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Vanishing of certain spaces of cusp forms with small weights and some applications

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0 Introduction

The aim of this note is to explain briefly the joint work [AB2] with S.Böcherer concerning the vanishing of certain spaces of modular forms with small weights ant its applications.

First we describe the motivation which made us to study such kind of problems. The starting point was the conjecture presented by K.Hashimoto about 1990 (published in [Ha2] in 1998). His conjecture was concerning the linear dependences of several kinds of certain theta series associated to Eichler orders of definite quaternion algebras over $\mathbb{Q}$. It was supported by many numerical computations by himself ([Ha2]). Böcherer and the author independently became interested in the conjecture, started a joint research about 1998, and succeeded in solving it by distinct methods ([AB2]). In the following we briefly explain a summary of the conjecture and describe how certain spaces of elliptic modular (or Jacobi) forms were introduced and how we were led to the solution. Moreover we explain some applications of our results.

1 Hashimoto's Conjecture

We abbreviate $\exp(2\pi i w)$ ($w \in \mathbb{C}$) to $e(w)$. Let B be a definite quaternion algebra over $\mathbb{Q}$ and $d(B)$ the product of prime integers that are ramified in B over $\mathbb{Q}$. Set $q := d(B)$.

For a positive integer $N = qN_2$ with $(N_2, q) = 1$, an order of B with level N is defined as an order of $\mathcal{O} \subset B$ satisfying the following conditions:

1. $\mathcal{O}_p := \mathcal{O} \bigotimes_{\mathbb{Z}_p} \mathbb{Z}$ is a maximal order of $B_p := B \bigotimes \mathbb{Q}_p$, if p is a prime integer dividing q.

2. $\mathcal{O}_p \cong \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ N\mathbb{Z}_p & Z_p \end{pmatrix}$ for any prime onteger p not dividing q.

If N is square free, such an order is called an Eichler order.

Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_T$ be all representatives of conjugacy classes of orders with level $N = qN_2$. The number $T = T(q, N_2)$ is called the type number of level qN_2 and has been computed explicitly by many authors. Among others Hasegawa-Hashimoto ([Ha1], [HH]) represented $T(q, N_2)$ as the sum of dimensions of certain spaces of cusp forms of weight 2.

Let $M_k(N) = M_k(\Gamma_0(N))$ (resp. $J_{k,1}(\Gamma_0(N))$) denote the space of modular forms of weight k (resp. Jacobi forms of weight k and index 1) on $\Gamma_0(N)$. Let $S_k(N) = S_k(\Gamma_0(N))$ be the subspace of $M_k(N)$ consisting of cusp forms.

We define the following four types of theta series attached to orders $\mathcal{O}_j$ $(1 \leq j \leq T)$ with level N:

$$\vartheta_j(\tau) := \sum_{a \in \mathcal{O}_j} e\big(N(a)\tau\big), \qquad \vartheta_j^J(\tau, z) := \sum_{a \in \mathcal{O}_j} e\big(N(a)\tau + \mathrm{tr}(a)z\big)$$

$$\vartheta_j^I(\tau) := \sum_{\substack{a \in \mathcal{O}_j \\ \mathrm{tr}(a)=0}} e\big(N(a)\tau\big), \qquad \vartheta_j^{II}(\tau) := \sum_{\substack{a \in \mathbb{Z}+2\mathcal{O}_j \\ \mathrm{tr}(a)=0}} e\big(N(a)\tau\big).$$

Then, $\vartheta_j(\tau) \in M_2(N)$, $\vartheta_j^J(\tau, z) \in J_{k,1}(\Gamma_0(N))$, and $\vartheta_j^I(\tau)$, $\vartheta_j^{II}(\tau)$ are modular forms of half-integral weight 3/2 with level $4N$.

Our concern is the linear (in)dependence of these theta series. Hashimoto presented the following conjecture, whose details were discussed in [Ha2] and we solved it in [AB2].

