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On certain modular functions derived from modular forms of weight 1/2 (重さ1/2のモジュラー形式から得られるある保型関数について)

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Abstract: In this article, we study certain modular functions $\eta_{\psi}(z)$ similar to the Dedekind eta function $\eta(z)$ and report some results on them. These functions are given by infinite products of Borcherds' type. As in the case of $\eta(z)$, it turns out that the functions $\eta_{\psi}(z)$ have good properties relating to some areas of number-theory. Especially, we investigate their automorphic properties in detail.
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1 Introduction

Let ψ be a Dirichlet character. We assume that ψ is even and *primitive*.

In this paper we study the function

$$\eta_{\psi}(z) := q^{-\frac{1}{2}L(-1,\psi)}\prod_{n=1}^{\infty}(1-q^{n})^{\psi(n)} \qquad (q=\exp(2\pi iz)),$$

where $L(s,\psi)$ denotes the Dirichlet L-function attached to ψ. We mainly consider the case of quadratic characters. In this case the definition is unambiguous, and a few examples already appeared in Ramanujan's Notebooks ([Be3], [Ra3]). We here take a systematic approach and show that the functions $\eta_{\psi}(z)$ have good properties similar to those of the Dedekind eta function

$$\eta(z)=\eta_{\mathbf{1}}(z)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^{n}) \qquad (\mathbf{1} = \text{the trivial character}).$$

There exists a major difference: the Dedekind eta function has weight 1/2, but the functions $\eta_{\psi}(z)$ for $\psi\neq\mathbf{1}$ have weight 0. We also remark that $\eta_{\psi}(z)$ is *not* a product nor a quotient of $\eta(z)$ when $\psi\neq\mathbf{1}$.

There have been various character analogues of the Dedekind eta function defined in the literature, cf. Berndt [Be1], [Be2], Goldstein and Razar [G-R], Meyer [Me] and others. Our functions $\eta_{\psi}(z)$ are also one of such analogues, and from some points of views they seem to be a natural generalization of $\eta(z)$ for the congruence subgroup $\Gamma_0(N)$ (see the notations) of $\mathrm{SL}_2(\mathbf{Z})$.

Notations.

We use the notation $\mathbf{e}(*) := \exp(2\pi i*)$.

We denote by $\mathcal{H}=\{z\in\mathbf{C}|\Im z>0\}$ the complex upper half plane. For $z\in\mathcal{H}$, we write

$$q:=\mathbf{e}(z)=\exp(2\pi iz).$$

For the logarithm $\log w$ ($w \neq 0$), we choose the principal branch with $-\pi < \arg w \leq \pi$.

For a Dirichlet character ψ, we denote by $r = r(\psi)$ the conductor of ψ and by

$$G(\psi) := \sum_{n=1}^{r(\psi)} \psi(n) \mathrm{e}\left(n/r(\psi)\right)$$

the Gauss sum attached to ψ.

We define the congruence subgroup of level N of $\mathrm{SL}_2(\mathbf{Z})$ by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

For each natural number t, we let χ_t be the primitive Dirichlet character associated to the quadratic field $\mathbf{Q}(\sqrt{t})$ (If t is a square number, we let $\chi_t := \mathbf{1}$ be the trivial character).

2 Construction of modular functions

Let ψ be an even primitive Dirichlet character. We define the theta series twisted by ψ:

$$\theta_\psi(z) := \sum_{n \in \mathbf{Z}} \psi(n) q^{n^2}, \qquad q = \exp(2\pi i z).$$

We know θ_ψ is a modular form of weight $1/2$ and character ψ with respect to $\Gamma_0(4r^2)$, where r is the conductor of ψ ([Sh, Proposition 2.2], see also [S-S]).

Let $D > 0$ be a fundamental discriminant and let $\chi_D = \left(\frac{D}{\cdot}\right)$ be the associated quadratic character. Following [Bo, Theorem 14.1], for $\theta_{\chi_D}(z) = \sum_{n \in \mathbf{Z}} \chi_D(n) q^{n^2}$, we define

$$\mathrm{Lift}(\theta_{\chi_D})(z) = \eta_{\chi_D}(z) := q^{-\frac{1}{2}L(-1,\chi_D)} \prod_{n=1}^{\infty} (1 - q^n)^{\chi_D(n)}. \tag{1}$$

Theorem 1 (transformation formula). *Let the notations be as above. Then, for $D > 1$, $\eta_{\chi_D}(z)$ is a (quasi) modular function for $\Gamma_0(D)$ with the transformation formula*

$$\eta_{\chi_D}\left(\frac{az+b}{cz+d}\right) = \nu_D(M) \chi_D^*(d) \eta_{\chi_D}(z)^{\chi_D(d)} \quad \text{for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D), \tag{2}$$

where χ_D^* stands for χ_D or $\mathbf{1}$ according as D is a prime or not, and $\nu_D(M)$ is a root of unity.

The number $\nu_D(M)$ is a 5-th root of 1 if $D = 5$, a 2-th root of 1 if $D = 8$ and always equals to 1 if $D > 8$, respectively (see Theorems 3, 7 *and* (22) *for an explicit form). In particular, $\eta_{\chi_5}(z)^5$, $\eta_{\chi_8}(z)^2$ and $\eta_{\chi_D}(z)$ ($D > 8$) are modular functions for respective $\Gamma_0(D, \chi_D)$, where*

$$\Gamma_0(D, \chi_D) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D) \mid \chi_D(d) = 1 \right\}$$

(which is a subgroup of index 2 in $\Gamma_0(D)$).

