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LOCAL INTEGRAL ESTIMATES OF GRADIENTS FOR DEGENERATE PARABOLIC SYSTEMS

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ABSTRACT: We consider L^q-estimates of gradients for degenerate $p-$Laplacian systems with discontinuous coefficients and external force of divergence of BMO-functions.

1. **INTRODUCTION** Let Ω be a domain in an Euclidean space R^m for $m \geq 2$ and T be a positive number. Suppose that $\frac{2m}{m+2} < p < \infty$. We consider the evolutional p-Laplacian system

$$\partial_t u^i - D_\alpha \left(|Du|_{gh}^{p-2} g^{\alpha\beta} D_\beta u^i \right) = \operatorname{div} \left(|F|^{p-2} F^i \right) \quad \text{in } Q = (0,T) \times \Omega, \quad i = 1, \cdots, n, \tag{1.1}$$

where $\left(g^{\alpha\beta}(z)\right)$ and $(h_{ij}(z))$ are symmetric matrices with measurable coefficients satisfying the uniform ellipticity and boundedness condition with positive constants γ, Γ

$$\gamma|\xi|^2 \leq g^{\alpha\beta}(z)\xi_\alpha^i \xi_\beta^j h_{ij}(z) \leq \Gamma|\xi|^2 \quad \text{for any } \xi = (\xi_\alpha^i) \in R^{mn} \text{ and almost every } z \in Q \tag{1.2}$$

and the notation $|\xi|_{gh}^2 = g^{\alpha\beta}\xi_\alpha^i \xi_\beta^j h_{ij}$ and $|\xi|^2 = (\xi_\alpha^i)^2$ is used. Here and what follows, the summation notation over repeated indices is adopted.

Given $F = (F_\alpha^i) \in L_{\text{loc}}^q(Q, R^{mn})$ for $q \geq p-1$, we define a weak solution of (1.1) to be a function $u \in L_{\text{loc}}^\infty \left((0,T); L_{\text{loc}}^2(\Omega, R^n)\right) \cap L_{\text{loc}}^p \left((0,T); W_{\text{loc}}^{1,p}(\Omega, R^n)\right)$ satisfying, for all $\phi \in C_0^\infty(Q, R^n)$,

$$\int_Q \left\{ -u \cdot \partial_t \phi + |Du|_{gh}^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\alpha \phi + |F|^{p-2} F \cdot D\phi \right\} dz = 0 \tag{1.3}$$

Such evolution systems as (1.1) describe the gradient flow of the $p-$energy functional with variable coefficients and lower order terms

$$E(u) = \int_\Omega \tfrac{1}{p} \left(g^{\alpha\beta}(x) D_\alpha u^i D_\beta u^j h_{ij}(x) \right)^{\frac{p}{2}} + |F|^{p-2} F^i \cdot Du^j h_{ij}(x)\, dx. \tag{1.4}$$

In this paper, we study how the regularity of a function F is reflected to the one of a solution under some assumption on the coefficients. Let us consider a L^q-regularity of the gradient of a solution. Such regularity on integrability is known to hold for linear elliptic and parabolic systems of divergence form (see [10, pp. 87–89], [2, 1] and the references in them). It is shown in [1] that, if $p = 2$ and the coefficients are of vanishing mean oscillation, called VMO-function, then L^q-estimate of the gradient holds for any L^q-functions F. In [9, 12],

