamma_u(E_{p,\mathbf{C}})} < \infty \right\},$$

and $\Gamma_u(E_{\mathbf{C}})$ be the (Fock) space of test functions, which is the projective limit of the family $\{\Gamma_u(E_{p,\mathbf{C}}) \ ; \ p \geq 0\}$. Hence it is easy to see that $\Gamma_u(E_{\mathbf{C}}) \subset \Gamma(H_{\mathbf{C}})$ and $\Gamma_u(E_{\mathbf{C}})$ is a nuclear space. The dual space $\Gamma_u(E_{\mathbf{C}})^*$ is called the space of generalized functions. By identifying $\Gamma(H_{\mathbf{C}})$ with its dual we get the following continuous inclusions:

$$\Gamma_u(E_{\mathbf{C}}) \hookrightarrow \Gamma_u(E_{p,\mathbf{C}}) \hookrightarrow \Gamma(H_{\mathbf{C}}) \hookrightarrow \Gamma_u(E_{p,\mathbf{C}})^* \hookrightarrow \Gamma_u(E_{\mathbf{C}})^*$$

and $\Gamma_u(E_{\mathbf{C}}) \subset \Gamma(H_{\mathbf{C}}) \subset \Gamma_u(E_{\mathbf{C}})^*$ is a Gel'fand triplet. Note that the we have used condition $(G2)$ in order to have the continuous inclusion $\Gamma_u(E_{p,\mathbf{C}}) \hookrightarrow$

$\Gamma_u(H_{\mathbb{C}})$. The canonical bilinear form on $\Gamma_u(E_{\mathbb{C}})^* \times \Gamma_u(E_{\mathbb{C}})$ is denoted by $\langle\langle\cdot,\cdot\rangle\rangle_\Gamma$. For each $\Phi \in \Gamma_u(E_{\mathbb{C}})^*$, there exists a unique $F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{symm}$ with

$$\|(F_n)\|_{\Gamma_u(E_{p,\mathbb{C}})^*} := \left(\sum_{n=0}^{\infty}(n!)^2\ell_u(n)|F_n|_{-p}^2\right)^{\frac{1}{2}} < \infty$$

for some $p \geq 0$ such that

$$\langle\langle(F_n),(f_n)\rangle\rangle_\Gamma = \sum_{n=0}^{\infty} n!\langle F_n, f_n\rangle.$$

We define

$$a_t\big(f_n\big)_{n=0}^{\infty} = \big(n\delta_t \otimes_1 f_n\big)_{n=1}^{\infty}, \quad a_t\Omega = 0, \quad f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}$$

where $\otimes_1$ is the contraction of tensor product and Ω is the Fock vacuum. It is easy to show that a_t is a continuous linear operator from $\Gamma_u(E_{\mathbb{C}})$ into itself. The adjoint operator a_t^* of a_t, given by

$$a_t^*\big(F_n\big)_{n=0}^{\infty} = \big(\delta_t\widehat{\otimes}F_n\big)_{n=0}^{\infty}, \quad F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{symm},$$

is a continous linear operator from $\Gamma_u(E_{\mathbb{C}})^*$ into itself. Therefore, a_t is called an *annihilation operator*, the dual operator a_t^* is called a *creation operator*, and $a_t^*a_t$ is called a *number operator* denoted by N_t. These operators satisfy the following canonical commutation relations:

$$[a_s, a_t] = 0, \quad [a_s^*, a_t^*] = 0, \quad [a_s, a_t^*] = \delta_s(t)I.$$

In quantum stochastic calculus (cf. [11][24]), $a_t + a_t^*$ is called the *quantum Gaussian white noise* and $a_t + a_t^* + N_t + I$ is called the *quantum Poisson white noise* acting on the same generalized functions space $\Gamma_u(E_{\mathbb{C}})^*$.

Since $u \in C_{+,1/2}$ satisfies $(G3)$, $(\frac{\xi^{\otimes n}}{n!}) \in \Gamma_u(E_{\mathbb{C}})$ for all $\xi \in E_{\mathbb{C}}$. So let us introduce the J-transform for the characterization of $\Gamma_u(E_{\mathbb{C}})$ and $\Gamma_u(E_{\mathbb{C}})^*$.

