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THE RESOLVENT TRACE FORMULA FOR RANK ONE LIE GROUPS

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1. Introduction

1.1. **Introduction.** Let $X = G/K$ be a Riemannian symmetric space of non-compact type with G a connected simple Lie group of real rank one and K a maximal compact subgroup of G. In the paper [18], Miatello-Wallach introduced a family of bi-K-invariaint functions Q_s, $s \in \mathbf{C}$ on G, which satisfies the same differential equation as the elementary spherical function ϕ_s of Harish-Chandra on the open set $G^+ = G - K$ but has singularities along K. By making the r-fold convolution of Q_s, they defined a function $Q_{r,s}$ on G, which is less singular than Q_s itself. Then, given a cofinite lattice Γ of G, they introduced the distribution $\mathbf{P}_{r,s}(\dot{x},\dot{y})$ by forming the Poincaré series

$$\mathbf{P}_{r,s}(\dot{x},\dot{y}) = c_r(s) \sum_{\gamma\in\Gamma} Q_{r,s}(x^{-1}\gamma y), \quad \dot{x},\ \dot{y} \in \Gamma\backslash X \tag{1}$$

with a suitable normalizing factor $c_r(s)$ and proved, among other things, that it is smooth on the complement of the diagonal in $(\Gamma\backslash X)\times(\Gamma\backslash X)$ and satisfies the differential equation

$$(\triangle + \rho_0^2 - s^2)^r \mathbf{P}_{r,s}(\dot{x}, -) = \delta(\dot{x}) \tag{2}$$

with $\triangle$ the Laplacian of $\Gamma\backslash X$, $\delta(\dot{x})$ the Dirac delta supported at $\dot{x}$. In the classical situation that X is the upper half plane, the distribution $\mathbf{P}_{1,s}(\dot{x},\dot{y})$, the resolvent kernel function of Laplacian for the Riemannian surface $\Gamma\backslash X$, was intensively investigated by several German mathematicians from the view point of real analytic automorphic forms ([3], [20]). Based on these works, J. Fischer deduced the resolvent trace formula by computing the integral

$$\int_{\Gamma\backslash X} (\mathbf{P}_{1,s}(\dot{x},\dot{x}) - \mathbf{P}_{1,s'}(\dot{x},\dot{x}))\, d\dot{x}, \quad s,\ s' \in \mathbf{C} \tag{3}$$

in two different ways ([5]).

In this paper, we show that the same type of procedure is possible for a higher dimensional X by considering the integral $\int_{\Gamma\backslash X} \mathbf{P}_{r,s}(\dot{x},\dot{x})d\dot{x}$ with r greater than a half of $\dim X$ instead of (3). As a result, following Fischer, we can obtain another proof of the meromorphic continuation of the Selberg zeta function for $\Gamma\backslash X$ and its functional equation, which was originally proved by Selberg, Gangolli and Gangolli-Warner ([7], [8], [21]).

Although a handy formula of $Q_{r,s}$ in the 'polar coordinate '(Cartan decomposition) is desirable for our purpose, it seems rather difficult to have such a formula directly from

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the definition of $Q_{r,s}$ recalled above. Our strategy is as follows. We first have an explicit formula of $Q_s = Q_{1,s}$ in terms of Gaussian hypergeometric series as in the classical case, and then use the system of differential equations among $Q_{r,s}$'s to show that $Q_{r,s}$ is obtained from Q_s by applying the differential operator $(\frac{-1}{2s}\frac{d}{ds})^r$ to it (Proposition 3.1.6, Theorem 3.2.1). Thus a formula of $Q_{r,s}$ in terms of a derivative of the hypergeometric series becomes available, which enables us to compute the integral $\int_{\Gamma\backslash X} \mathbf{P}_{r,s}(\dot{x},\dot{x})d\dot{x}$ by dividing it into the local contributions for Γ-conjugacy classes and by using various formulas involving the beta function and the hypergeometric series. Consequently we can evaluate the integral by means of the logarithmic derivative of the Selberg zeta function for $\Gamma\backslash X$. On the other hand, by the spectral expansion of $\mathbf{P}_{r,s}(\dot{x},-)$ given in [18], we compute the same integral in terms of the eigenvalues of Laplacian on $L^2(\Gamma\backslash X)$. Combining these two expressions of $\int_{\Gamma\backslash X} \mathbf{P}_{r,s}(\dot{x},\dot{x})d\dot{x}$, we arrive at the resolvent trace formula, which was studied in [5, Theorem 2.5.2, p.108] for $G = PSL_2(\mathbf{R})$, in [4] for $G = PSL_2(\mathbf{C})$ and in [1] for Jacobi forms.

Finally, we would like to say a few words on the status of our results. The resolvent trace formula (RTF for short) for a general compact locally symmetric space $\Gamma\backslash X$ with rank one X is more or less known, because it is essentially the same as the determinant expression of the Selberg zeta function obtained already in [16] together with its explicit gamma factor. But we believe that our method, that is a slight extension of Fischer's, provides a more direct and elementary way to have the RTF than the traditional method employed in [21], [7] and [8], which necessitates difficult tools such as the Paley-Wiener theorem and the Plancherel formula for $X = G/K$. We also believe that our Theorem 3.2.1, that gives an expression of $Q_{r,s}$ in terms of the derivative of the hypergeometric series, is new and is interesting itself.