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the similar L^q−estimate as in the case $p = 2$ is shown to hold for stationary p−Laplacian systems. For a function F of bounded mean oscillation, called BMO-function, a BMO-regularity of the gradient may be expected to hold under some assumption for the coefficients. In fact, in the case $p = 2$, such BMO-estimate holds for VMO-coefficients and this condition for the coefficient is the optimal one for BMO estimates (see [1]). For stationary p−Laplacian systems, related results are known to hold in the degenerate case $p > 2$ [9]. The above L^q and BMO estimates of the gradient are accomplished by using the growth estimates for the mean oscillation of the gradient of solutions to homogeneous systems with constant coefficients in the perturbation arguments. In the just linear case $p = 2$, we can combine the duality and interpolation arguments with L^2 and BMO estimates to have L^q−estimates of the gradient (see [1, 18]). In the p−Laplacian case, since the interpolation argument can not directly adopted, we have to study L^q and BMO estimates, separately. Here, anyway, the growth estimates for the mean oscillation of the gradient for homogeneous systems with constant coefficients obtained in [9], also play a important role. However, since the estimate in [9] is for the mean oscillation of the gradient in $L^2 - norm$, but not in $L^p - norm$, it does work only for BMO-estimates of the gradient in the degenerate case, but not in the singular case $1 < p < 2$. For L^q−estimates of the gradient for stationary p−Laplacian systems, somewhat rough estimate for the mean oscillation is well-worked by combination with L^∞−estimate of the gradient and the localized Fefferman and Stein's inequality for a sharp maximal function (see [12, 9]). On the other hand, it is not known whether such growth estimate for the mean oscillation holds for evolutional p−Laplacian systems with constant coefficients and only principal part or not (except the "scalar" case). The inconvenience of estimates gives some difficulties in L^q and BMO estimates of the gradient in the evolutional case and it comes from the non-homogeneity of the evolutional p−Laplacian operator, the situation of which is completely different from the stationary case. In Hölder estimates of the gradient for a given Hölder continuous function F, similar problem as above appears and thus, a technical device is needed even to obtain a Hölder estimate with "lower" Hölder exponent than the one of a given function F (see [15]). The method in [15] does not seem to be applied for L^q and BMO estimates of the gradient. In this paper, we show that, for evolutional p−Laplacian systems with a BMO-function F and VMO-coefficients, a L^q−estimate for the gradient holds for any $q \geq p$. Of cource, it seems more natural to consider L^q−regularity of the gradient for a given L^q−function F. However, as described above, we are now faced with some difficulty, which concerns the estimate for L^∞−norm and the mean oscillation in L^p−norm of the gradient for evolutional p−Laplacian systems.

To state our main result, we recall the definition of the function spaces: Let $G \subset Q$ be a domain. Then we define that an integrable function $f \in L^q(G)$, $q \geq 1$, is of bounded mean oscillation in G (with respect to L^q−norm), referred as BMO (in G), if, for a positive number ρ,

$$[f]_{*,q,\rho,G} \overset{\text{def.}}{=} \sup_{\substack{P \subset G \\ \text{diam}(P) \leq \rho}} \left(\frac{1}{|P|} \int_P |f - (f)_P|^q \, dz \right)^{\frac{1}{q}} < \infty \tag{1.5}$$

holds for a parabolic cylinder $P = Q_{r,\tau} = (t_0 - \tau, t_0) \times B_r(x_0)$ with a vertex $(t_0, x_0) \in \overline{G}$ and $r, \tau > 0$ and the integral mean $(f)_P$ in P, where $\mathrm{diam}(P) = \sqrt{(2r)^2 + \tau^2}$ is the diameter of a region P measured by the Euclidean metric in R^{m+1}. Recall that Lebesgue's differential theorem (see (2.7) below) holds for any $f \in L^q_{\mathrm{loc}}(Q), q > 1$ (refer to [19, 5.3 (c), pp. 23-24]), which motivates the definition (1.5). If (1.5) holds for all $P \subset G$, we abbreviate (1.5) to $[f]_{*,q,G}$. If (1.5) converges to zero as $\rho \searrow 0$, then we say that f is of vanishing mean oscillation in G (with respect to L^q−norm), referred as VMO (in G). A locally integrable function in Q is said to be of locally bounded or vanishing mean oscillation in Q, if the above conditions hold for all domains G compactly contained in Q. Locally continuous functions are of locally vanishing mean oscillation, but, in general, functions of local VMO functions need not to be locally continuous.