Definition 2.1 (J-transform). For $(F_n)_{n=0}^{\infty} \in \Gamma_u(E_{\mathbb{C}})^*$, *$J$-transform* is defined to be the function

$$(J(F_n))(\xi) = \left\langle\left\langle(F_n),\left(\frac{\xi^{\otimes n}}{n!}\right)\right\rangle\right\rangle_\Gamma, \quad \xi \in E_{\mathbb{C}}. \tag{2.4}$$

The next Theorem 2.2 claims that $\Gamma_u(E_{\mathbb{C}})^*$ can be characterized in terms of analyticity and growth order of J-transforms.

Theorem 2.2. *Suppose $u \in C_{+,1/2}$ satisfies conditions $(G1)(G2)(G3)$. Then a $\mathbb{C}$-valued function F on $E_{\mathbb{C}}$ is the J-transform of a generalized function in $\Gamma_u(E_{\mathbb{C}})^*$ if and only if it satisfies the conditions:*

(a) For any $\xi, \eta \in E_{\mathbb{C}}$, the function $F(z\xi + \eta)$ is an entire holomorphic function of $z \in \mathbb{C}$.

(b) There exist constants $K, a, p \geq 0$ such that

$$|F(\xi)| \leq K u^*(a|\xi|_p^2)^{1/2}, \quad \forall \xi \in E_{\mathbb{C}}.$$

Since Theorems 2.2 can be obtained with the same technique given in [7], we omit the proof. The characterization of $\Gamma_u(E_{\mathbb{C}})$ will be stated in Theorem A.1. The well-known examples will be given later in Example 3.4, Remarks 3.3 and A.3.

One could notice that the prior knowledge of the measure theory may not be required to understand the introduced materials in this section.

3 Gel'fand Triplets in Terms of Multiple Wiener-Itô Integrals Associated with Gaussian and Poisson Measures

In this section, in order to make a bridge between a tangent vector $\dot{\Phi}_t$ given later in Section 4 and white noises, we shall construct the Gel'fand triplet in terms of multiple Wiener-Itô integrals associated with Gaussian and Poisson measures on E^*. Then one can see the correspondences between the tangent vecor $\dot{\Phi}_t$, classical and quantum white noises of Gaussian and Poisson types in Section 4.

In the following, we quickly summarize the essence of Gaussian white noise theory from [19][20][23] and Poisson white noise theory from [13].

Let μ_G be the standard Gaussian measure on E^* given by

$$\int_{E^*} \exp[i\langle x, \xi\rangle] d\mu_G(x) = \exp\Big[-\frac{1}{2}\int_{\mathbb{R}} |\xi(t)|^2 dt\Big], \quad \xi \in E$$

and μ_P be the Poisson measure on E^* by

$$\int_{E^*} \exp[i\langle x, \xi\rangle] d\mu_P(x) = \exp\Big[\int_{\mathbb{R}} (e^{i\xi(t)} - 1) dt\Big], \quad \xi \in E.$$

Let us denote the complex Hilbert space $L^2(E^*, \mu_X)$ by $(L^2)_X$, $X = G,\ P$, and the multiple Wiener-Itô integrals with respect to a measure μ_X by $I_n^X(f_n)$ for $f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$. Then each $\varphi \in (L^2)_X$ is uniquely decomposed as

$$\varphi(x) = \sum_{n=0}^{\infty} I_n^X(f_n), \quad f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}.$$

It is important to notice that there exist unitary isomorphisms U_X between $(L^2)_X$, $X = G, P$, and $\Gamma(H_{\mathbb{C}})$ determined uniquely by the *exponential functions (vectors)*

$$\phi_\xi^G(x) \equiv \exp\Big[\langle x, \xi\rangle - \frac{1}{2}|\xi|_0^2\Big] \longleftrightarrow \left(\frac{\xi^{\otimes n}}{n!}\right)_{n=0}^{\infty} \equiv e(\xi), \quad \xi \in E_{\mathbb{C}} \tag{3.1}$$

when $X = G$ and

$$\phi_\xi^P(x) \equiv \exp\Big[\langle x, \log(1+\xi)\rangle - \int_{\mathbb{R}} \xi(t)dt\Big] \longleftrightarrow e(\xi), \quad \xi \in E_{\mathbb{C}} \tag{3.2}$$