2. Preliminaries

In this section we introduce basic objects and fix notations.

2.1. **Notations.** We denote by $\mathbf{N}$ the set of natural numbers, i.e. $\mathbf{N} = \{1,2,3,\dots\}$. Put $\mathbf{N}_0 = \mathbf{N}\cup\{0\}$. The cardinality of a finite set S is denoted by $\#S$.

2.2. **Lie groups and Lie algebras.** Let G be a connected semisimple Lie group of real rank one with finite center. Put $\mathfrak{g} = \mathrm{Lie}(G)$, the real Lie algebra of G. Let K be a maximal compact subgroup of G and θ the Cartan involution of $\mathfrak{g}$ corresponding to K, then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{k} = \mathrm{Lie}(K)$. We fix an Iwasawa decomposition $G = NAK$ of G; A is a maximal split torus in G whose Lie algebra $\mathfrak{a}$ is orthogonal to $\mathfrak{k}$ with respect to the Killing form B of G and N a maximal unipotent subgroup of G normalized by A. Since $\dim A = 1$ by assumption, there exists a unique root $\alpha \in \mathfrak{a}^*$ such that $\mathfrak{n}_{j\alpha} = \{X \in \mathfrak{g} | \ \mathrm{ad}(H)X = j\cdot\alpha(H)X,\ H \in \mathfrak{a}\}$ with $j \in \mathbf{Z}$ is zero if $|j| > 2$, and $\mathrm{Lie}(N) = \mathfrak{n} = \mathfrak{n}_\alpha + \mathfrak{n}_{2\alpha}$.

Let H_0 be the unique element of $\mathfrak{a}$ such that $\alpha(H_0) = 1$. Let $\langle\ ,\ \rangle : \mathfrak{a}\times\mathfrak{a} \to \mathbf{R}$ be the inner product induced by B; it gives the identification $\mathfrak{a} \cong \mathfrak{a}^*$. The dual inner product of $\mathfrak{a}^*$ is also denoted by $\langle\ ,\ \rangle$. Put $p = \dim_{\mathbf{R}} \mathfrak{n}_\alpha$, $q = \dim_{\mathbf{R}} \mathfrak{n}_{2\alpha}$, $\rho_0 = 2^{-1}(p+2q)$, $c_0 = (2p+8q)^{-1}$ and $m = 2^{-1}\dim(G/K)$. Then by the classification, we have the list:

$\mathfrak{g}$	p	q	ρ_0	m
$\mathfrak{su}(l,1)$	$2l-2$	1	l	l
$\mathfrak{so}(l,1)$	$l-1$	0	$2^{-1}(l-1)$	$2^{-1}l$
$\mathfrak{sp}(l,1)$	$4l-4$	1	$2l-1$	$2l-1$
$\mathfrak{f}_{4(-20)}$	8	7	11	8

(l is a natural number greater than one.)

Lemma 2.2.1. *We have*

$$2m = p+q+1, \quad \langle H_0, H_0\rangle = c_0^{-1}, \quad \langle \alpha, \alpha\rangle = c_0.$$

From now on we assume that $m \in \mathbf{N}$, $m \geqslant 2$. In other words, we exclude the case of $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbf{R})$ or $\mathfrak{so}(2l+1,1)$ with $l \geqslant 1$.

2.3. **Haar measures.** Let dk be the Haar measure of the compact group K with total mass one. Let dt be the standard Lebesgue measure of $\mathbf{R}$; by the identification $\mathbf{R} \cong A = \exp \mathfrak{a}$, $t \mapsto \exp(tH_0)$, it gives the Haar measure of the torus A. Denote by $C_c^0(N)$ the space of compactly supported continuous functions on N. Since $N = \exp(\mathfrak{n}_\alpha + \mathfrak{n}_{2\alpha})$ is a unipotent Lie group we can take its Haar measure dn such that the formula

$$\int_N f(n)dn = \int_{\mathfrak{n}_\alpha}\int_{\mathfrak{n}_{2\alpha}} f(\exp(X+Y))\, dX dY, \quad f \in C_c^0(N)$$

holds with dX (resp. dY) the Euclidian measure of $\mathfrak{n}_\alpha$ (resp. $\mathfrak{n}_{2\alpha}$). (We regard $\mathfrak{n}_{j\alpha}$ as a Euclidean space by the inner product $-B(Z, \theta Z)$.)

Then we fix the Haar measure dg of $G = ANK$ by $dg = da \cdot dn \cdot dk$. To handle various bi-$K$-invariant functions (distributions) on G, the Cartan decomposition $G = K\exp([0,\infty)H_0)K$ is indispensable. We put

$$G^+ = G - K = KA^+K \text{ with } A^+ = \{\exp(tH_0) |\ t > 0\}.$$

If $g \in G^+$, and $g = k_1(g)a(g)k_2(g)$, with $k_1(g), k_2(g) \in K$ and $a(g) \in A^+$, then $a(g)$ is uniquely determined by g. We choose the Riemannian metric dx on $X = G/K$, induced by the restriction $B|_{\mathfrak{p}}$ of B to $\mathfrak{p}$. We then have that the hyperbolic distance $d(xK, yK) = B(tH_0, tH_0)^{1/2} = t$ if $x, y \in G$ and $a(x^{-1}y) = \exp(tH_0)$, with $t > 0$.