Theorem 1 *Suppose that the coefficients are of locally vanishing mean oscillation in Q and that the function F is of locally bounded mean oscillation in Q. Let u be a weak solution of (1.1). Then Du is also locally L^q−integrable in Q for any $q \geq p$ and, for any $q \geq p$ and $z_0 \in Q$, there exist positive constants C and d depending only on m, p, q, γ, Γ, $dist_p(z_0, \partial_p Q)$ and the VMO-norm of the coefficients such that*

$$\begin{aligned}\int_{Q_{\frac{d}{2}}} |Du|^q dz \;\leq\; & C\,(r_0)^{\left(\frac{p}{\beta_0} - 1\right)\frac{2\epsilon(q-p)}{p(\epsilon+2)}} \left(\frac{1}{|\overline{Q}|}\int_{\overline{Q}} |Du|^p dz\right)^{1 + \left(1 + \frac{\epsilon}{\beta_0}\right)\frac{q-p}{\epsilon+2}} \\ & + C\,(r_0)^{\left(\frac{p}{\alpha_0} - 1\right)\frac{\epsilon(q-p)}{p(\epsilon+2)}} \left(\frac{1}{|\overline{Q}|}\int_{\overline{Q}} |Du|^p dz\right)^{1 + \left(1 + \frac{\epsilon}{\alpha_0}\right)\frac{q-p}{\epsilon+2}} \\ & + C\left(1 + (r_0)^{\frac{-2(p+\epsilon)(q-p)}{p(\epsilon+2)}} \left(\frac{1}{|\overline{Q}|}\int_{\overline{Q}} |F|^p dz\right)^{\frac{q-p}{\epsilon+2}}\right. \\ & \left. + \left(\frac{1}{|\widetilde{Q}|}\right)^{\frac{q-p}{p+\epsilon}} \left([|F|^{p-2}F]_{*,\frac{p}{p-1},\widetilde{Q}}\right)^{\frac{q-p}{p-1}}\right) \int_{Q_d} |Du|^p dz, \qquad (1.6)\end{aligned}$$

holds for $r_0 = \frac{1}{2} dist_2(z_0, \partial_p Q)$, $\widetilde{Q} = Q_{(r_0)^2, r_0}(z_0)$ and $\overline{Q} = Q_{2^p (r_0)^2, 2 (r_0)^{\frac{2}{p}}}(z_0)$.

For the proof, we use the perturbation argument with the p−Laplacian systems with constant coefficients and only principal part. The main task is to choose the "good" parabolic cylinders on which the L^∞ and Hölder estimates of gradients for such p−Laplacian systems can be improved to be well-worked in the perturbation arguments. The approach is similar to the one of Kinnunen and Lewis ([11], also see [14]), who obtained the higher integrability of the gradients for evolutional p-Laplacian systems. Employ Whitney decomposition with the covering argument and sum up the resulting integral estimates on such "good" local cylinders as above to obtain the estimation for the upper level set of gradients of solutions. Then, we apply the usual integral formula to arrive at the desirable L^q−estimates.

2. **PRELIMINARY** In this section, we gather the local estimates needed in the proof of our main theorem. Take arbitrarily and fix $z_0 = (t_0, x_0) \in Q = (0, T) \times \Omega$ and put

$r_0 = \frac{1}{2}\mathrm{dist}_2(z_0, \partial_p Q)$. Let $h > 1$ be determined later and $Q_d(z_0) = (t_0 - d^2, t_0) \times B_d(x_0)$ for a positive constant $d \leq \frac{1}{4h}\mathrm{dist}_2(z_0, \partial_p Q) = \frac{r_0}{2h}$. Then $Q_d(z_0) \subset Q_{r_0}(z_0)$ be compactly contained in Q.