when $X = P$, respectively. We remark that the Poisson case is not addressed in [3][4][5][7][9][10][18]. It is known that $\{\phi_\xi^X\ ; \xi \in E_{\mathbb{C}}\}$ spans a dense subspace of $[E]_{u,X}$ and $\{e(\xi); \xi \in E_{\mathbb{C}}\}$ does the same for $\Gamma_u(E_{\mathbb{C}})$. In those cases, it holds that the $(L^2)_X$-norm of φ is given by

$$\|\varphi\|_0^2 = \int_{E^*} |\varphi(x)|^2 d\mu_X(x) = \sum_{n=0}^{\infty} n!|f_n|_0^2 = \|(f_n)\|_{\Gamma(H_{\mathbb{C}})}^2.$$

The S_X-transform of $\varphi \in (L^2)_X$, given by

$$(S_X\varphi)(\xi) := \int_{E^*} \varphi(x)\phi_\xi^X(x)d\mu_X(x), \quad \xi \in E_{\mathbb{C}},$$

is an isomorphism from $(L^2)_X$ onto the Hilbert space $\mathcal{K}$ of holomorphic functions F on $E_{\mathbb{C}}$ with a reproducing kernel $\exp[\langle \xi, \eta\rangle]$, $\xi, \eta \in E$.

Let $[E_p]_{u,X} = \{\varphi \in (L^2)_X\ ;\ \|\varphi\|_p := \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)}|f|_p^2 < \infty\}$ and $[E]_{u,X}$ be the space of test functions, which is the projective limit of the family $\{[E_p]_{u,X}\ ;\ p \geq 0\}$. Hence it can be shown easily that $[E]_{u,X} \subset (L^2)_X$ by the condition $(G2)$, and $[E]_{u,X}$ is a nuclear space. The dual space $[E]_{u,X}^*$ is called the space of generalized functions. Then we obtain the following continuous inclusions:

$$[E]_{u,X} \hookrightarrow [E_p]_{u,X} \hookrightarrow (L^2)_X \hookrightarrow [E_p]_u^* \hookrightarrow [E]_{u,X}^*,$$

and $[E]_{u,X} \subset (L^2)_X \subset [E]_{u,X}^*$ is a Gel'fand triplet.

Let $\partial_{t,G}$ be the Gâteaux derivative in the direction of δ_t, so-called *Hida derivative*, and $\partial_{t,P}$ be the difference operator $\Delta_t\varphi = \varphi(x+\delta_t) - \varphi(x)$. Let $\partial_{t,G}^*$ and $\partial_{t,P}^*$ be the adjoint operators of $\partial_{t,G}$ and $\partial_{t,P}$, respectively.

Since $u \in C_{+,1/2}$ satisfies $(G3)$, the exponential function $\phi_\xi^X(x) \in [E]_{u,X}$ for any $\xi \in E_{\mathbb{C}}$. Hence the S_X-transform can be extended to a continuous linear functional on $[E]_{u,X}^*$ as follows.

Definition 3.1 (**S_X-transform**). For $\Phi \in [E]_{u,X}^*$, S_X*-transform* is defined by

$$(S_X\Phi)(\xi) = \langle\langle \Phi, \phi_\xi^X \rangle\rangle, \quad \xi \in E_{\mathbb{C}}, \tag{3.3}$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the bilinear pairing of $[E]_{u,X}^*$ and $[E]_{u,X}$.

Now we come to the characterization of $[E]_{u,X}^*$ associated with μ_X, $X = G, P$, in a single statement. The proof is almost the same as that in [7], but it is under $(G1)(G2)^*(G3)$ only with μ_G. The condition $(G2)^*$ is given in Appendix A.2.

Theorem 3.2. *Let a measure μ_X on E^* be given. Suppose $u \in C_{+,1/2}$ satisfies conditions $(G1)(G2)(G3)$. Then a $\mathbb{C}$-valued function F on $E_{\mathbb{C}}$ is the S_X-transform of a generalized function in $[E]_{u,X}^*$ if and only if it satisfies the conditions:*

(a) For any $\xi, \eta \in E_{\mathbb{C}}$, the function $F(z\xi + \eta)$ is an entire holomorphic function of $z \in \mathbb{C}$.