The function η_{χ_D} does not have zeros nor poles in the complex upper half plane $\mathcal{H}$.

We emphasize that our functions have a transformation foumula for $\Gamma_0(D)$. We also remark that there exists a major difference between the case $D = 1$ (weight $1/2$) and $D > 1$ (weight 0).

Except for the determination of the root of unity, the theorem is obtained from one of

(1) Weil's converse theorem for weight 0,

(2) Kronecker's limit formulae,

(3) Klein forms (This method determines the root of unity, and is discussed in detail in Sects. 4–8).

Proof. In this section, we sketch the method (1). We prepare the *Weil's converse theorem for weight* 0. This method is closely related to the work by L. J. Goldstein and M. Razar [G-R] which treats "Hecke integrals" in a general setting.

The functions

$$f^*(z) := \log \eta_{\chi_D}(z) = -L(-1,\chi_D)\pi i z - \sum_{m,n\geq 1} \frac{\chi_D(m)}{n}\exp(2\pi i m n z)$$

and

$$g(z) := -\frac{G(\chi_D)}{2}L(1,\bar{\chi}_D) - G(\chi_D)\sum_{m,n\geq 1}\frac{\bar{\chi}_D(m)}{m}\exp(2\pi i m n z) \tag{3}$$

satisfy

$$f^*(-1/Dz) = g(z), \tag{4}$$

which can be shown as in the proof of [Mi, Lemma 4.4.1]. Then, we can apply to $f^*(z)$ and $g(z)$ the Weil's converse theorem for weight 0, which is a direct analogue of [Mi, Theorem 4.3.15] (weight > 0) for the case of weight 0. Thus we obtain the statement except for the determination of the root of unity. □

We shall determine the root of unity (automorphic factor) by two different methods in Sects. 3 and 8.

Next, we mention some interesting relations between η_{χ_D} and the Dedekind eta function. S. Ramanujan (see Entry 10 (iii), (viii) and Entry 11 (iii) of Chap. 19 in [Be3] or [Ra3]) discovered the formulas

$$\begin{aligned} \frac{1}{\eta_{\chi_5}(z)} - 1 - \eta_{\chi_5}(z) &= \frac{1}{q^{1/5}}\prod_{n=1}^{\infty}(1-q^n)^{-\chi_5(n)} - 1 - q^{1/5}\prod_{n=1}^{\infty}(1-q^n)^{\chi_5(n)} \\ &= \frac{1}{q^{1/5}}\prod_{n=1}^{\infty}\frac{1-q^{n/5}}{1-q^{5n}} = \frac{\eta(z/5)}{\eta(5z)}, \end{aligned} \tag{5}$$

$$\frac{1}{(\eta_{\chi_5}(z))^5} + 11 - (\eta_{\chi_5}(z))^5 = \left(\frac{\eta(z)}{\eta(5z)}\right)^6.$$

The first proof of these famous formulas was given by G. N. Watson. See also B. C. Berndt's book [Be3, pp. 265-267], where a short proof based on other formulas of Ramanujan, is given. There are other proofs in the literature: K. G. Ramanathan [Ra2, pp. 696-697] gave another proof for (5); M. D. Hirschhorn has given more than one proof.

In the case of other D, we have similar results:

Theorem 2. *For $D = 13$, we have*

$$\frac{1}{\eta_{\chi_{13}}(z)} - 3 - \eta_{\chi_{13}}(z) = \left(\frac{\eta(z)}{\eta(13z)}\right)^2.$$

For $D = 8$, we have

$$\frac{1}{\eta_{\chi_8}(z)^2} - 6 + \eta_{\chi_8}(z)^2 = \frac{\eta(z)^4\eta(4z)^2}{\eta(2z)^2\eta(8z)^4}.$$

For $D = 12$, we have

$$\frac{1}{\eta_{\chi_{12}}(z)} - 4 + \eta_{\chi_{12}}(z) = \frac{\eta(z)^3\eta(4z)\eta(6z)^2}{\eta(2z)^2\eta(3z)\eta(12z)^3}.$$

Proof. Both sides are modular functions for $\Gamma_0(D)$, which is easily seen by the transformation formula (2) (for the left-hand side) and by [C-N, pp.331-333] (for the right-hand side). We evaluate both modular functions at the cusps, and obtain the results. □

3 Automorphy and reciprocity

We shall carry out the determination of the root of unity $\nu_D(M)$ in Theorem 1. Note that for fundamental discriminants $D = 1,\ 5,\ 8,\ 12,\ 13,\ 17,\ 21,\ 24,\ 28,\ 29,\ 33,\ldots$ we have

$$-\frac{1}{2}L(-1,\chi_D) = \frac{1}{24},\ \frac{1}{5},\ \frac{1}{2},\ 1,\ 1,\ 2,\ 2,\ 3,\ 4,\ 3,\ 6,\ldots \qquad \text{(cf. [Za, Sect. 7])}.$$

For $D = 1,\ 5,\ 8$, η_{χ_D} has a non-trivial multiplier system $\nu_D(M)$.