Employ the Whitney decomposition to divide $P_{1,1} = (-1, 1) \times B_1(0)$ into a family $\{P_i\}$ of cylinders $P_i = P_{(\rho_i)^2,\rho_i}(z_i) = (t_i - (\rho_i)^2, t_i + (\rho_i)^2) \times B_{\rho_i}(x_i)$ with center $z_i = (t_i, x_i) \in P_{2^p,2}$, $i = 1, 2, \cdots$. We can refer to [20, p. 339] for the precise way of the Whitney decomposition. (also see [10, pp.126–128]). This family of cylinders has the following properties: The cylinders P_i, $i = 1, 2, \cdots$, are of uniformly bounded overlap and each $P_{(5\rho_i)^2,5\rho_i}(z_i)$ $1 = 1, 2, \cdots$, is totally contained in $P_{1,1}$. To make the decomposition result to be well-worked in our setting, we divide each P_i into two cylinders $Q_{\rho_i}(t_i + (\rho_i)^2, x_i)$ and $Q_{\rho_i}(t_i, x_i)$ and then relabel the resulting cylinders to be $Q_i = Q_{\rho_i}(t_i, x_i)$, $i = 1, 2, \cdots$ so that cylinders Q_i, $i = 1, 2, \cdots$, have the uniformly finite overlaps each other and each $Q_{(5\rho_i)^2,5\rho_i}$ $i = 1, 2, \cdots$, is totally contained in Q_1.

By the parallel and scaling transformation, $t = t_0 + d^2 s, x = x_0 + dy$, we transform $Q_d(z_0)$ into Q_1, use the decomposition as above and make a scaling back to divide $Q_d(z_0)$ into Whitney type cylinders

$$Q_i = Q_{r_i,r_i^2}(z_i),\ z_i = (t_i, x_i) \in Q_d(z_0),\ r_i = \tfrac{1}{5}\mathrm{dist}_2(z_i, \partial\widetilde{Q}) \quad (i = 1, \cdots). \tag{2.1}$$

Then, clearly, $Q_{(5r_i)^2,5r_i}(z_i) \subset Q_d(z_0)$.

In the followings, by a parallel transformation, we assume that z_0 is the origin and put $Q_d = Q_d(z_0)$ and $\widetilde{Q} = Q_{r_0}(z_0)$. We will divide the arguments into the degenerate case $p > 2$ and the singular case $\frac{2m}{m+2} < p < 2$. First we treat the degenerate case. Let $s > p$ be stipulated later and λ_0 be a positive number such that

$$\lambda_0 \geq \max\left\{1,\ \left(\frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} |Du|^s dz\right)^{\frac{1}{s-p+2}}\right\}. \tag{2.2}$$

For all $z_0 \in Q_d$, we can choose a cylinder Q_{i_0} with some $i_0 = 1, 2, \cdots$ such that

$$|Q_{i_0}| = \min_{z_0 \in Q_i} |Q_i|, \tag{2.3}$$

since cylinders Q_i, $i = 1, 2, \cdots$, are of uniformly bounded overlap.

For any $z_0 \in Q_d$ and all $\lambda \geq 0$, we take a positive integer i_0 in (2.3) and set a positive number $\mu = \mu(z_0, \lambda)$

$$\mu(z_0, \lambda) = \lambda\, |Q_{i_0}|^{-\frac{1}{s-p+2}}. \tag{2.4}$$

Then there exists a positive constant $\overline{C}$ depending only on m and p such that

$$\left(\frac{1}{|Q_{\mu^{2-p}\rho^2,\rho}|}\int_{Q_{\mu^{2-p}\rho^2,\rho}(z_0)} |Du|^p dz\right)^{\frac{s}{p}} \leq \frac{1}{|Q_{\mu^{2-p}\rho^2,\rho}|}\int_{Q_{\mu^{2-p}\rho^2,\rho}(z_0)} |Du|^s dz$$

$$\leq C|Q_{i_0}|^{-\frac{s}{s-p+2}}\lambda^{p-2}\int_{\widetilde{Q}}|Du|^s dz$$

$$\leq \overline{C}^s\left|\widetilde{Q}\right|\left(\lambda|Q_{i_0}|^{-\frac{1}{s-p+2}}\right)^s = \overline{C}^s\left|\widetilde{Q}\right|\mu^s \qquad (2.5)$$