Conjecture 1 (Hashimoto, 1990) *Assume N is square free.*

$$(A)\ \sum_{j=1}^T c_j\vartheta_j(\tau) = 0 \iff (B)\ \sum_{j=1}^T c_j\vartheta_j^{II}(\tau) = 0 \iff (C)\ \sum_{j=1}^T c_j\vartheta_j^{I}(\tau) = 0$$

$$\Updownarrow$$

$$(B')\ \sum_{j=1}^T c_j\vartheta_j^J(\tau, z) = 0.$$

Here c_j's are some constants.

We make some remarks.

Remark **1.** The equivalences of the assertions above are easily proved except the case of $(A) \Longrightarrow (B)$ (or $(A) \Longrightarrow (B')$).

2. If N is not square free, as is discussed in [Ha3] Hashimoto also conjectured that the assertions $(A) \iff (B) \iff (B')$ will hold true.

3. If $4 \mid N$, then the assertion $(C) \Longrightarrow (B)$ is not true, though the opposite one $(B) \Longrightarrow (C)$ always holds.

We explain the background of this conjecture. Gross [Gr] was the first who recoginizec the significance of the problem of linear (in)dependences of the theta series concerned. He showed that in case of $q = p$, a prime integer, and $N = q$, a linear relation of $\{\vartheta_j^{II}(\tau)\}$ implies the existence of an eigen form $f \in S_2(q)$ with $L(f,1) = 0$. Then Böcherer, Schulze-Pillot generalized this result to the case of Eichler orders (i.e., N is square free) in a more concrete fashion. We exhibit here only a part of their results. Let $\Theta^{II}(q, N_2)$ be the $\mathbb{C}$-linear span of the theta series $\vartheta_j^{II}(\tau)$ $(1 \le j \le T(q, N_2))$.

Theorem 2 (Gross [Gr], and Böcherer, Schulze-Pillot [BS]) *Assume that N is a square free positive integer. Let $g(\tau) \in S_{3/2}^q(N)$, a new form, and $f(\tau) \in S_2(\Gamma_0(N))$ a normalized new form corresponding to g by the Shimura correspondence. Then,*

$$g \in \Theta^{II}(q, N_2) \iff L(f,1) \neq 0$$

and moreover

$$L(f,1)g(\tau) = c \cdot \sum_{j=1}^{H} \frac{\langle g, \vartheta_j^{II}\rangle}{e_j} \vartheta_j^{II}(\tau),$$

where $c \neq 0$ is a constant depending only on q, N_2, and $H = H(q, N_2)$ is the class number of the Eichler order with level $N = qN_2$. Here $\vartheta_j^{II}(\tau)$ $(1 \le j \le T)$ are the same as above, while $\vartheta_j^{II}(\tau)$ $(T+1 \le j \le H)$ are some repetitions of the theta series for $1 \le j \le T$. Moreover $S_{3/2}^q(N)$ is a certain space of cusp forms or weight $3/2$ with level $4N$ introduced by Kohnen [Ko] *(for the precise definition we refer to* [Ko]*).*

In the rest of this subsection we give a proof of the easier parts $((B) \iff (C)$, $(B) \iff (B'))$ of the conjecture.

For $r = 0$, 1 and $(\tau, z) \in \mathfrak{H} \times \mathbb{C}$, we define the ordinary theta series by

$$\theta_r(\tau, z) = \sum_{\lambda \in \mathbb{Z}} e\Big(\Big(\lambda + \frac{r}{2}\Big)^2 \tau + 2\Big(\lambda + \frac{r}{2}\Big) z\Big).$$

Then the theta transformation formula in this case is well known and given by

$$(1) \qquad \begin{pmatrix} \theta_0(M(\tau,z)) \\ \theta_1(M(\tau,z)) \end{pmatrix} = (c\tau + d)^{1/2} e\Big(\frac{cz^2}{c\tau + d}\Big) U(M) \begin{pmatrix} \theta_0(\tau,z) \\ \theta_1(\tau,z) \end{pmatrix},$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $M(\tau, z) = \Big(M\tau, \frac{z}{c\tau+d}\Big)$, and $U(M)$ is a unitary matrix of degree 2 depending only on M. Here we choose the branch of the holomorphic function $z^{1/2}$ $(z \neq 0)$ with $-\pi < \arg z \le \pi$.