The measure dg on G is decomposed along the Cartan decomposition as follows.

Lemma 2.3.1. *For any positive measurable function φ on G, the formula*

$$\int_G \varphi(g)dg = c_G \int_K \int_0^\infty \int_K \varphi(k_1 \exp(tH_0)k_2)\mu(t)dk_1 dt dk_2 \tag{4}$$

holds. Here

$$\mu(t) = (\sinh t)^{p+q}(\cosh t)^q,$$
$$c_G = 2\Gamma(m)^{-1}(2^{-1}c_0)^{-m+1/2}\pi^m.$$

3. Spherical functions

In the first subsection, after recalling the standard properties of zonal spherical functions for G/K, we introduce a bi-K-invariant function $\phi_s^{(2)}$ on G^+ with singularities along K, which is called the secondary spherical function by T. Oda ([19]). We investigate its properties in some detail to show that its r-times derivative with respect to s^2 gives the function $Q_{r,s}$ of Miatello-Wallach ([18]).

3.1. **The spherical function with singularities.** For $s \in \mathbf{C}$, the zonal spherical function ϕ_s for G/K is defined by the integral

$$\phi_s(g) = \int_K e^{(s+\rho_0)\alpha(H(kg))}dk, \quad g \in G.$$

Here for $g \in G$, $H(g)$ denotes the unique vector in $\mathfrak{a}$ such that $g \in N\exp(H(g))K$. The basic property of ϕ_s is listed below.

(a) It is bi-K-invariant C^∞-function on G, i.e., $\phi_s \in C^\infty(K\backslash G/K)$.
(b) It satisfies the differential equation

$$\Omega\phi_s(g) = (s^2 - \rho_0^2)\phi_s(g), \quad g \in G$$

with Ω the Casimir element of G corresponding to c_0B.
(c) If $\mathrm{Re}(s) > 0$, then

$$\lim_{t\to+\infty} e^{t(\rho_0-s)}\phi_s(\exp(tH_0)) = c(s)$$

with $c(s)$ the c-function for G/K given by

$$c(s) = 2^{\rho_0-s}\Gamma(m)\Gamma(s)\Gamma\left(\frac{s+\rho_0}{2}\right)^{-1}\Gamma\left(\frac{s-\rho_0+2m}{2}\right)^{-1}.$$

Put $u_s^{(1)}(t) = \phi_s(\exp(tH_0))$, $t \in \mathbf{R}$. Then by (a), $u_s^{(1)}(t)$ is a C^∞-function on $\mathbf{R}$ which determines ϕ_s uniquely, and by (b) it satisfies the ordinary second order differential equation

$$(D)_s: \qquad \frac{d^2u}{dt^2} + \left(\frac{p}{\tanh t} + \frac{q}{\tanh(2t)}\right)\frac{du}{dt} + (\rho_0^2 - s^2)u = 0$$

which has the regular singularity at $t = 0$ with characteristic exponents $\{0, 2-2m\}$. Change the variable by $z = \tanh^2 t$ and consider the function $w(z) = (\cosh t)^{\rho_0-s}u_s^{(1)}(t)$. Then it turns out that w is a solution of the Gaussian hypergeometric differential equation

$z(1-z)\frac{d^2w}{dz^2}+\{c-(1+a+b)z\}\frac{dw}{dz}-abw=0$ with $a=2^{-1}(-s+\rho_0)$, $b=2^{-1}(-s+\rho_0-q+1)$ and $c=m$. Thus we have

$$u_s^{(1)}(t)=(\cosh t)^{s-\rho_0}{}_2F_1\left(\frac{-s+\rho_0}{2},\frac{-s+\rho_0-q+1}{2};m;\tanh^2 t\right),\quad t\in\mathbf{R}.$$

We are interested in another class of solutions of $(D)_s$ which admit a singularity at $t=0$. Among them, the one with the fastest decay at infinity, which we now define, is of particular importance: For $s\in\mathbf{C}-\{-1,-2,-3,\dots\}$, put

$$u_s^{(2)}(t)=\gamma(s)(\cosh t)^{-(s+\rho_0)}{}_2F_1\left(\frac{s+\rho_0}{2},\frac{s-\rho_0+2m}{2};s+1;\frac{1}{\cosh^2 t}\right),\quad t\in\mathbf{R}-\{0\},$$

$$\gamma(s)=\Gamma\left(\frac{s+\rho_0}{2}\right)\Gamma\left(\frac{s-\rho_0+2m}{2}\right)\Gamma(s+1)^{-1}\Gamma(m-1)^{-1}=2^{-s+\rho_0}m(s\mathrm{c}(s))^{-1}.$$

Proposition 3.1.1. (i) *If $s\in\mathbf{C}$ is not a pole of $\gamma(s)$ and $\gamma(s)\neq 0$, then the family $\{u_s^{(1)},u_s^{(2)}\}$ gives a system of fundamental solutions of $(D)_s$ around $t=0$.*

(ii) *There exists a unique family of functions $\{\phi_s^{(2)}\mid \mathrm{Re}(s)\geqslant 0\}$ in $C^\infty(K\backslash G^+/K)$ such that*

(a)

$$\Omega\phi_s^{(2)}(g)=(s^2-\rho_0^2)\phi_s^{(2)}(g),\quad g\in G^+.$$

(b)

$$\phi_s^{(2)}(\exp(tH_0))=O(e^{-t(\mathrm{Re}(s)+\rho_0)}),\quad (t\to+\infty),$$

(c)

$$\lim_{t\to+0}t^{2m-2}\phi_s^{(2)}(\exp(tH_0))=1.$$

For a given $g\in G^+$ the function $s\mapsto\phi_s^{(2)}(g)$ is holomorphic on $\mathrm{Re}(s)\geqslant 0$. We have $\phi_s^{(2)}(\exp(tH_0))=u_s^{(2)}(t)$ for $t\in\mathbf{R}-\{0\}$.