holds for all $\lambda \geq \lambda_0$, any $z_0 \in Q_d(z_0)$ and all positive numbers ρ,

$$\tfrac{1}{2}\rho_{i_0} \leq \rho \leq r_0. \qquad (2.6)$$

Here we note the followings: Since $|Q_i| < 1$ for all $i = 1, 2, \cdots$, we have that $\mu^{2-p} \leq (\lambda_0)^{2-p}$ holds for any $\lambda \geq \lambda_0$, and then, $Q_{\mu^{2-p}\rho^2,\rho}(z_0) \subset Q_{(r_0)^2,r_0}(z_0) \subset Q_{d^2,d}$ for all ρ satisfying (2.6). We see from Lebesgue's differentiation theorem (refer to [19, 5.3 (c), pp. 23-24]) that

$$\lim_{\rho\searrow 0}\left(\frac{1}{|Q_{\mu^{2-p}\rho^2,\rho}|}\int_{Q_{\mu^{2-p}\rho^2,\rho}(z_0)}|Du|^p dz\right) = |Du(z_0)|^p > \overline{C}^p\left|\widetilde{Q}\right|^{\frac{p}{s}}\mu^p \qquad (2.7)$$

holds for almost every $z_0 \in Q_{d^2,d}(0)$ satisfying $|Du(z_0)|^p > \overline{C}^p\left|\widetilde{Q}\right|^{\frac{p}{s}}\mu^p$.

For all $\lambda \geq \lambda_0$ and any $z_0 \in \{|Du(z_0)| > \overline{C}\left|\widetilde{Q}\right|^{\frac{1}{s}}\mu\}$, put $\mu = \lambda|Q_{i_0}|^{-\frac{1}{s-p+2}}$, where i_0 and μ are determined in (2.3) and (2.4). Then, noting the continuity of integral, we find from (2.5) and (2.7) that, for all $\lambda \geq \lambda_0$ and any $z_0 \in \{|Du(z_0)| > \overline{C}\left|\widetilde{Q}\right|^{\frac{1}{s}}\mu\}$, there exists a positive number ρ_0, $0 < \rho_0 \leq \frac{1}{2}\rho_{i_0}$, such that

$$\frac{1}{|Q_{\mu^{2-p}(\rho_0)^2,\rho_0}|}\int_{Q_{\mu^{2-p}(\rho_0)^2,\rho_0}(z_0)}|Du|^p dz = \overline{C}^p\left|\widetilde{Q}\right|^{\frac{p}{s}}\mu^p \qquad (2.8)$$

and

$$\frac{1}{|Q_{\mu^{2-p}\rho^2,\rho}|}\int_{Q_{\mu^{2-p}\rho^2,\rho}(z_0)}|Du|^p dz \leq \left(\frac{1}{|Q_{\mu^{2-p}\rho^2,\rho}|}\int_{Q_{\mu^{2-p}\rho^2,\rho}(z_0)}|Du|^s dz\right)^{\frac{p}{s}}$$

$$\leq \overline{C}^p\left|\widetilde{Q}\right|^{\frac{p}{s}}\mu^p \qquad (2.9)$$

holds for all ρ, $\rho_0 \leq \rho \leq 2hd$, and, in particular, for $\rho = h\rho_0$ and $\rho = 2h\rho_0$.

Now, let $z_0 = (t_0, x_0) \in Q_d$ satisfying $|Du(z_0)|^p > \overline{C}^p\left|\widetilde{Q}\right|\mu^p$ for $\mu = \lambda\,|Q_{i_0}|^{-\frac{1}{s-p+2}}$, $\lambda \geq \lambda_0$ and i_0 is as in (2.3). For brevity, we assume that z_0 is the origin. Put $\Lambda = \mu^{2-p}$ and $R = h\rho_0$. From now on, we proceed to local estimates on $Q_{\Lambda R^2,R}$ and $Q_{\Lambda(\rho_0)^2,\rho_0}$. We observe to improve the L^∞−estimate and Hölder estimate for the p−Laplacian system with constant coefficients and only principal part in $Q_{\Lambda R^2,R}$ and $Q_{\Lambda(\rho_0)^2,\rho_0}$, referred as "good " parabolic cylinders.