As a map, $\nu_D(*)$ is a quasi multiplier system in the sense that

Definition. We call a map $\nu_D(M) : \Gamma_0(D) \to \mathbf{C}^1$ (=the unit circle) a *quasi multiplier system* with character χ_D, if for any $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma_0(D)$,

$$\nu_D(M_1M_2) = \nu_D(M_1)\nu_D(M_2)^{\chi_D(d_1)}.$$

Theorem 3. *The number $\nu_D(M)$ in Theorem 1, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$, is explicitly given as follows. For $D = 5$, we have*

$$\nu_5(M) = \mathbf{e}\left(\frac{1}{5}\chi_5(d)bd\right) = \mathbf{e}\left(\frac{1}{5}ab\right).$$

For $D = 8$, we have

$$\nu_8(M) = \mathbf{e}\left(\frac{1}{2}\left(\frac{a-d+bc}{8} - b\right)\right).$$

For $D > 8$, we have

$$\nu_D(M) = 1.$$

Proof. The proof uses what is called "the method by Riemann and Dedekind". First we find that a certain Dedekind's type sums $S(\chi_D; c, a)$ appear in the transformation formula of $\log\eta_{\chi_D}$. Next we find an explicit formula for $\nu_D(M)$ in terms of the components of any matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$ and prove it by induction. In the case of $D = 1$, $\nu_1(M)$ is nothing but the multiplier system of the Dedekind eta function $\eta(z)$ and this analysis is well-known (cf. [Ra1, Chap. 9]). □

From the first part of the proof of Theorem 3, we derive a remarkable formula:

Theorem 4. *We define the number $\phi_D(M) \in \mathbf{Q}$ by*

$$\log\eta_{\chi_D}(Mz) = \chi_D(d)\log\eta_{\chi_D}(z) + 2\pi i\phi_D(M), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D).$$

The map $\phi_D : \Gamma_0(D) \to \mathbf{Q}$ satisfies

$$\phi_D(M_1M_2) = \phi_D(M_1) + \chi_D(d_1)\phi_D(M_2) \quad (\forall M_1,\ \forall M_2 \in \Gamma_0(D)),$$

where d_1 is the right lower entry of M_1. Then, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$ with $c \neq 0$, we have

$$\phi_D(M) = -\frac{1}{2}L(-1,\chi_D)\frac{a+\chi_D(d)d}{c} - \frac{1}{2}S(\chi_D;c,a).$$

Here we define the "twisted Dedekind sum" by

$$S(\chi_D;c,a) := \mathrm{sgn}(c) \sum_{m(\bmod c)} \chi_D(m) \left(\left(\frac{m}{c}\right)\right)\left(\left(\frac{am}{c}\right)\right), \tag{6}$$

where $((x)) = x - [x] - 1/2$ is the (modified) fractional part of x and $\mathrm{sgn}(c) := c/|c|$. □

Remark.
(i) Many authors studied various character analogues of the logarithm of the Dedekind eta function, cf. [Be1], [Be2], [G-R], [Me], [Na]. In each case, certain generalizations of Dedekind sums attached to Dirichlet characters arise in the transformation formulas. In particular, Theoerem 4 was essentially obtained in [Be2, (6.5)] by a different method. We mention that the definition of character analogues of $\log\eta(z)$ in [Be2] omitted the terms involving special values of the Dirichlet L-function.
(ii) We can regard our twisted Dedekind sums $S(\chi_D;c,a)$ as "periods" of Eisenstein series of weight 2.

Now, we find a *reciprocity formula* for twisted Dedekind sums.

Theorem 5. *For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$ with $bc \neq 0$, the following reciprocity formula holds:*

$$S(\chi_D;c,a) + S^*(\chi_D;Db,d) = -L(-1,\chi_D)\frac{a+\chi_D(d)d}{c},$$

where we define the other "twisted Dedekind sum" by

$$S^*(\chi_D;c,a) := \frac{G(\chi_D)}{4c} \sum_{0 \neq m(\bmod c)} \chi_D(m)\cot\left(\pi\frac{m}{c}\right)\cot\left(\pi\frac{am}{c}\right). \tag{7}$$

Proof. For $g(z) = \log\eta_{\chi_D}(-1/Dz)$ defined by (3), in a similar way to the proof of Theorem 4 (the method by Riemann and Dedekind), we can prove

$$\phi_D^*(M) = -\frac{1}{2}S^*(\chi_D;c,a).$$

Therefore, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$, we have

$$\log\eta_{\chi_D}\left(\frac{-a/Dz+b}{-c/Dz+d}\right) = \chi_D(d)\log\eta_{\chi_D}(-1/Dz) - \pi i L(-1,\chi_D)\frac{a+\chi_D(d)d}{c} - \pi i S(\chi_D;c,a)$$

(by Theotem 4) on the one hand and

$$\begin{aligned}
\log\eta_{\chi_D}\left(\frac{-a/Dz+b}{-c/Dz+d}\right) &= \log\eta_{\chi_D}\left(\frac{Dbz-a}{Ddz-c}\right) \\
&= \log\eta_{\chi_D}(-1/Dz') \quad \left(z' := \frac{dz-c/D}{-Dbz+a}\right) \\
&= g(z') = g\left(\frac{dz-c/D}{-Dbz+a}\right) \quad \text{(by (4))} \\
&= \chi_D(a)g(z) - 2\pi i\frac{1}{2}S^*(\chi_D;-Db,d)
\end{aligned}$$

on the other hand. By comparing these two expressions, we get the result. □

Remark.
(i) For $D > 1$, two twisted Dedekind sums $S(\chi_D; c, a)$ and $S^*(\chi_D; c, a)$ defined by (6) and (7) do *not* coincide with each other, in general.
(ii) Related results for generalized Dedekind sums attached to Dirichlet characters were obtained by Berndt [Be1, Theorems 4, 7], [Be2, Theorem 7] and Nagasaka [Na, Theorem 2].