Proposition 3.1.2. *Put*

$$\mathrm{c}(s)=\frac{(-1)^m\pi}{\Gamma(m)\Gamma(m-1)}\cdot\prod_{j=1}^{m-1}\left\{\left(\frac{s}{2}\right)^2-\left(\frac{\rho_0}{2}-j\right)^2\right\}\cdot\left\{\cot\left(\frac{s-\rho_0}{2}\pi\right)+\cot\left(\frac{s+\rho_0}{2}\pi\right)\right\}.$$

Then for $s\in\mathbf{C}$ with $|\mathrm{Re}(s)|<1$, we have

$$\phi_{-s}^{(2)}(g)=\phi_s^{(2)}(g)+\mathrm{c}(s)\phi_s(g),\quad g\in G^+. \tag{5}$$

The 'bad 'behavior of the function $\phi_s^{(2)}(\exp(tH_0))$ near $t=0$ is controlled by a simple function. Indeed, we have

Proposition 3.1.3. *There exists a function $(s,t)\mapsto \mathrm{Y}_s(t)$ on $\mathbf{C}\times(\mathbf{R}-\{0\})$ with the following properties*

(a) *We can write*

$$\mathsf{Y}_s(t) = \sum_{j=0}^{m-1} \frac{\mathsf{a}_j(s)}{(\sinh t)^{2j}} + \mathsf{b}(s)\log(\sinh^2 t)$$
$$+ \frac{\mathsf{b}(s)}{\Gamma(m)\Gamma(m-1)}\left\{\psi\left(\frac{s+\rho_0}{2}\right) + \psi\left(\frac{s-\rho_0}{2}+1\right)\right\}$$

with polynomial functions $\mathsf{a}_j(s)$ *and* $\mathsf{b}(s)$ *such that*

$$\mathsf{a}_j(-s) = \mathsf{a}_j(s), \quad \deg(\mathsf{a}_j(s)) \leqslant 2(m-j-1), \quad j = 0, \dots, m-1,$$
$$\mathsf{a}_{m-1}(s) = 1,$$
$$\mathsf{b}(s) = (-1)^m \prod_{j=1}^{m-1}\left\{\left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2\right\}.$$

Here $\psi(s)$ *is the digamma function, i.e., the logarithmic derivative of the Gamma function.*

(b) *There exists a family of polynomial functions* $\{\mathsf{c}_n(s)\}_{n\geqslant 1}$ *and* $\{\mathsf{d}_n(s)\}_{n\geqslant 1}$ *such that* $\sum_{n=1}^{\infty} \mathsf{c}_n(s)t^n$ *and* $\sum_{n=1}^{\infty} \mathsf{d}_n(s)t^n$ *have positive radius of convergence and such that*

$$\phi_s^{(2)}(\exp(tH_0)) = \mathsf{Y}_s(t) + \sum_{n=1}^{\infty} \mathsf{c}_n(s)t^n + \log(t)\sum_{n=1}^{\infty} \mathsf{d}_n(s)t^n$$

on $0 < t < \epsilon$ *with a small* $\epsilon > 0$.

We introduce a family of functions $\phi_s^{[r]}$ ($\mathrm{Re}(s) > -1$, $r \in \mathbf{N}_0$) as

Definition 3.1.4. *For* $r \in \mathbf{N}_0$, *we put*

$$\phi_s^{[r]}(g) = \frac{c_G^{-1}}{(2-2m)r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r \phi_s^{(2)}(g), \quad g \in G^+, \quad \mathrm{Re}(s) > -1.$$

The basic property of $\phi_s^{[r]}$ we need is as follows.

Proposition 3.1.5. *Let* $r \in \mathbf{N}_0$ *and* $s \in \mathbf{C}$ *with* $\mathrm{Re}(s) > 0$.

(i) *The function* $\phi_s^{[r]}$ *belongs to* $C^\infty(K\backslash G^+/K)$.

(ii) *We have*

$$\phi_s^{[r]}(\exp(tH_0)) = O(e^{-t(\mathrm{Re}(s)+\rho_0)}),$$

on $t > R$ *with a large* $R > 0$.

(iii) *If* $r \geqslant m$, *then the function* $\phi_s^{[r]}$ *has a continuous extension to all of* G. *We have*

$$\lim_{g\to e, g\in G^+} \phi_s^{[r]}(g)$$
$$= \frac{-c_G^{-1}}{2r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r\left[\frac{\mathsf{b}(s)}{\Gamma(m)^2}\left\{\psi\left(\frac{s+\rho_0}{2}\right) + \psi\left(\frac{s-\rho_0}{2}+1\right)\right\}\right].$$

(iv) *Let* $\mathrm{Re}(s) > \rho_0$. *Then we have*

$$(\Omega - s^2 + \rho_0^2)\phi_s^{[r+1]} = \phi_s^{[r]}, \quad r \in \mathbf{N}_0,$$
$$(\Omega - s^2 + \rho_0)\phi_s^{[0]} = \delta$$

in the sense of distributions on G/K *with* δ *the Dirac delta supported at the origin of* G/K.