(b) There exist constants $K, a, p \geq 0$ such that

$$|F(\xi)| \leq Ku^*(a|\xi|_p^2)^{1/2}, \quad \forall \xi \in E_{\mathbb{C}}.$$

Remark 3.3. Theorem 3.2 was first proved by Potthoff-Streit [25] in case of $X = G$ and $u(r) = e^r$. It was extended to the case of $X = G$ and $u(r) = \exp[(1-\beta)r^{\frac{1}{1-\beta}}]$ by Kondratiev-Streit [16][17]. Moreover, Cochran et al. [9] proved the case when $X = G$ and the growth condition (b) is determined by the exponential generating function $G_\alpha(r) = \sum \frac{\alpha(n)}{n!} r^n$. Asai et al. [4][6][7] minimized conditions on sequences $\{\alpha(n)\}$ of positive real numbers in such a way that Theorem 3.2 holds.

Example 3.4. The Gel'fand triplet $[E]_{u,X} \subset (L^2)_X \subset [E]_{u,X}^*$ becomes
(1) the *Hida-Kubo-Takenaka space* [19][20][23] if $X = G$ and $u(r) = e^r$, and the *Ito-Kubo space* [13] if $X = P$ and $u(r) = e^r$,
(2) the *Kondratiev-Streit space* [17] if $X = G$ and $u(r) = \exp[(1+\beta)r^{\frac{1}{1+\beta}}]$ for $0 \leq \beta < 1$,

(3) the *Cochran-Kuo-Sengupta (CKS) space* of Bell numbers with degree k if $X = G$ and $u^*(r) = \exp_k(r)/\exp_k(0)$, where $\exp_k(r)$ is the k-th iterated exponential function [9]. Consult papers [3][4][5][6][9] for more general construction of CKS space and [2][18] for more details on Bell numbers.

Remark 3.5. We exclude the $\beta = 1$ space of Kondratiev-Streit [15]. It is because the function $u(r) = \exp[2\sqrt{r}]$ does not satisfy Equation (2.1).

4 Main Results

In the previous two sections, we have constructed the Gel'fand triplets in terms of Fock space and Schrödinger representations with associated transformations. Notice that in the Section 2, we did not introduce any probability measure on E^* as the standard white noise theory [19][20][23]. Hence, the property of a measure on E^* plays virtually no rôle in the definition of J-transform. In fact, the essential tools to prove Theorems 2.2 and A.1 are the Cauchy integral formula for entire holomorphic functions of several variables, Legendre transform, dual function, Schwartz kernel theorem, and properties of the nuclear space. So, does it imply that the considerations of the growth order of holomorphic funcions and associated topologies are sufficient to examine stochastic processes by the generalized function theory on infinite dimensional space ?. The answer is completely *No* even for the study of fundamental stochastic objects such as Brownian motion and Poisson process. Let us emphasize the following point. It is easy to see that the *"flow"* Φ_t, given by

$$\Phi_t = \begin{cases} (0, 1_{[0,t]}, 0, \cdots) & \text{if } t \geq 0 \\ (0, -1_{[t,0]}, 0, \cdots) & \text{if } t < 0, \end{cases}$$

is an element of $\Gamma(H_{\mathbf{C}})$. Then the *"tangent vector"* $\dot{\Phi}_t$ is $(0, \delta_t, 0, \cdots)$ and belongs to $\Gamma_u(E_{\mathbf{C}})^*$ with $u(r) = e^r$ and $J\dot{\Phi}_t(\xi) = \xi(t)$.

On the other hand, it is known that the Brownian motion $B(t)$ is represented by

$$B(t) = \begin{cases} I_1^G(1_{[0,t]}) & \text{if } t \geq 0 \\ -I_1^G(1_{[t,0]}) & \text{if } t < 0. \end{cases} \tag{4.1}$$

Similarly, the compensated Poisson process is given by

$$P(t) - t = \begin{cases} I_1^P(1_{[0,t]}) & \text{if } t \geq 0 \\ -I_1^P(1_{[t,0]}) & \text{if } t < 0. \end{cases} \tag{4.2}$$