4 The Legendre relation

We need so-called the Legendre relation (Riemann condition, CCR=canonical commutative relation, etc.). First, we define the notion of Eisenstein summation. For some one-variable function $f(x)$, we define the Eisenstein sum of f by

$$\sum_{m}{}_{\mathrm{e}} f(m) = \lim_{M\to\infty} \sum_{m=-M}^{M} f(m).$$

For the case of two-variable functions, we define $\sum_{m,n}{}_{\mathrm{e}} = \sum_{m}{}_{\mathrm{e}} \sum_{n}{}_{\mathrm{e}}$. We define the Eisenstein product by a similar way. If we write $\sum_{\mathrm{e}}'$, then it means that we remove $m = 0$ or $(m, n) = (0, 0)$ from the summation. The infinite product of the sine function is written by

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right) = x \prod_{n}{}'_{\mathrm{e}} \left(1 - \frac{x}{\pi n}\right). \tag{8}$$

From this formula, we deduce (for $x \notin \pi\mathbf{Z}$)

$$\frac{\sin(x-t)}{\sin(x)} = \prod_{n}{}_{\mathrm{e}} \left(1 - \frac{t}{x + n\pi}\right). \tag{9}$$

Next, we consider Eisenstein series for a posivie integer k and a basis $\varpi = \begin{pmatrix} \varpi_2 \\ \varpi_1 \end{pmatrix}$ of a lattice in $\mathbf{C}$. We define

$$e_k(\varpi) = \sum_{n}{}'_{\mathrm{e}} \frac{1}{n\varpi^k} = \sum_{n_1,n_2}{}'_{\mathrm{e}} \frac{1}{(n_1\varpi_2 + n_2\varpi_1)^k}$$

and

$$E_k(u; \varpi) = \sum_{n}{}'_{\mathrm{e}} \frac{1}{(u + n\varpi)^k} = \sum_{n_1,n_2}{}_{\mathrm{e}} \frac{1}{(u + n_1\varpi_2 + n_2\varpi_1)^k},$$

where $n = (\,n_1 \ n_2\,)$. For larger k (≥ 3), these series are independent of the summation process. We take another basis ϖ' of some sublattice of the lattice which is generated by ϖ. Then, e_{2k} satisfies the *distribution relation*

$$e_{2k}(\varpi') + \sum_{r\in R'} E_{2k}(r; \varpi') = \begin{cases} e_2(\varpi) - \dfrac{2\pi i\delta c}{\varpi_1\varpi_1'}, & \text{if } 2k = 2, \\ e_{2k}(\varpi), & \text{if } 2k \geq 4. \end{cases} \tag{10}$$

Here we write $\varpi' = \gamma\varpi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\varpi$, take R a set of representatives of the lattice generated by ϖ modulo the sublattice generated by ϖ' and put $R' := R - \{0\}$. The number $\delta = \pm 1$ is the sign of the orientation of ϖ, i.e, it is equal to $+1$ if $z = \varpi_2/\varpi_1$ is on the upper half plane $\mathcal{H}$, and -1 otherwise.

Especially, for $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Im z > 0$, then $\delta = 1$, R' is empty and (10) becomes

$$e_2(\varpi') - e_2(\varpi) = -\frac{2\pi i}{\varpi_1 \varpi_2}. \tag{11}$$

By comparing the Fourier development of e_2 and the eta constant, we know

$$e_2(\varpi) = \frac{\eta_1(\varpi)}{\varpi_1}.$$

Therefore, the equation (11) is nothing but the *Legendre relation*

$$\eta_1\varpi_2 - \eta_2\varpi_1 = 2\pi i, \tag{12}$$

where $\eta_2 := \eta_1(\gamma\varpi)$.

Remark.
All the facts in this section are found in the text book of Weil [We2], in particular Chap. III.

5 Some properties of the sigma function

We define the Weierstrass sigma function by the infinite product

$$\sigma(u;\varpi) = u\prod_n{}' \left(1 - \frac{u}{n\varpi}\right)\exp\left(\frac{u}{n\varpi} + \frac{u^2}{2n\varpi^2}\right), \tag{13}$$

where $u \in \mathbf{C}$, $\varpi = \begin{pmatrix} \varpi_2 \\ \varpi_1 \end{pmatrix}$ and we denote $n = (\, n_1 \; n_2 \,) \in \mathbf{Z}^2$.