We have a characterization of the family $\{\phi_s^{[r]}\}$.

Proposition 3.1.6. *Let* $\{\varphi_{r,s} | \; r \in \mathbf{N}_0, \; \mathrm{Re}(s) > \rho_0\}$ *be a family of bi-*K*-invariant distributions on* G *with the following properties.*

(i) *For* $r \in \mathbf{N}_0$, $\mathrm{Re}(s) > \rho_0$ *the distribution* $\varphi_{r,s}$ *is represented by a* C^∞*-function on* G^+.

(ii) *For* $\mathrm{Re}(s) > \rho_0$,

$$\lim_{t \to +0} t^{2m-2}\varphi_{0,s}(\exp(tH_0)) = 1.$$

(iii) *For* $r \in \mathbf{N}_0$, $\mathrm{Re}(s) > \rho_0$,

$$\varphi_{r,s}(\exp(tH_0)) = O(e^{-t(\mathrm{Re}(s)+\rho_0)}), \quad t \to +\infty.$$

(iv) *Let* $\mathrm{Re}(s) > \rho_0$. *If we regard* $\varphi_{r,s}$*'s as distributions on* G/K, *they satisfy the differential equations*

$$(\Omega - s^2 + \rho_0^2)\varphi_{r+1,s} = \varphi_{r,s}, \quad r \in \mathbf{N}_0,$$
$$(\Omega - s^2 + \rho_0^2)\varphi_{0,s} = \delta.$$

Then for $r \in \mathbf{N}_0$ *and* $s \in \mathbf{C}$, $\mathrm{Re}(s) > \rho_0$ *we have* $\varphi_{r,s}(g) = \phi_s^{[r]}(g)$ *on* G/K *in the sense of distributions.*

3.2. **Miatello-Wallach's spherical functions.** We recall some basic properties of the functions $Q_{r,s}$, $r \in \mathbf{N}$ which Miatello-Wallach introduced and studied in detail ([18]).

(i) For $s \in \mathbf{C}$, $\mathrm{Re}(s) > 0$, $Q_{1,s} \in C^\infty(K\backslash G^+/K)$ ([18, Theorem 1.1 (a)]) .

(ii) For a fixed $g \in G^+$, the function $s \mapsto Q_{1,s}(g)$ is holomorphic on $\mathrm{Re}(s) > 0$ and has a meromorphic continuation to $\mathbf{C}$ ([18, Theorem 1.1,(b)]).

(iii)

$$Q_{1,s}(\exp(tH_0)) \sim \frac{c_G^{-1} sc(s)}{m-1} \cdot t^{2-2m}, \quad t \to +0$$

([18, Theorem 1.1, (d)]).

(iv) Let $\mathrm{Re}(s) > \rho_0$ and $r \in \mathbf{N}$. Then $Q_{r,s}$ is bi-K-invariant and integrable function on G satisfying the formula

$$Q_{r+1,s} = Q_{1,s} \star Q_{r,s}.$$

Here $\star$ means the convolution on G with respect to the measure dg. (see [18, page 678]).

(v) Let $\mathrm{Re}(s) > \rho_0$ and $r \in \mathbf{N}$. Then

$$Q_{r,s}(\exp(tH_0)) = O(e^{-t(\mathrm{Re}(s)+\rho_0)}), \quad t \to +\infty$$

([18, Lemma 2.4]).

(vi) Let $\mathrm{Re}(s) > \rho_0$. Then the distributions $Q_{r,s}$ on G/K satisfy the differential equations

$$(\Omega - s^2 + \rho_0^2)Q_{r+1,s} = -2sc(s)Q_{r,s}$$

for $r \in \mathbf{N}_0$ with the convention that $Q_{0,s} = \delta$, the Dirac delta supported at the origin of G/K ([18, Lemma 2.2, Lemma 2.6]).

Thus the family $\{(-2sc(s))^{-(r+1)}Q_{r+1,s} | \ r \in \mathbf{N}_0, \ \mathrm{Re}(s) > \rho_0\}$ possesses all the properties (i) to (iv) in Proposition 3.1.6. Hence applying that proposition, we have the following theorem, which is one of the main results of this article.

Theorem 3.2.1. *Let* $\mathrm{Re}(s) > \rho_0$ *and* $r \in \mathbf{N}_0$. *Then as distributions on* G/K *the equality*

$$\phi_s^{[r]}(g) = \left(\frac{-1}{2sc(s)}\right)^{r+1} Q_{r+1,s}(g)$$

holds.