Since characteristic functions $1_{[0,t]}$ and $1_{[t,0]}$ are elements of H, $B(t)$ and $P(t)-t$ are in $(L^2)_G$ and $(L^2)_P$, respectively. Hence we obtain $U_G\Phi_tU_G^{-1} = B(t)$ and $U_P\Phi_tU_P^{-1} = P(t)-t$. So the distributional derivative of $B(t)$ with respect to t, so-called ***Gaussian white noise*** $\dot{B}(t)$, has the form $\dot{B}(t) = I_1^G(\delta_t)$ for each $t \in \mathbb{R}$. Similarly, the ***Poisson white noise*** $\dot{P}(t)$ has the expression $\dot{P}(t) - 1 = I_1^P(\delta_t)$ for each $t \in \mathbb{R}$. In those cases, since δ_t is in E^*, $\dot{B}(t)$ and $\dot{P}(t)-1$ belong to $[E]^*_{u,G}$ and $[E]^*_{u,P}$, respectively (A function u will be chosen in the proof of Theorem 4.2). Thus, we get the relationship between the vector $\dot{\Phi}_t$, classical Gaussian and Poisson white noises as follows.

Proposition 4.1. *It holds that*
(1) $U_G\dot{\Phi}_tU_G^{-1} = \dot{B}(t)$,
(2) $U_P\dot{\Phi}_tU_P^{-1} = \dot{P}(t) - 1$.

It is very convenient to check the holomorphy and the growth order of $(J\dot{\Phi}_t)(\xi)$ to show that the tangent vector $\dot{\Phi}_t$ of the flow Φ_t belongs to $\Gamma_u(E_{\mathbb{C}})^*$ with the appropriate choice of a function u. However, Theorem 2.2 cannot distinguish wheather $\dot{\Phi}_t$ is the tangent vector corresponding to Brownian motion or (compensated) Poisson process due to the lack of isomorphism U_X.

In the next theorem, connections between classical and quantum white noises of Gaussian and Poisson types will be shown.

Theorem 4.2. *It holds that*
(1) $U_G(a_t + a_t^*)U_G^{-1} = \dot{B}(t)$, *where* $\dot{B}(t)$ *is considered as a multiplication operator.*
(2) $U_P(a_t + a_t^* + N_t + I)U_P^{-1} = \dot{P}(t)$, *where* $\dot{P}(t)$ *is considered as a multiplication operator.*

Proof. In the proof, the exponential functions (vectors) given by Equations (3.1) (3.2) will play essential roles to distinguish the type of white noise.

Let us start the proof with the Gaussian case $X = G$, first.

$$\begin{aligned}(J(a_t + a_t^*)e(\xi))(\eta) &= \langle\langle (a_t + a_t^*)e(\xi), e(\eta)\rangle\rangle_\Gamma, \quad \xi, \eta \in E_{\mathbb{C}} \\ &= (\xi(t) + \eta(t))e^{\langle \xi,\eta\rangle}. \end{aligned} \tag{4.3}$$

On the other hand, since the multiplication by $\dot{B}(t)$ is described by $\partial_{t,G} + \partial_{t,G}^*$ (see [19]), we have

$$\begin{aligned}(S_G\dot{B}\phi_\xi^G)(\eta) &= (S_G[\partial_{t,G} + \partial_{t,G}^*]\phi_\xi^G)(\eta) \\ &= \langle\langle [\partial_{t,G} + \partial_{t,G}^*]\phi_\xi^G, \phi_\eta^G\rangle\rangle_G \\ &= (\xi(t) + \eta(t))e^{\langle \xi,\eta\rangle}. \end{aligned} \tag{4.4}$$

Due to Equations (4.3)(4.4), we have $U_G(a_t + a_t^*)U_G^{-1} = \dot{B}(t)$, where $\dot{B}(t)$ is considered as a multiplication operator. In fact, $(S_G\dot{B})(\xi) = \langle\delta_t, \xi\rangle = \xi(t)$ satisfies the condition (b) with $u(r) = \exp(r)$ in Theorem 3.2. Hence we get $\dot{B}(t) \in (E)^*_G$. Therefore, we have finished to prove our first assertion.