We apply the Eisenstein product for this product (we call this operation "observation"):

$$\begin{aligned}
\sigma(u;\varpi) &= u\prod_{n}{}'_{\mathrm{e}} \left(1 - \frac{u}{n\varpi}\right)\exp\left(\frac{u^2}{2n\varpi^2}\right) \\
&= u e^{\frac{1}{2}u^2 e_2(\varpi)} \prod_{n}{}'_{\mathrm{e}} \left(1 - \frac{u}{n\varpi}\right).
\end{aligned}$$

Moreover, by using (8) and (9), we have

$$\begin{aligned}
\sigma(u;\varpi) &= \frac{\varpi_1}{\pi}\exp\left(\frac{1}{2}u^2 e_2(\varpi)\right)\sin\left(\frac{\pi u}{\varpi_1}\right)\prod_{n_1}{}'_{\mathrm{e}} \frac{\sin\left(\frac{\pi(n_1\varpi_2 - u)}{\varpi_1}\right)}{\sin\left(\frac{\pi n_1 \varpi_2}{\varpi_1}\right)} \\
&= \frac{\varpi_1}{2\pi i}\exp\left(\frac{1}{2}\varpi_1\eta_1(\varpi)v^2\right)\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)\prod_{n=1}^{\infty}\frac{(1 - x^{-1}q^n)(1 - xq^n)}{(1 - q^n)^2},
\end{aligned} \tag{14}$$

where $z = \varpi_2/\varpi_1$, $v = u/\varpi_1$, $q = \mathrm{e}(z)$ and $x = \mathrm{e}(v)$. This is almost equal to the Jacobi theta function. In fact ϑ_{11} has the triple product expansion

$$\vartheta_{11}(v;z) = -iq^{\frac{1}{8}}\,(x^{\frac{1}{2}} - x^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1 - q^n)(1 - x^{-1}q^n)(1 - xq^n). \tag{15}$$

Combining (14) and (15), we have

$$\sigma(u;\varpi) = -\frac{\varpi_1}{2\pi}\exp\left(\frac{1}{2}\varpi_1\eta_1(\varpi)v^2\right)\frac{\vartheta_{11}(v;z)}{\eta(z)^3}. \tag{16}$$

This $\eta(z) = q^{1/24}\prod(1-q^n)$ is the Dedekind eta function.

Next, we consider the automorphic properties of the sigma function. We restrict our consideration to only the action by $\mathrm{SL}_2(\mathbf{Z})$. Assume that u is independent of ϖ and $\alpha = (\alpha_1\ \alpha_2) \in \mathbf{R}^2$, then the action by $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ on ϖ is as the following:

$$\sigma(u + \alpha\gamma\varpi; \gamma\varpi) = \sigma(u + \alpha\gamma\varpi; \varpi).$$

In particular, if $\alpha = 0$, by (16), we have

$$1 = \frac{\sigma(u+\alpha\varpi;\gamma\varpi)}{\sigma(u+\alpha\varpi;\varpi)} = \frac{\varpi_1'}{\varpi_1}\frac{\vartheta_{11}(v';z')}{\vartheta_{11}(v;z)}\frac{\eta(z)^3}{\eta(z')^3}\exp\left(\frac{u^2}{2}\left(\frac{\eta_1'}{\varpi_1'} - \frac{\eta_1}{\varpi_1}\right)\right),$$

where $\varpi_1' = c\varpi_2 + d\varpi_1$, $\eta_1' = c\eta_2 + d\eta_1$, $v' = v/\varpi_1'$ and $z' = (az+b)/(cz+d)$. Note that

$$\frac{\eta_1'}{\varpi_1'} - \frac{\eta_1}{\varpi_1} = \frac{2\pi i c}{\varpi_1(c\varpi_2 + d\varpi_1)},$$

by the distribution relation (10). Now, we shall *use* the automorphy of the Dedekind eta function

$$\eta(z') = \varepsilon_\gamma (cz+d)^{\frac{1}{2}}\eta(z), \quad |\varepsilon_\gamma| = 1.$$

If we put $\varpi = Z = \begin{pmatrix} z \\ 1 \end{pmatrix}$, then we obtain the automorphy of the Jacobi theta function

$$\vartheta_{11}\left(\frac{v}{cz+d}; \frac{az+b}{cz+d}\right) = \varepsilon_\gamma^3 \mathbf{e}\left(\frac{cv^2}{2(cz+d)}\right)(cz+d)^{\frac{1}{2}}\vartheta_{11}(v;z). \tag{17}$$

Put $\beta_{11} = (1/2\ 1/2)$. For $\beta = (\beta_1\ \beta_2) = \alpha + \beta_{11} = (\alpha_1\ \alpha_2) + (1/2\ 1/2) \in \mathbf{R}^2$, the characterized theta functions are defined by

$$\vartheta_\beta(v;z) = \mathbf{e}\left(\frac{1}{2}\alpha_1^2 z + \alpha_1 v + \alpha_1\beta_2\right)\vartheta_{11}(v + \alpha Z; z)$$