Each $\phi(\tau, z) \in J_{k,1}(\Gamma_0(N))$ has the expression

$$(2) \qquad \phi(\tau, z) = h_0(\tau)\theta_0(\tau, z) + h_1(\tau)\theta_1(\tau, z)$$

with the transformation formula

$$\big(h_0(M\tau), h_1(M\tau)\big)U(M) = (c\tau+d)^{k-1/2}\big(h_0(\tau), h_1(\tau)\big) \qquad (M \in SL_2(\mathbb{Z})).$$

Moreover it is known by Kramer [Kr] that the expresiion (2) for the theta series $\vartheta_j^J(\tau, z)$ is given by

$$\vartheta_j^J(\tau,z) = \vartheta_j^I(\tau)\theta_0(\tau,z) + \big(\vartheta_j^{II}(\tau/4) - \vartheta_j^I(\tau)\big)\theta_1(\tau,z).$$

Lemma 3 *Let* $\phi \in J_{k,1}(\Gamma_0(N))$.

(a) *If* N *is not divisible by* 4 *and* $h_0(\tau) = 0$*, then,* $h_1(\tau) = 0$ *and hence* $\phi = 0$.

(b) *If* $h_0(\tau) + h_1(\tau) = 0$*, then* $\phi = 0$ *(we do not need the condition* $4 \nmid N$*).*

2 How to prove (A) $\Longrightarrow$ (B')

In the case of N being square free we shall give a sketch of a proof of (A) $\Longrightarrow$ (B') instead of proving (A) $\Longrightarrow$ (B). The details were discussed in [AB2].

We consider the linear map

(3) $$D_0 : J_{k,1}(\Gamma_0(N)) \ni \phi(\tau,z) \longrightarrow \phi(\tau,0) \in M_k(\Gamma_0(N)).$$

Set

$$\omega(M) = \det U(M) \qquad (M \in SL_2(\mathbb{Z})),$$

which forms a unitary character of $SL_2(\mathbb{Z})$. This ω actually coincides with the character obtained from η^6, η denoting the Dedekind eta function:

$$\eta^6(M\tau) = \omega(M)(c\tau+d)^3\eta^6(\tau), \qquad (M \in SL_2(\mathbb{Z})).$$

For any unitary character ψ of $\Gamma_0(N)$ we denote by $M_k(\Gamma_0(N),\psi)$ (resp. $S_k(\Gamma_0(N),\psi)$) the space of modular forms (resp. cusp forms) of weight k with the character ψ on $\Gamma_0(N)$.

We write $J_{k,1}(\Gamma_0(N))^0$ for the kernel of the map D_0. This kernel is described as follows.

Proposition 4 *We have the isomorphism*

$$\iota : J_{k,1}(\Gamma_0(N))^0 \cong M_{k-1}(\Gamma_0(N), \overline{\omega})$$

given by $\phi \longrightarrow \varphi(\tau) := -\dfrac{h_1(\tau)}{\theta_0(\tau)}$*, where* $\theta_r(\tau) = \theta_r(\tau, 0)$ $(r = 0,\ 1)$ *and* $\overline{\omega}$ *is the complex conjugate of* ω*. Moreover* $\phi \in J_{k,1}(\Gamma_0(N))^0$ *has the expression*

(4) $$\phi(\tau,z) = \varphi(\tau)\big(\theta_1(\tau)\theta_0(\tau,z) - \theta_0(\tau)\theta_1(\tau,z)\big).$$

We consider one more linear map

$$D_2 : J_{k,1}(\Gamma_0(N)) \longrightarrow M_{k+2}(\Gamma_0(N)), \tag{5}$$

given by

$$\phi(\tau, z) \longmapsto \left(\frac{k}{2\pi i}\frac{\partial^2}{\partial z^2} - 2\frac{\partial}{\partial \tau}\right)\phi \Big|_{z=0}.$$

One remarkable thing of this map is that, if $\phi \in J_{k,1}(\Gamma_0(N))^0$, then

$$D_2\phi = 4k\varphi\xi \qquad \text{with } \xi = \theta_1\theta_0' - \theta_0\theta_1' = -\pi i\eta^6,$$

$\eta(\tau)$ being the Dedekind eta function.