Next consider the Poisson case $X = P$.

$$\begin{aligned}(J(a_t + a_t^* + a_t^*a_t + I)e(\xi))(\eta) &= \langle\langle(a_t + a_t^* + N_t + I)e(\xi), e(\eta)\rangle\rangle_\Gamma, \quad \xi, \eta \in E_{\mathbb{C}} \\ &= (\xi(t) + \eta(t) + \eta(t)\xi(t) + 1)e^{\langle\xi,\eta\rangle} \qquad (4.5)\end{aligned}$$

Note that the function $\eta\xi$ above makes sense as a member of $E_{\mathbb{C}}$ due to $(H4)$. On the other hand, since the multiplication by $\dot{P}(t)$ is described as $\partial_{t,P} + \partial_{t,P}^* + \partial_{t,P}^*\partial_{t,P} + I$ (see [13]), we have

$$\begin{aligned}(S_P\dot{P}\phi_\xi^P)(\eta) &= (S_P[\partial_{t,P} + \partial_{t,P}^* + \partial_{t,P}^*\partial_{t,P} + I]\phi_\xi^P)(\eta) \\ &= \langle\langle[\partial_{t,P} + \partial_{t,P}^* + \partial_{t,P}^*\partial_{t,P} + I]\phi_\xi^P, \phi_\eta^P\rangle\rangle_P \\ &= (\xi(t) + \eta(t) + \eta(t)\xi(t) + 1)e^{\langle\xi,\eta\rangle}. \qquad (4.6)\end{aligned}$$

By Equations (4.5)(4.6), we have $U_P(a_t + a_t^* + N_t + I)U_P^{-1} = \dot{P}(t)$, where $\dot{P}(t)$ is considered as a multiplication operator. In fact, $(S_P\dot{P})(\xi) = \langle\delta_t, \xi\rangle + 1 = \xi(t) + 1$ satisfies the condition (b) with $u(r) = \exp(r)$ in Theorem 3.2. Hence we get $\dot{P}(t) \in (E)^*_P$. Thus we have proved the second claim. □

Remark 4.3. The Laplace transform $\mathcal{L}$ is used in Gannoun et al. [10]. We point out that they studied the Gel'fand triplets in terms of holomorphic functions' space and Fock space, and shows that both triplets are topologically equivalnt. There are no considerations about operator theory on their triplets and the exponential vector associated with Poisson meaure.

We know the following relationships between $\mathcal{L}$ and our S_G, S_P-transforms:

$$S_G\Phi(\xi) = \exp\Big(-\frac{1}{2}|\xi|_0^2\Big)\mathcal{L}\Phi(\xi), \quad S_P\Phi(e^\xi - 1) = \exp\Big(-\int_{\mathbb{R}}(e^{\xi(t)} - 1)dt\Big)\mathcal{L}\Phi(\xi)$$

$\xi \in E_{\mathbb{C}}$ for $\Phi \in [E]_u^*((F_n) \in \Gamma_u(E_{\mathbb{C}})^*)$. In this situation [12] (see also Theorem 6.1 by Hudson-Parthasarathy [11]), S_P is an isomorphism from $(L^2)_P$ onto a reproducing kernel Hilbert space with kernel

$$\exp\langle e^\xi - 1, e^\eta - 1\rangle, \quad \xi, \eta \in E.$$

This reproducing kernel Hilbert space is different from $\mathcal{K}$. In addition, it is easy to see that

$$(S_P\dot{P}(t))(e^\xi - 1) = \exp[e^{\xi(t)} - 1]. \qquad (4.7)$$

So if the exponential function

$$\frac{e^{\langle x,\xi\rangle}}{E_{\mu_P}[e^{\langle x,\xi\rangle}]} = \exp\left[\langle x,\xi\rangle - \left(\int_{\mathbb{R}}(e^{\xi(t)}-1)dt\right)\right] \tag{4.8}$$

is considered, the quantum Poisson white noise $a_t + a_t^* + a_t^* a_t + I$ acts on $\Gamma_u(E_{\mathbb{C}})^*$ with $u(r) = \exp[e^r - 1]$ due to (4.7), i.e., the CKS-space of Bell numbers with degree 2. Hence the exponential function given by (4.8) is not an appropriate choice to represent quantum Gaussian and Poisson white noises on the common Fock space $\Gamma_u(E_{\mathbb{C}})^*$ with $u(r) = e^r$.

A Appendix

A.1 Characterizations of Test Functions

The characterization of $\Gamma_u(E_{\mathbb{C}})$ is stated as follows. The proof is almost the same as those in [6][7], which is under $(G1)(G2)^*(G3)$.