(hence $\vartheta_{\beta_{11}} = \vartheta_{11}$). We deduce the automorphy of these functions by (17):

$$\vartheta_\beta\left(\frac{v}{cz+d}; \frac{az+b}{cz+d}\right) = \varepsilon_\gamma^3 (cz+d)^{\frac{1}{2}}\mathbf{e}\left(\frac{cv^2}{2(cz+d)}\right)\delta_\gamma(\alpha)\vartheta_{\beta'}(v;z), \tag{18}$$

$$\delta_\gamma(\alpha) = \mathbf{e}\left(\alpha_1'\alpha_2' - \alpha_1\alpha_2 + \alpha_1\left(\alpha_1 + \frac{1}{2}\right) - \alpha_1'\left(\alpha_1' + \frac{1}{2}\right)\right),$$

where $\alpha' = \alpha\gamma$ and $\beta' = \alpha' + \beta_{11}$. Specially, if we take $\beta \in ((1/2)\mathbf{Z})/\mathbf{Z})^2$, we obtain the automorphy of the Jacobi theta functions:

$$\delta_\gamma(\alpha) = \begin{cases} 1, & \text{if } \beta = (1/2\ 1/2) = \beta_{11}, \\ ((a+c)(b+d-a-c+1)-1)/4, & \text{if } \beta = (0\ 0), \\ a(b-a+1)/4, & \text{if } \beta = (1/2\ 0), \\ c(d-c+1)/4, & \text{if } \beta = (0\ 1/2), \end{cases}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.

For $\alpha \neq 0$, the automorphy is not within one theta function, and some authors describe it under the restriction to suitable subgroups of $\mathrm{SL}_2(\mathbf{Z})$.

A direct proof of the automorphy (18) is given by Ibukiyama [Ib]. Our proof is the same as Weber [We1], Weil [We2], Rademacher [Ra1] and others, but we avoid the calculation of ε_γ.

The sigma function has a factorization into automorphic funcitons. Put $f(x) = (1-x)e^{x+x^2/2}$. We then define the n-th part of the sigma function:

$$\sigma^{(n)}(u;\varpi) := \prod_{\gcd(m_1,m_2)=n} f\left(\frac{u}{m\varpi}\right) = \sigma^{(1)}\left(\frac{u}{n};\varpi\right).$$

We define $\langle n\rangle g(\varpi) := g\begin{pmatrix} n\varpi_2 \\ n\varpi_1 \end{pmatrix}$ and put $T := \sum_{n=1}^{\infty} \langle n\rangle$. Then the original sigma funciton is written by

$$\sigma(u;\varpi) = u\left(\sigma^{(1)}(u;\varpi)\right)^{T} = u\left(\sigma^{(1)}(u;\varpi)\right)^{\zeta(-1,\lambda)},$$

where we put $\lambda_p := \langle p\rangle$ and write $\zeta(s,\lambda) := \prod_p (1-\lambda_p^{-s})$. This argument is the same for Eisenstein series.

We need a slightly different automorphy from above. We consider it in the next section.

In the rest of this section, we shall prove the quasi-periodicity of the sigma funciton. It is well-known and simply obtained in the system of Weierstrass's elliptic functions. But we would like to get it directly from our starting definiton (13). Shortly saying, the translation of ϖ_1's direction has no effect in the Eisenstein product (because $\varpi_1 \to 1$ and q is an invariant of translation of integers), and the effect appears in the exponential factor. The translation of ϖ_2's direction appears in the Eisenstein product as a finite quotient. Above two are not independent. Finally, the Legendre relation (12) decides their twist to the quasi-periodicity as a well-known form.

For $n' = (\, n_1' \; n_2' \,) \in \mathbf{Z}^2$, we have the *quasi-periodicity*

$$\sigma(u+n'\varpi;\varpi) = (-1)^{n_1'+n_2'+n_1'n_2'} e^{\frac{1}{2}n'\eta(2u+n'\varpi)}\sigma(u;\varpi), \tag{19}$$

where $\eta = \begin{pmatrix} \eta_2 \\ \eta_1 \end{pmatrix}$ is the eta constants with respect to the basis ϖ.

In fact, we see

$$\begin{aligned}
&\frac{\sigma(u+n'\varpi;\varpi)}{\sigma(u;\varpi)} \\
= \;& \frac{u+n'\varpi}{u}\prod_n{}' \frac{1-\frac{u+n'\varpi}{n\varpi}}{1-\frac{u}{n\varpi}} e^{\frac{n'\varpi}{n\varpi}+\frac{1}{2(n\varpi)^2}\{2un'\varpi+(n'\varpi)^2\}} \\
= \;& \prod\nolimits_{\mathrm{e}} \left(1-\frac{-n'\varpi}{u+n\varpi}\right) e^{\frac{c_2}{2}\{2un'\varpi+(n'\varpi)^2\}} \\
= \;& e^{\left(u+\frac{1}{2}n'\varpi\right)n'\varpi e_2} \prod_{n_1}{}_{\mathrm{e}} \prod_{n_2}{}_{\mathrm{e}} \left(1-\frac{-n'\varpi}{u+n_1\varpi_2+n_2\varpi_1}\right) \\
= \;& e^{\left(u+\frac{1}{2}n'\varpi\right)\left(n'\eta+\frac{2\pi i}{\varpi_1}n_1'\right)} \prod_{n_1}{}_{\mathrm{e}} \frac{\sin\pi(v+n_1z-n_1'z-n_2')}{\sin\pi(v+n_1z)} \\
= \;& e^{\left(u+\frac{1}{2}n'\varpi\right)n'\eta+2\pi i v n_1+\pi i(n_1'z+n_2')n_1'}(-1)^{n_2'}\frac{\sin\pi(v-n_1'z)}{\sin\pi v} \\
& \times \lim_{N\to\infty}\prod_{n_1=1}^{N} \frac{\sin\pi(v+(n_1'+n_1)z)\sin\pi(v+(n_1'-n_1)z)}{\sin\pi(v+n_1z)\sin\pi(v-n_1z)} \\
= \;& (-1)^{n_2'} e^{\left(u+\frac{1}{2}n'\varpi\right)n'\eta} x^{n_1'} q^{\frac{n_1'^2}{2}} (-1)^{n_1'n_2'} q^{\frac{n_1'}{2}} \frac{1-xq^{-n_1'}}{1-x}
\end{aligned}$$