Let $v \in L^\infty(-\Lambda(2R)^2, 0 : L^2(B_{2R}, R^n)) \cap L^p(-\Lambda(2R)^2, 0 : W^{1,p}(B_{2R}, R^n))$ be a weak solution of

$$\partial_t v = D_\alpha\left(|Dv|_{\bar{g}\bar{h}}^{p-2}\bar{g}^{\alpha\beta}D_\beta v\right) \quad \text{in } Q_{\Lambda(2R)^2,2R}$$

$$v = u \quad \text{on } \partial_p Q_{\Lambda(2R)^2,2R}, \qquad (2.10)$$

$$\bar{g}^{\alpha\beta} = \frac{1}{|Q_{\Lambda(2R)^2,2R}|}\int_{Q_{\Lambda(2R)^2,2R}} g^{\alpha\beta}dz, \quad \bar{h}_{ij} = \frac{1}{|Q_{\Lambda(2R)^2,2R}|}\int_{Q_{\Lambda(2R)^2,2R}} h_{ij}dz,$$
$$\bar{g} = \left(\bar{g}^{\alpha\beta}\right), \quad \bar{h} = \left(\bar{h}_{ij}\right). \tag{2.11}$$

First we state the L^∞–estimate in the "good" parabolic cylinders.

Lemma 2 *(L^∞–estimate) There exists a positive constant C depending only on $m, p, \gamma, \Gamma, \overline{C}$ and $\left|\tilde{Q}\right|$ such that*

$$\sup_{Q_{\Lambda R^2,R}} |Dv|^p \le C\frac{1}{|Q_{\Lambda(\rho_0)^2,\rho_0}|}\int_{Q_{\Lambda(\rho_0)^2,\rho_0}} |Du|^p dz \tag{2.12}$$

Proof. We have the L^∞–estimate [3, Proposition 3.1, p. 109]

$$\sup_{Q_{\Lambda R^2,R}} |Dv|^p \le C\left(\frac{\Lambda}{|Q_{\Lambda(2R)^2,2R}|}\int_{Q_{\Lambda(2R)^2,2R}} |Dv|^p dz\right)^{\frac{p}{2}} + C\Lambda^{\frac{p}{2-p}}. \tag{2.13}$$

The perturbation estimates (2.14) gives

$$\frac{1}{|Q_{\Lambda(2R)^2,2R}|}\int_{Q_{\Lambda(2R)^2,2R}} |Dv|^p dz \le C\frac{1}{|Q_{\Lambda(2R)^2,2R}|}\int_{Q_{\Lambda(2R)^2,2R}} |Du|^p dz + C. \tag{2.14}$$

Substitute (2.14) into (2.13) to have

$$\sup_{Q_{\Lambda R^2,R}} |Dv|^p \le C\left(\frac{\Lambda}{|Q_{\Lambda(2R)^2,2R}|}\int_{Q_{\Lambda(2R)^2,2R}} |Du|^p dz\right)^{\frac{p}{2}} + C\left(\Lambda^{\frac{p}{2}} + \Lambda^{\frac{p}{2-p}}\right). \tag{2.15}$$

We now make estimation of each term in the right hand side of (2.15). Since $\lambda \ge 1$ and $|Q_{i_0}| \le 1$, we find by (2.8) that

$$\Lambda^{\frac{p}{2}} \le 1 \le \left(\overline{C}^p\left|\tilde{Q}\right|^{\frac{p}{s}}\right)^{-1}\frac{1}{|Q_{\Lambda(\rho_0)^2,\rho_0}|}\int_{Q_{\Lambda(\rho_0)^2,\rho_0}} |Du|^p dz. \tag{2.16}$$