Theorem A.1. *Suppose $u \in C_{+,1/2}$ satisfies conditions $(G1)(G2)(G3)$. Then a $\mathbb{C}$-valued function F on $E_{\mathbb{C}}$ is the J-transform of a generalized function in $\Gamma_u(E_{\mathbb{C}})$ if and only if it satisfies the conditions:*

(a) For any $\xi, \eta \in E_{\mathbb{C}}$, the function $F(z\xi + \eta)$ is an entire holomorphic function of $z \in \mathbb{C}$.

(b)' For any constants $a, p \geq 0$, there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq K u(a|\xi|_{-p}^2)^{1/2}, \quad \forall \xi \in E_{\mathbb{C}}.$$

The characterization of $[E]_{u,X}$ associated with μ_X, $X = G, P$, is stated below in a single statement. The proof is almost the same as that in [7], which is under $(G1)(G2)^*(G3)$ only with μ_G .

Theorem A.2. *Let a measure μ_X on E^* be given. Suppose $u \in C_{+,1/2}$ satisfies conditions $(G1)(G2)(G3)$. Then a $\mathbb{C}$-valued function F on $E_{\mathbb{C}}$ is the S_X-transform of a generalized function in $[E]_{u,X}$ if and only if it satisfies the conditions:*

(a) For any $\xi, \eta \in E_{\mathbb{C}}$, the function $F(z\xi + \eta)$ is an entire holomorphic function of $z \in \mathbb{C}$.

(b)' For any constants $a, p \geq 0$, there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq K u(a|\xi|_{-p}^2)^{1/2}, \quad \forall \xi \in E_{\mathbb{C}}.$$

Remark A.3. Theorem A.2 was proved by Kuo et al. [21] in case of $X = G$ and $u(r) = e^r$. It was extended to the case of $X = G$ and $u(r) = \exp[(1+\beta)r^{\frac{1}{1+\beta}}]$ by Kondratiev-Streit [17]. Moreover, Asai et al. [3] proved the case when $X = G$ and the growth condition (b)' is determined by the exponential generating function $G_{1/\alpha}(r) = \sum \frac{1}{\alpha(n)n!}r^n$. Asai et al. [4][6][7] minimized conditions on sequences $\{\alpha(n)\}$ of positive real numbers in such a way that Theorem A.2 holds.

A.2 Relationships with the Work by Gannoun et al. [10]

In the rest of this paper, we shall point out some of similarities and differences between series of our papers [6][7] and Gannoun et al. [10]. We refer the readers to consult the papers [4][6][7][8] for more technical and delicate differences, which will not be mentioned in this paper.

First, the basic equalities are

$$u(r) = e^{2\theta(\sqrt{r})}, \quad u^*(r) = e^{2\theta^*(\sqrt{r})}$$

where $\theta^*(s) = \sup_{t>0}\{st - \theta(t)\}$ is adopted in GHOR. In the following table we give the correspondence between our G-conditions and their θ-conditions.

	u	θ
$C_{+,1/2}$	$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty$	$\lim_{r\to\infty} \frac{\theta(r)}{r} = \infty$
$(G1)$	$\inf_{r\geq 0} u(r) = 1$	
$(G1)^*$	u is increasing and $u(0) = 1$	θ is increasing and $\theta(0) = 0$
$(G2)$	$\limsup_{r\to\infty} \frac{\log u(r)}{r} < \infty$	
$(G2)^*$	$\lim_{r\to\infty} \frac{\log u(r)}{r} < \infty$	$\lim_{r\to\infty} \frac{\theta(r)}{r^2} < \infty$
$(G3)$	u is $(\log, x^2)$-convex	θ is convex

Actually, $(G2)^*$ is slightly stronger than $(G2)$. However, $(G2)$ is strong enough to guarantee that the nuclear spaces $\Gamma_u(E_{\mathbb{C}})$ and $[E]_{u,X}$ are the subspaces of $\Gamma(H_{\mathbb{C}})$ and $(L^2)_X$, respectively. Moreover, although $(G1)$ is weaker than $(G1)^*$, by Lemma 3.1 in [7] we can construct an equivalent function satisfying $(G1)^*$ even if we begin with $(G1)$. θ-conditions corresponding to $(G1)(G2)$ conditions are not considered in [10].

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