4. Miatello-Wallach's function $\mathbf{P}_{r,s}$ and its spectral expansion

4.1. **The function $\mathbf{P}_{r,s}$.** Let $X = G/K$. Let Γ be a neat co-finite lattice of G, that is a discrete torsion-free subgroup of G such that $\Gamma\backslash G$ has finite volume. We assume that if Γ is not cocompact then it satisfies the Langlands' axiom. Here is a notational convention: A point of the double coset space $\Gamma\backslash X$ is denoted by a letter with a dot and any one of the lifts of that point to G is by the same letter without a dot. For example if $x \in G$ then the corresponding coset $\Gamma xK \in \Gamma\backslash X$ is $\dot{x}$.

Let Δ be the Laplacian of $\Gamma\backslash X$ corresponding to $-\Omega$.

In [18], Miatello-Wallach introduced the functions $\mathbf{P}_{r,s}$ ($r \in \mathbf{N}_0$, $\mathrm{Re}(s) > \rho_0$) by

$$\mathbf{P}_{r,s}(\dot{x}, \dot{y}) = \left(\frac{-1}{2sc(s)}\right)^{r} \sum_{\gamma\in\Gamma} Q_{r,s}(x^{-1}\gamma y), \quad \dot{x}, \ \dot{y} \in \Gamma\backslash X$$

with $Q_{r,s}$ the spherical function which we recalled in 3.2. Among other things, they proved that

(a) the series $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$ converges absolutely and defines $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$ holomorphic in s on $\mathrm{Re}(s) > \rho_0$ and smooth in $\dot{x}, \dot{y}$ in the complement of the diagonal of $(\Gamma\backslash X) \times (\Gamma\backslash X)$;

(b) for each $\dot{x} \in \Gamma\backslash X$, as a distribution on $\Gamma\backslash X$ $\mathbf{P}_{r,s}(\dot{x}, -)$ satisfies

$$(\Delta + s^2 - \rho_0^2)^r \mathbf{P}_{r,s}(\dot{x}, -) = \delta(\dot{x})$$

with $\delta(\dot{x})$ the Dirac delta on $\Gamma\backslash X$ supported at $\dot{x}$

([18, page 685, Theorem 3.4], [2, page 621, Theorem 3.2]).

Proposition 4.1.1. *Let* $s \in \mathbf{C}$ *with* $\mathrm{Re}(s) > \rho_0$. *If* $r > m$, *then* $\mathbf{P}_{r,s}(\dot{x}, \dot{y})$ *has a unique continuous extension to all of* $(\Gamma\backslash X) \times (\Gamma\backslash X)$.

By this proposition, we can consider the restriction of $\mathbf{P}_{r,s}(\dot{x},\dot{y})$ to the diagonal $\dot{x}=\dot{y}$ of $(\Gamma\backslash X)\times(\Gamma\backslash X)$. *From now on we assume that* Γ *is cocompact.* Then $\mathbf{P}_{r,s}(\dot{x},\dot{y})$ becomes bounded on $(\Gamma\backslash X)\times(\Gamma\backslash X)$ if $r>m$; in particular the function $\mathbf{P}_{r,s}(\dot{x},\dot{x})$ is integrable on $\Gamma\backslash X$. We want to evaluate the integral

$$\int_{\Gamma\backslash X}\mathbf{P}_{r+1,s}(\dot{x},\dot{x})\,d\dot{x} \tag{6}$$

with $r\geqslant m$ explicitly.

4.2. **Spectral expansion of $\mathbf{P}_{r,s}(\dot{x},\dot{y})$.** In this subsection we compute the integral (6) by using the spectral expansion of $\mathbf{P}_{r+1,s}(\dot{x},\dot{y})$. Since we assume that Γ is cocompact the Laplacian Δ has no continuous spectrum on $L^2(\Gamma\backslash X)$. The eigenvalues of Δ forms a countable subset of non-negative reals enumerated as

$$0=\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_n\leqslant\ldots$$

so that each eigenvalue occurs in this sequence with its multiplicity. Let $\{\varphi_n\}_{n\geqslant 0}$ be the orthonormal basis of $L^2(\Gamma\backslash X)$ such that $\varphi_n\in C^\infty(\Gamma\backslash X)$ and $\Delta\varphi_n=\lambda_n\varphi_n$. For each n we fix a complex number s_n such that $\lambda_n=\rho_0^2-s_n^2$.

Proposition 4.2.1. *Let* $r\in\mathbf{N}_0$ *and* $s\in\mathbf{C}$ *be such that* $r\geqslant m$ *and* $\mathrm{Re}(s)>\rho_0$. *Then*

$$\mathbf{P}_{r+1,s}(\dot{x},\dot{y})=\sum_{n=0}^{\infty}\frac{\overline{\varphi_n(\dot{x})}\varphi_n(\dot{y})}{(s^2-s_n^2)^{r+1}},\quad \dot{x},\ \dot{y}\in\Gamma\backslash X. \tag{7}$$

Here the infinite series in the right-hand side of this identity converges uniformly in $(\dot{x},\dot{y})\in(\Gamma\backslash X)\times(\Gamma\backslash X)$.

By this proposition we can compute the integral (6) in terms of the eigenvalues of Δ.