Also the direct calculation with (2.8) and (2.9) gives

$$\Lambda^{\frac{p}{2-p}} = \left(\lambda|Q_{i_0}|^{-\frac{1}{s-p+2}}\right)^p = \left(\overline{C}^p\left|\tilde{Q}\right|^{\frac{p}{s}}\right)^{-1}\frac{1}{|Q_{\Lambda(\rho_0)^2,\rho_0}|}\int_{Q_{\Lambda(\rho_0)^2,\rho_0}} |Du|^p dz, \tag{2.17}$$

$$\Lambda^{\frac{p}{2}}\left(\frac{1}{|Q_{2R,\Lambda(2R)^2}|}\int_{Q_{2R,\Lambda(2R)^2}} |Du|^p dz\right)^{\frac{p}{2}-1} \le \left(\overline{C}^p\left|\tilde{Q}\right|^{\frac{p}{s}}\right)^{\frac{p}{2}-1}. \tag{2.18}$$

Combine (2.15), (2.16) and (2.18) with (2.13) and note (2.8) and (2.9) to have the desired estimate (2.12).

The next estimate concerns the Hölder estimates in the "good" parabolic cylinders.

Lemma 3 *(Hölder estimate) There exist positive constants C depending only on $m, p, \gamma, \Gamma, \overline{C}$ and $\left|\tilde{Q}\right|$ and α, $0 < \alpha < 1$, depending only on m and p such that*

$$\underset{Q_{\Lambda(\rho_0)^2,\rho_0}}{osc} (Dv) \leq Ch^{-\alpha} \left(\frac{1}{\left|Q_{\Lambda(\rho_0)^2,\rho_0}\right|} \int_{Q_{\Lambda(\rho_0)^2,\rho_0}} |Du|^p dz \right)^{\frac{1}{p}} \tag{2.19}$$

Proof.

From (2.12) with (2.17), we see that

$$\sup_{Q_{\Lambda R^2,R}} |Dv| \leq C\mu = C\Lambda^{\frac{1}{2-p}} . \tag{2.20}$$

Let M be a positive number such that $M = C\mu$ in (2.20). Note that the transformed map $\tilde{v}(t,x) = \frac{1}{M} v\left(\frac{t}{M^{p-2}}, x\right)$ is also a weak solution of (2.10). Apply the Hölder estimate for the map $\tilde{v}$ [5, Theorem 1.1″, p. 258] to have

$$\underset{Q_{M^{p-2}\Lambda(\rho_0)^2,\rho_0}}{\operatorname{osc}} (D\tilde{v})$$

$$\leq C \sup_{Q_{M^{p-2}\Lambda R^2,R}} |D\tilde{v}| \left(\frac{\rho_0 + (M^{p-2}\Lambda(\rho_0)^2)^{\frac{1}{2}} \max\left\{1, \left(\sup\limits_{Q_{M^{p-2}\Lambda(\rho_0)^2,\rho_0}} |D\tilde{v}|\right)^{\frac{p-2}{2}}\right\}}{\operatorname{dist}\left(Q_{M^{p-2}\Lambda(\rho_0)^2,\rho_0}, \partial_p Q_{M^{p-2}\Lambda R^2,R}\right)} \right)^{\alpha} \tag{2.21}$$