Theorem 5 *We have the following commutative diagram:*

$$\begin{array}{ccc} & J_{k,1}(\Gamma_0(N))^0 & \\ \iota \swarrow & & \searrow D_2 \\ M_{k-1}(\Gamma_0(N), \overline{\omega}) & \underset{\times 4k\xi}{\longrightarrow} & S_{k+2}(\Gamma_0(N))^0 \end{array}$$

In the diagram

$$S_{k+2}(\Gamma_0(N))^0 := \left\{ f \in S_{k+2}(\Gamma_0(N)) \,\Big|\, \frac{f}{\xi} \in M_{k-1}(\Gamma_0(N), \overline{\omega}) \right\}$$

and all the arrows are isomorphisms.

To prove the assertion $(A) \Longrightarrow (B')$ we set $\phi(\tau, z) = \sum_{j=1}^{T} c_j\vartheta_j^J(\tau, z)$. Then by the assumption of (A), $\phi(\tau, 0) = \sum_{j=1}^{T} c_j\vartheta_j(\tau) = 0$; namely, $\phi \in J_{2,1}(\Gamma_0(N))^0$. Therefore it is sufficient to prove $J_{2,1}(\Gamma_0(N))^0 = \{0\}$ if N is square free. In view of Theorem 5 we have only to prove

$$M_1(\Gamma_0(N), \overline{\omega}) = \{0\} \quad \text{or} \quad S_4(\Gamma_0(N))^0 = \{0\}.$$

Böcherer proved the latter identity by using the notion of Weierstrass subspaces. On this occasion we shall give an outline of the proof of vanishing of $M_1(\Gamma_0(N), \overline{\omega})$.

3 Vanishing of $M_1(\Gamma_0(N), \omega^{\pm 1})$

First we note that $\overline{\omega} = \omega^{-1} = \omega^3$, since ω^4 is the identity character of $SL_2(\mathbb{Z})$. We assume that N is square free. Then

$$M_1(\Gamma_0(N), \omega^{\pm 1}) = S_1(\Gamma_0(N), \omega^{\pm 1}) \qquad \text{(see [AB2, Proposition 1.2]).}$$

$$\widetilde{N} = \begin{cases} N & \text{if } (N,2) = 1, \\ N/2 & \text{if } (N,2) > 1. \end{cases}$$

We see easily from the property of ω ([AB2, Lemma 1.1 and Lemma 3.1]) that the following injective linear map exists:

$$S_1(\Gamma_0(N), \omega^{\pm 1}) \ni \varphi \longrightarrow \varphi | V(4) \in S_1(\Gamma_0(16\widetilde{N}), \chi),$$

where $(\varphi|V(4))(\tau) = \varphi(4\tau)$ and χ is a Dirichlet character mod$16\widetilde{N}$ derived from the unique non-trivial character χ_0 mod 4 (namely, $m_\chi = 4$).

The following lemma plays an key role.

Lemma 6 *Let M be a square free positive integer coprime to 2 and ν a positive integer ≥ 2. Then for any Dirichlet character χ defined modulo $2^\nu M$ with conductor $m_\chi = 4$, we have*

$$S_1(\Gamma_0(2^\nu M, \chi) = \{0\}.$$

Outline of Proof. Assume that there exists a normalized new form $f \in S_1^{new}(\Gamma(2^\nu M, \chi)$. Let $L(s, f)$ denote the L-function attached to f. By the famous theorem of Deligne-Serre [Se] there exists a Galois extension $K/\mathbb{Q}$ and an irreducible two-dimensional representation $\rho : Gal(K/\mathbb{Q}) \longrightarrow GL_2(\mathbb{C})$ such that

$$L(s, f) = L(s, \rho), \tag{6}$$

where $Gal(K/\mathbb{Q})$ is the Galois group of the extension $K/\mathbb{Q}$ and $L(s,\rho)$ is the Artin L-function associated to ρ. For the Artin L-function we refer to [Ma]. Moreover by the theorem $\chi = \det \rho$ and the Artin conductor of ρ coincides with $2^\nu M$. We compute the local factor of the both hand sides of (6) for each prime p dividing M.