Proposition 4.2.2. *If* $r\geqslant m$ *and* $\mathrm{Re}(s)>\rho_0$, *then*

$$\int_{\Gamma\backslash X}\mathbf{P}_{r+1,s}(\dot{x},\dot{x})\,d\dot{x}=\sum_{n=0}^{\infty}\frac{1}{(s^2-s_n^2)^{r+1}}. \tag{8}$$

5. COMPUTATION OF THE INTEGRAL $\int_{\Gamma\backslash X}\mathbf{P}_{r,s}(\dot{x},\dot{x})d\dot{x}$ AND THE RESOLVENT TRACE FORMULA

5.1. **Computation of hyperbolic term.** Let Γ be as in the previous section. Then an element $\gamma\in\Gamma-\{e\}$ is G-conjugate to an element h_γ of A^+M with $A^+=\exp((0,+\infty)H_0)$ and M the centralizer of A in K; h_γ is not uniquely determined by γ, but its ambiguity is unimportant for our purpose. We can write

$$h_\gamma=\exp(t_\gamma H_0)m_\gamma,\quad t_\gamma>0,\ m_\gamma\in M.$$

Let G_γ be the centralizer of γ in G and put $\Gamma_\gamma=\Gamma\cap G_\gamma$. Then G_γ is reductive and $\Gamma_\gamma\backslash G_\gamma$ is compact. We fix a Haar measure dg_γ on G_γ in a manner analogous to the manner in which the Haar measure on G was fixed, following the Iwasawa decomposition of G_γ, and put $d\dot{g}_\gamma$ for the invariant measure on $\Gamma_\gamma\backslash G_\gamma$. The group Γ_γ is known to be isomorphic to $\mathbf{Z}$. Hence there exists a unique generator γ_0 of Γ_γ and a positive integer $j(\gamma)$ (the multiplicity of γ) such that $\gamma = \gamma_0^{j(\gamma)}$. Let $\mathcal{H}(\Gamma)$ be the set of Γ-conjugacy classes in $\Gamma - \{e\}$. We first calculate the orbital integral of $\phi_s^{[r]}$ associated with a hyperbolic conjugacy class.

Proposition 5.1.1. *Let $r \in \mathbf{N}_0$ and $\mathrm{Re}(s) > \rho_0$. For $[\gamma] \in \mathcal{H}(\Gamma)$, put*

$$J^{[r]}([\gamma]\,;s) = \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} \phi_s^{[r]}(g^{-1}\gamma g)\, dg_\gamma^*,$$

where dg_γ^* is the G-invariant measure on $G_\gamma\backslash G$ normalized so that $dg = dg_\gamma dg_\gamma^*$. Then the integral $J^{[r]}([\gamma]\,;s)$ converges absolutely and uniformly on $\mathrm{Re}(s) \geqslant \rho_0 + \epsilon$ for any $\epsilon > 0$ and is evaluated as

$$J^{[r]}([\gamma]\,;s) = \frac{1}{r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r\left\{-j(\gamma)^{-1}|\det(1 - \mathrm{Ad}(h_\gamma)^{-1}|_{\mathfrak{n}})|^{-1}\frac{t_\gamma e^{-(s+\rho_0)t_\gamma}}{2s}\right\}.$$

Recall the integral (6), which is expressed by eigenvalues of Laplacian in Proposition 4.2.2. Now we obtain another expression of that integral.

Proposition 5.1.2. (a) *The infinite series*

$$J_{\mathrm{hyp}}(s) = -\sum_{\gamma\in\mathcal{H}(\Gamma)} j(\gamma)^{-1}|\det(1 - \mathrm{Ad}(h_\gamma^{-1})|_{\mathfrak{n}})|^{-1}\frac{t_\gamma e^{-(s+\rho_0)t_\gamma}}{2s}$$

converges absolutely and uniformly on $\mathrm{Re}(s) \geqslant \rho_0 + \epsilon$ for any $\epsilon > 0$.
(b) *If $r \geqslant m$ and $\mathrm{Re}(s) > \rho_0$, then we have*

$$\int_{\Gamma\backslash X} \mathbf{P}_{r+1,s}(\dot{x},\dot{x})d\dot{x} = \mathrm{vol}(\Gamma\backslash G)\left(\lim_{g\to e,\ g\in G+}\phi_s^{[r]}(g)\right) + \sum_{\gamma\in\mathcal{H}(\Gamma)} J^{[r]}([\gamma]\,;s), \tag{9}$$

where the series in the right-hand side of (9) converges absolutely and uniformly on $\mathrm{Re}(s) \geqslant \rho_0 + \epsilon$ for any $\epsilon > 0$.

The assertion is ensured by the next lemma.

Lemma 5.1.3. *Suppose that Γ is a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Then the counting function*

$$\pi_0(T) := \#\{\{\gamma\}_\Gamma \in \mathcal{H}(\Gamma)\,|\,N(\gamma) = e^{t_\gamma} \leqslant T\},\quad T > 0$$

satisfies the growth condition

$$\pi_0(T) = O(T^{2\rho_0}) \quad as \quad T \to \infty.$$

5.2. **The resolvent trace formula.** From Proposotion 4.2.2, Proposition 3.1.5 (iii) and Proposition 5.1.2, we arrive at the formula.