$$
\begin{aligned}
& \times \lim_{N\to\infty} \prod_{n_1=1}^{N} \frac{(1-xq^{n_1-n'_1})(1-x^{-1}q^{n_1-n'_1})}{(1-xq^{n_1})(1-x^{-1}q^{n_1})} \\
= \quad & (-1)^{n'_2+n'_1n'_2} e^{(u+\frac{1}{2}n'\varpi)n'\eta} x^{n'_1} q^{\frac{1}{2}n'_1(n'_1+1)} \frac{1-xq^{-n'_1}}{1-x} \\
& \times \lim_{N\to\infty} \prod_{l=1}^{N} \frac{(1-xq^{N+l})(1-x^{-1}q^{1-l})}{(1-xq^{l})(1-x^{-1}q^{N-l})} \\
= \quad & (-1)^{n'_2+n'_1n'_2} e^{(u+\frac{1}{2}n'\varpi)n'\eta} x^{n'_1} q^{\frac{1}{2}n'_1(n'_1+1)} \frac{1-xq^{-n'_1}}{1-x} \frac{1-x^{-1}}{1-x^{-1}q^{-n'_1}} \prod_{l=1}^{n'_1} (-x^{-1}q^{-l}) \\
= \quad & (-1)^{n'_1+n'_2+n'_1n'_2} e^{(u+\frac{1}{2}n'\varpi)n'\eta}.
\end{aligned}
$$

As mentioned above, we used the Legendre relation (12).

Remark.

The first infinite product (13) of the Weierstrass sigma function has the most higher symmetric shape. When we fix a specified generator of the lattice, this symmetry breaks. But the Legendre relation restores it in many scenes. Symmetry means "indetermination", therefore we named the process of "fix basis" and "apply the Eisensten summation (product)" "observation". We consider that a relation of two observations has a right to be called canonical commutative relations as quantum mechanics. But we do not know the exact correspondence between these terminology.

Indetermination has two faces for the sigma function: the first is indetermination between ϖ and $\gamma\varpi$ (where γ is in $\mathrm{GL}_2(\mathbf{Z})$), and the second is that between u and $u + n\varpi$. The first means the automorphy, and the second means the periodicity.

Observation breaks the symmetrical ϖ-expression to a special q-expression. Therefore, trivial things on ϖ may be complex on q.

Needless to say, there are many remaining problems. In particular, we did not consider the automorphy of the original Dedekind eta function. Does the eta function $\eta(z)$ have a homogeneous ϖ-expression ? Our only knowledge is that η^2 is a product of 7 Klein forms.

We see the sigma function as a solution of the multiplicative Cousin problem for lattices on $\mathbf{C}$. Therefore, we can imagine that if we find an explicit solusion of the multiplacative Cousin problem for lattices on $\mathbf{C}^g$ then it is an automorphic function of degree g.

6 Klein forms

In this section we introduce the Klein forms and show their automorphy. Fix the notations: $\mathcal{A}(M) = [0,1)^2$ (resp. $\mathbf{Q}^2 \cap [0,1)^2$ or $((1/N)\mathbf{Z}/\mathbf{Z})^2 \cap [0,1)^2$) for $M = \mathbf{R}$ (resp. $\mathbf{Q}$ or $(1/N)\mathbf{Z}$). For $\alpha = (\,\alpha_1\ \alpha_2\,) \in \mathcal{A}(\mathbf{R})$ and ϖ, we define the Klein form

$$\mathbf{k}_\alpha(\varpi) = e^{-\frac{1}{2}\alpha\eta\alpha\varpi}\sigma(\alpha\varpi;\varpi).$$

We see these functions are automorphic functions for $\mathrm{SL}_2(\mathbf{Z})$ in the sense as follows:

Theorem 6. *For* $\gamma \in \mathrm{SL}_2(\mathbf{Z})$

$$\mathbf{k}_\alpha(\gamma\varpi) = (-1)^{n_1+n_2+n_1n_2}\mathbf{e}\left(\frac{1}{2}\alpha\gamma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}{}^t\alpha'\right)\mathbf{k}_{\alpha'}(\varpi), \tag{20}$$

where $\alpha' \in \mathcal{A}(\mathbf{R})$ *with* $\alpha\gamma - \alpha' = n = (\,n_1\ n_2\,) \in \mathbf{Z}^2$.