It is known in [Mi, Theorem 4.6.17, (2)] that, if $p|M$, then

$$L_p(s, f) = (1 - a_p p^{-s})^{-1} \quad \text{with } (a_p^2 = \chi(p)p^{-1}), \tag{7}$$

since M is square free and $m_\chi = 4$. On the other hand we have, for the local factor $L_p(s, \rho)$,

$$L_p(s, \rho) = \det\Big(id_V - \rho(\sigma_p)\,\big|_{V^{I_p}} \cdot p^{-s}\Big)^{-1},$$

where $V = \mathbb{C}^2$ is the representation space of ρ, I_p (resp. D_p) is the inertia (resp. decomposition) subgroup of $Gal(K/\mathbb{Q})$ at the prime p, and σ_p is the Frobenius element of p. Moreover $V^{I_p} = \{v \in V \mid \rho(g)v = v \ \forall g \in I_p\}$. We may divide into three cases by the dimension of the space V^{I_p}.

(i) $V^{I_p} = \{0\}$: In this case we have $L_p(s,\rho) = 1$, which contradicts the form (7).

(ii) $V^{I_p} = V$: Then $L_p(s,\rho)$ is a polynomial of degree two in p^{-s}, which does not occur.

(iii) $\dim V^{I_p} = \{0\}$: This is the case we have $L_p(s,\rho) = (1-\zeta p^{-s})^{-1}$ with $\zeta = \rho(\sigma_p)$ a root of unity. This expression also contradicts the form (7).

Anyway all these three cases contradict (7). ∎

With the help of this lemma, Lemma 4.6.9 of [Mi] and the multiplicity one theorem ([Mi, Theorem 4.6.19]) we have the decomposition of the space $S_1(\Gamma_0(16\widetilde{N}),\chi)$:

$$S_1(\Gamma_0(16\widetilde{N}),\chi) = V_0 \bigoplus V_1 \bigoplus V_2, \tag{8}$$

where, for $0 \leq \delta \leq 2$,

$$V_\delta = \bigoplus_{l|2^\delta \widetilde{N}} \{f(lz) \mid f \in S_1^{new}(2^{4-\delta},\chi_\delta)\},$$

χ_δ being a Dirichlet character modulo $2^{4-\delta}$ with conductor 4.

The final ingredient to the proof is the following vanishing of the spaces of modular forms:

$$S_1^{new}(2^{4-\delta},\chi) = \{0\} \quad (\delta = 0,\ 1,\ 2), \tag{9}$$

where χ is a Dirichlet character mod$2^{4-\delta}$ with conductor 4. The proof of (9) is based on the following computation of the dimensions:

$$\dim S_2(\Gamma_0(2^{4-\delta})) = 0 \quad (\delta = 0,\ 1,\ 2).$$

Therefore we obtain our main results.

Theorem 7 *If N is square free, then $M_1(\Gamma_0(N),\omega^{\pm 1}) = \{0\}$.*

Theorem 8 *The Hashimoto conjecture for square free levels N holds true.*

4 Applications

The most important application of our Theorem 7 is to obtain a solution of the Hashimoto conjecture for square free levels. In this section some more applications will be explained.

We write $U(d)$ for the group consisting of unitary matrices of degree d. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$, let $J(M,z) := cz + d$ be the ordinary factor of automorphy. A cocycle $\sigma_\lambda(A,B)$ for $\lambda \in \mathbb{R}$ and $A,\ B \in SL_2(\mathbb{R})$ is defined by

$$\sigma_\lambda(A,B) := \frac{J(A,Bz)^\lambda J(B,z)^\lambda}{J(AB,z)^\lambda}.$$

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ containing -1_2. A map $\chi : \Gamma \longrightarrow U(d)$ is called a unitary **multiplier system** of Γ of weight $2k$ ($k \in \mathbb{R}$), dimension d, if it satisfies the following two conditions (i), (ii):

(i) $\chi(-1_2) = e^{-2\pi i k} 1_d$, 1_d denoting the identity matrix of size d.