Theorem 5.2.1. *If* $r \geqslant m$ *and* $\operatorname{Re}(s) > \rho_0$, *then we have*

$$\sum_{n=0}^{\infty} \frac{1}{(s^2 - s_n^2)^{r+1}} = \frac{1}{r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r \left(J_{\mathrm{id}}(s) + J_{\mathrm{hyp}}(s)\right)$$

with

$$J_{\mathrm{id}}(s) = \operatorname{vol}(\Gamma\backslash G)\frac{2^{-m-3/2}\pi^{-m}c_0^{m-1/2}}{\Gamma(m)}(-1)^{m+1}$$
$$\times \prod_{j=1}^{m-1}\left\{\left(\frac{s}{2}\right)^2 - \left(\frac{\rho_0}{2} - j\right)^2\right\} \cdot \left\{\psi\left(\frac{s+\rho_0}{2}\right) + \psi\left(\frac{s-\rho_0}{2} + 1\right)\right\},$$
$$J_{\mathrm{hyp}}(s) = -\sum_{\gamma \in \mathcal{H}(\Gamma)} j(\gamma)^{-1} |\det(1 - \operatorname{Ad}(h_\gamma^{-1})|_{\mathfrak{n}})|^{-1} \frac{t_\gamma e^{-(s+\rho_0)t_\gamma}}{2s}.$$

6. Selberg zeta function

6.1. **Analytic continuation of the Selberg zeta function.** We recall the definition of the Selberg zeta function for $\Gamma\backslash X$ with Γ as in the previous section. Let H be a θ-stable Cartan subgroup of G containing A. Then $H = AH^-$ with $H^- = H \cap K$. Let P be the set of those root β for $(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ with $\beta(H_0) > 0$, and Λ the set of linear forms on $\mathfrak{h}_{\mathbf{C}}$ of the form

$$\lambda = \sum_{\beta \in P} n_\beta \beta, \qquad n_\beta \in \mathbf{N}_0. \tag{10}$$

For $\lambda \in \Lambda$ let m_λ denote the number of the ways to express it in the form (10).

Let $\operatorname{Prim}(\Gamma)$ be the set of primitive conjugacy classes in $\mathcal{H}(\Gamma)$,i.e., the set of non-trivial Γ-conjugacy class which is not a power of any other Γ-conjugacy class. Then for $[\gamma] \in \mathcal{H}(\Gamma)$ there exists a unique $[\gamma_0] \in \operatorname{Prim}(\Gamma)$ such that $[\gamma] = [\gamma_0^{j(\gamma)}]$ with $j(\gamma)$ the multiplicity of γ.

Since H^- is a Cartan subgroup of the compact group M, any element of M is M-conjugate to an element of H^-. Hence the G-conjugacy class of a $[\gamma] \in \mathcal{H}(\Gamma)$ contains an element of H expressed as

$$h_\gamma = \exp(t_\gamma H_0)h_\gamma^-, \qquad t_\gamma > 0,\ h_\gamma^- \in H^-.$$

For $\lambda \in \Lambda$ the associated character of H is denoted by $\xi_\lambda : H \to \mathbf{C}^*$. With these notations, the Selberg zeta function for $\Gamma\backslash X$ is defined as the Euler product

$$Z_\Gamma(s) = \prod_{[\gamma] \in \operatorname{Prim}(\Gamma)} \prod_{\lambda \in \Lambda} (1 - \xi_{-\lambda}(h_\gamma)e^{-st_\gamma})^{m_\lambda}. \tag{11}$$

It is easy to see that the logarithmic derivative of $Z_\Gamma(s)$ is related to the function $J_{\text{hyp}}(s)$ by the formula

$$-\frac{1}{2s}\frac{d}{ds}\log Z_\Gamma(s+\rho_0) = J_{\text{hyp}}(s) \tag{12}$$

Hence by Proposition 5.1.2 (a), the infinite product (11) converges absolutely and locally uniformly on $\mathrm{Re}(s) > 2\rho_0$ defining $Z_\Gamma(s)$ holomorphic in s on that half-plane.

Corollary 6.1.1. *The Selberg zeta function $Z_\Gamma(s)$, defined for $\mathrm{Re}(s) > 2\rho_0$, has the analytic continuation as a meromorphic function on the whole complex plane. $Z_\Gamma(s)$ has zeros located at $s = \rho_0 \pm s_n, n \geqslant 0$. If $\lambda_n \neq \rho_0^2$, the order of the zeros at $s = \rho_0 \pm s_n$ equals the multiplicity of the eigenvalue λ_n. If ρ_0^2 is an eigenvalue of the Laplacian Δ, then the order of the zero at $s = \rho_0$ equals twice the multiplicity of the eigenvalue $\lambda_k = \rho_0^2$.*

Remark. (1) For almost all $n \geqslant 0$, s_n is purely imaginary.
(2) We can also show that there exists a meromorphic function $Z_{\text{id}}(s)$ such that

$$-\frac{1}{2s}\frac{d}{ds}\log Z_{\text{id}}(s+\rho_0) = J_{\text{id}}(s).$$

Since the left-hand side of the formula in Theorem 5.2.1 is invariant under $s \to -s$, the completed Selberg zeta function $\widehat{Z}_\Gamma(s) := Z_\Gamma(s)Z_{\text{id}}(s)$ satisfy the symmetric functional equation

$$\widehat{Z}_\Gamma(2\rho_0 - s) = \widehat{Z}_\Gamma(s).$$

The function $Z_{\text{id}}(s)$ is called *gamma factors* (or identity factor) of $Z_\Gamma(s)$. It is known that $Z_{\text{id}}(s)$ is described by the multiple gamma functions. We refer [16], [22] and [9] for this topic.

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