Proof. In fact, by the quasi-periodicity (19), we have

$$\begin{aligned}
\mathbf{k}_\alpha(\gamma\varpi) &= e^{-\frac{1}{2}\alpha\gamma\eta\alpha\gamma\varpi}\sigma(\alpha\gamma\varpi;\gamma\varpi) \\
&= (-1)^{n_1+n_2+n_1n_2}e^{\frac{1}{2}(\alpha\gamma\eta\alpha'\varpi-\alpha'\eta\alpha\gamma\varpi)}\mathbf{k}_{\alpha'}(\varpi) \\
&= (-1)^{n_1+n_2+n_1n_2}\mathbf{e}\left(\frac{1}{2}\alpha\gamma\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}{}^t\alpha'\right)\mathbf{k}_{\alpha'}(\varpi),
\end{aligned}$$

where we used the Legendre relation (12) in the last equality. □

The automorphic factor $\varepsilon(\gamma,\alpha) := \mathbf{k}_\alpha(\gamma\varpi)/\mathbf{k}_{\alpha'}(\varpi)$ has complex absolute value 1. Furthermore, by the Legendre relation (12) again, the Klein forms are written by

$$\mathbf{k}_\alpha(\varpi) = -\frac{\varpi_1}{2\pi i\eta(z)^2}q^{\frac{1}{2}B_2(\alpha_1)}\mathbf{e}\left(\frac{1}{2}\alpha_2(\alpha_1-1)\right)\prod_{n=1}^{\infty}(1-x^{-1}q^n)(1-xq^{n-1}). \tag{21}$$

If we choose α in $\mathcal{A}(\mathbf{Q})$ [resp. $\mathcal{A}((1/N)\mathbf{Z})$], then α' is in the same set for every $\gamma\in \mathrm{SL}_2(\mathbf{Z})$. Therefore, if we consider suitable products or quotients of Klein forms, then they become true modular functions of $\mathrm{SL}_2(\mathbf{Z})$ for some congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$.

7 Integrality of $L(-1,\chi_D)/2$

Theorem 7. *For every fundamental discriminant $D>8$, $L(-1,\chi_D)/2$ is an integer.*

Proof. The proof is devided into two parts. The first part shows the theorem for prime discriminants. The second part reduces the general case to the case of prime discriminant (which is possibly negative).

Part one. Let $D=p>5$ be an odd prime number. We see

$$\begin{aligned}
\frac{1}{2}L(-1,\chi_p) &\equiv -\frac{p}{2}\sum_{n=1}^{p'}\chi_p(n)B_2(n/p) \pmod{\mathbf{Z}} \\
&= -\frac{1}{2p}\sum\chi_p(n)n^2+\frac{1}{2}\sum\chi_p(n)n \quad (\in \mathbf{Z}_2,\mathbf{Z}_3,\mathbf{Z}_5) \\
&\equiv -\frac{1}{4p}\sum(n^4-rn^4)+\frac{1}{2}\sum n \pmod{\mathbf{Z}} \\
&= \frac{r-1}{120p}p'(p'+1)p(3p'^2+3p'-1)+\frac{p^2-1}{16} \\
&= \left(\frac{r-1}{2}\cdot\frac{3p^2-7}{60}+1\right)\frac{p^2-1}{16} \quad (\in \mathbf{Z}_p),
\end{aligned}$$

where $p'=(p-1)/2$ and r is the square of an odd primitive root modulo p. Hence we obtain the result.

Part two. Write $D=D_1D_2$ with (possibly negative) fundamental discriminants D_1 and D_2 such that $|D_i|>1$ ($i=1,\ 2$) and they are prime to each other. By the Chinese remainder theorem, we can show

$$\frac{1}{2}L(-1,\chi_D)\equiv D_1Cm_1^2L(-1,\chi_{D2}) \pmod{\mathbf{Z}},$$

with some integers C and m_1. If $D_2<0$, then $L(-1,\chi_{D_2})$ vanishes by the gamma factor. Only exceptional discriminant is 40 and we calculate directly in this case. In the other case, we can put $D_2=p>5$ and reduce to the part one. □

8 Proof of the automorphy

We are now ready to complete the proof of the automorphy of η_{χ_D} (Theorems 1 and 3) by using those of the Klein forms (Theorem 6). Put $\alpha = (\,\alpha_1/D\ 0\,)$ and $\varpi = \begin{pmatrix} Dz \\ 1 \end{pmatrix}$, then by (21)

$$\mathrm{k}_\alpha(\varpi) = -\frac{q^{\frac{1}{2}B_2(\alpha_1)}}{2\pi i\eta(Dz)^2}(1-q^{\alpha_1})\prod_n(1-q^{Dn-\alpha_1})(1-q^{Dn+\alpha_1}).$$

Therefore, we can express our η_{χ_D} as a product of Klein forms:

$$\eta_{\chi_D}(z) = \prod_{n\in(\mathbf{Z}/D\mathbf{Z})^\times/\pm1}\left(\mathrm{k}_{(n/D\ 0)}\begin{pmatrix} Dz \\ 1 \end{pmatrix}\right)^{\chi_D(n)}.$$

Then we deduce from Theorem 6 that the automorphic factor of η_{χ_D} for $\gamma\in\Gamma_0(D)$ is

$$\frac{\eta_{\chi_D}(\gamma z)}{\eta_{\chi_D}(z)^{\chi_D(a)}} = \mathbf{e}\left(-\frac{ab}{2}L(-1,\chi_D)\right)\times\begin{cases}\chi_D(a), & \text{if } D \text{ is a prime number},\\ 1, & \text{otherwise},\end{cases} \tag{22}$$

when $D \neq 8$. By Theorem 7 in the previous section, the exponential factor is 1 except for the cases $D = 5$ or 8. In these cases, we calculate directly. This gives a complete proof of Theorems 1 and 3. □

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