(ii) For any $A,\ B \in \Gamma$, $\chi(AB) = \sigma_{2k}(A,B)\chi(A)\chi(B)$.

We define the Selberg zeta function $Z_{\Gamma,\chi}(s)$ attached to Γ and a unitary multiplier system χ of Γ:

$$Z_{\Gamma,\chi}(s) := \prod_{\{P_0\}_\Gamma,\, \mathrm{tr} P_0 > 2} \ \prod_{m=0}^{\infty} \det\left(id_V - \chi(P_0) N(P_0)^{-s-m}\right),$$

where $\{P_0\}_\Gamma$ runs over all the Γ-conjugacy classes of primitive hyperbolic elements of Γ with $\mathrm{tr} P_0 > 2$. The infinite products on the right hand side converge absolutely for $\mathrm{Re}(s) > 1$. Here $N(P_0)$ (called the norm of P_0) denotes the square of the eigen value of P_0 which is larger than one. Via the Selberg trace formula $Z_{\Gamma,\chi}(s)$ can be continued to a meromorphic function in the whole s plane which is holomorphic in the interval $1/2 \le s \le 1$.

We exclusively discuss the cases of $\chi = \omega^{\pm 1}$ and $\chi = \overline{U}$, the complex conjugate of U. For a holomorphic function f at $a \in \mathbb{C}$, $\mathrm{Ord}_{s=a} f$ denotes the order of zero at $s = a$ of f. By using the resolvent trace formula in [Fi], [He] we have the following theorem (see [AB2] for the proof and also [AB1]).

Theorem 9 *If N is square free, then*

$$\dim S_1(\Gamma_0(N), \omega^{\pm 1}) = \frac{1}{2} \mathrm{Ord}_{s=1/2}\Big(Z_{\Gamma_0(N),\omega^{\pm 1}}(s)\Big).$$

As an immediate corollary of this theorem and Theorem 7 we have

Corollary 10 *If N is square free, then*

$$Z_{\Gamma_0(N),\omega^{\pm 1}}(1/2) \neq 0.$$

One more application is that the injectivity of D_0 implies the bijectivity of D_0.

Proposition 11 *If N is square free, then the linear map D_0 is bijective.*

This is an immediate consequence of the following theorem of Kramer and the injectivity of D_0.

Theorem 12 (Kramer [Kr]) *If N is square free, then*

$$\dim J_{2,1}(\Gamma_0(N)) = \dim M_2(\Gamma_0(N)).$$

Then by Proposition 11 the map D_0 induces a bijection from the subspace $J_{2,1}^{cusp}(\Gamma_0(N))$ to $S_2(\Gamma_0(N))$, where $J_{2,1}^{cusp}(\Gamma_0(N))$ is the subspace of cusp forms of $J_{2,1}(\Gamma_0(N))$. In particular

$$\dim J_{2,1}^{cusp}(\Gamma_0(N)) = \dim S_2(\Gamma_0(N)). \tag{10}$$

Let $\chi = \overline{U}$, the complex conjugate of the unitary multiplier system U of $SL_2(\mathbb{Z})$ given by (1). We have proved in [Ar1, Theorem 5.2, (iii)] that

$$\begin{aligned}\dim J_{2,1}^{cusp}(\Gamma_0(N)) &= \operatorname{Ord}_{s=3/4}(Z_{\Gamma_0(N),\chi}(s)) + \lambda_N,\\ \dim J_{1,1}^{skew}(\Gamma_0(N)) &= \operatorname{Ord}_{s=3/4}(Z_{\Gamma_0(N),\chi}(s)),\end{aligned}$$

where $J_{1,1}^{skew}(\Gamma_0(N))$ is the space of skew-holomorphic Jacobi forms of weight 1 and index 1 with respect to $\Gamma_0(N)$. Here $\lambda_N = \lambda_{\Gamma_0(N)}(2;1)$ is a rational number defined in [Ar1]. Precisely in our situation it is given by

$$\lambda_N = \frac{1}{4\pi}v(\Gamma_0(N)\backslash\mathfrak{H}) - \frac{\nu_2}{4}\epsilon_2(2;1) - \frac{\nu_3}{3}\epsilon_3(2;1) + \nu_\infty - \sum_{0<v|N}\beta_v - t_\infty. \tag{11}$$

We explain the notations used in (11). First ν_∞ is the number of the $\Gamma_0(N)$-equivalence classes of cusps of $\Gamma_0(N)$. The $\Gamma_0(N)$-inequivalent cusps are represented by $\xi = \frac{1}{v}$ $(v|N, v>0)$. The number of such ξ equals $\nu_\infty = 2^r$, r being the number of prime integers dividing N. For each $0 < v \mid N$, β_v is given by

$$\beta_v = \langle -\frac{N}{4v}\rangle,$$

where $\langle x\rangle$ for $x \in \mathbb{R}$ denotes the real number with $x-\langle x\rangle \in \mathbb{Z}$, $0 \le \langle x\rangle < 1$. The number t_∞ is defined to be the number of linearly independent Eisenstein series attached to the multiplier system $(\Gamma_0(N),\chi)$ (see [Ar1, p.192]). In our situation we have $t_\infty = 2^r$. Moreover,

$$\begin{aligned}\epsilon_2(2;1) &= -G_2(1)\cos\Big((2+\frac{1}{2})\frac{\pi}{2}\Big) = 0 \qquad (G_2(1) = \frac{1}{\sqrt{2}}\Big(1+e(\frac{1}{2})\Big) = 0),\\ \epsilon_2(2;1) &= \frac{\sin(2\pi/3)}{2\sin(\pi/3)} = 1 \qquad (\text{since } G_3(1) = \frac{1}{\sqrt{3}}\Big(1+2e(\frac{1}{3})\Big) = 0).\end{aligned}$$

ν_2 (resp. ν_3) is the number of $\Gamma_0(N)$-equivalence classes of all elliptic points of $\Gamma_0(N)$ of order 2 (resp. 3). Finally we set

$$\mu = N\prod_{p|N}\left(1+\frac{1}{p}\right)$$

and $v(\Gamma_0(N)\backslash\mathfrak{H})$ is the volume of $\Gamma_0(N)\backslash\mathfrak{H}$ with respect to $\frac{dxdy}{y^2}$ which is actually given by

$$v(\Gamma_0(N)\backslash\mathfrak{H}) = \frac{\pi}{3}\mu.$$

Summing up, we have

$$\dim J_{2,1}^{cusp}(\Gamma_0(N)) = \dim J_{1,1}^{skew}(\Gamma_0(N)) + \lambda_N$$

with

$$\begin{aligned}\lambda_N &= \frac{\mu}{12} - \frac{\nu_3}{3} + 2^r - \sum_{0<v|N}\left\langle -\frac{N}{4v}\right\rangle - 2^r \\ &= \frac{\mu}{12} - \frac{\nu_3}{3} - \sum_{0<v|N}\left(1 - \left\langle \frac{N}{4v}\right\rangle\right)\end{aligned}$$

Note that

$$\sum_{0<v|N}\left(\frac{1}{2} - \left\langle \frac{N}{4v}\right\rangle\right) = \frac{1}{4}\prod_{p|N}\left(1+\left(\frac{-1}{p}\right)\right) = \frac{\nu_2}{4}.$$

Let g_N denote the dimension of the space $S_2(\Gamma_0(N))$ which is actually given by

$$g_N = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}$$

(for instance see [Sh]). Hence we have

$$\lambda_N = g_N - 1.$$

Therefore

$$\dim J_{2,1}^{cusp}(\Gamma_0(N)) = \dim J_{1,1}^{skew}(\Gamma_0(N)) + g_N - 1 = \dim J_{1,1}^{skew}(\Gamma_0(N)) - 1 + \dim S_2(\Gamma_0(N)).$$

Since we have the equality (10) of the dimensions, we conclude that

$$\dim J_{1,1}^{skew}(\Gamma_0(N)) = \mathrm{Ord}_{s=3/4}(Z_{\Gamma_0(N),\chi}(s)) = 1.$$

For other applications of the bijectivity of D_0 we refer the reader to [AB2].

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