We can evaluate the denominator in the right hand side of (2.21)

$$\begin{aligned} &\operatorname{dist}\left(Q_{M^{p-2}\Lambda(\rho_0)^2,\rho_0}, \partial_p Q_{M^{p-2}\Lambda R^2,R}\right) \\ &= \min_{\substack{(t,x)\in Q_{(\rho_0)^2,\rho_0},\\ (s,y)\in\partial_p Q_{R^2,R}}} \left(|x-y| + M^{\frac{p-2}{2}}\Lambda^{\frac{1}{2}}|t-s|^{\frac{1}{2}}\right) \\ &= \min\{\rho_0(h-1),\ M^{\frac{p-2}{2}}\Lambda^{\frac{1}{2}}\rho_0(h^2-1)^{\frac{1}{2}}\} \\ &\geq C\rho_0(h-1) \geq C\rho_0 h, \end{aligned} \tag{2.22}$$

since, by the definition of Λ and (2.20), we have

$$M^{\frac{p-2}{2}}\Lambda^{\frac{1}{2}} = C \tag{2.23}$$

and the positive constant h is sufficiently large. Transforming back and noting (2.22) and the fact that $\sup_{Q_{M^{p-2}\Lambda R^2,R}} |D\tilde{v}| \leq 1$, we obtain, from (2.21),

$$\frac{1}{M} \underset{Q_{\Lambda(\rho_0)^2,\rho_0}}{\operatorname{osc}} |Dv| \leq Ch^{-\alpha}. \tag{2.24}$$

We now estimate the difference of u from v in the local L^p-norm. Let us use the notation in (2.11) for the integral average.

Lemma 4 *There exists a positive constant C depending only on m, p, γ, Γ such that*

$$\int_{Q_{\Lambda(2R)^2,2R}} |Dv - Du|^p dz \le C \left|Q_{\Lambda(2R)^2,2R}\right|^{\frac{s-p}{s}} \left(|g|_{*,hd}^{\frac{s-p}{s}} + |h|_{*,hd}^{\frac{s-p}{s}}\right) \left(\int_{Q_{\Lambda(2R)^2,2R}} |Du|^s dz\right)^{\frac{p}{s}}$$
$$+ \int_{Q_{\Lambda(2R)^2,2R}} \left||F|^{p-2}F - \overline{|F|^{p-2}F}\right|^{\frac{p}{p-1}} dz, \tag{2.25}$$

where a positive number $s > p$ is the same one as in (2.2).

Proof. Subtract (1.1) from (2.10) and use a test function $\bar{h}_{ij}(v^j - u^j)$, which is shown to be admissible by the usual approximation argument, in the resulting equation. We utilize algebraic inequalities

$$\bar{g}^{\alpha\beta}\left(|P|_{\bar{g}\bar{h}}^{p-2}P_\alpha^i - |Q|_{\bar{g}\bar{h}}^{p-2}Q_\alpha^i\right)\left(P_\beta^j - Q_\beta^j\right)\bar{h}_{ij} \ge C\,|P-Q|^p, \tag{2.26}$$
$$\left|\left(|P|_{gh}^{p-2}g^{\alpha\beta}P_\beta^j h_{ij} - |P|_{\bar{g}\bar{h}}^{p-2}\bar{g}^{\alpha\beta}P_\beta^j\bar{h}_{ij}\right)\right| \le C\,|gh - \bar{g}\bar{h}|\,|P|^{p-1},$$

which hold for any $P = (P_\alpha^i), Q = (Q_\alpha^i) \in R^{mn}$ with a positive constant C depending only on p, λ and Λ. We use Young's inequality to have, for any $\epsilon > 0$,

$$\int_{Q_{\Lambda(2R)^2,2R}} \partial_t \tfrac{1}{2}\left(\bar{h}_{ij}(v^j - u^j)(v^i - u^i)\right)$$
$$+\bar{g}^{\alpha\beta}\left(|Dv|_{\bar{g}\bar{h}}^{p-2}D_\beta v^i - |Du|_{\bar{g}\bar{h}}^{p-2}D_\beta u^i\right)\left(D_\alpha v^j - D_\alpha u^j\right)\bar{h}_{ij}dz$$
$$= \int_{Q_{\Lambda(2R)^2,2R}} \left(|F|^{p-2}F^i - \overline{|F|^{p-2}F^i}\right)\left(Dv^j - Du^j\right)\bar{h}_{ij}dz$$
$$- \int_{Q_{\Lambda(2R)^2,2R}} \left(|Du|_{\bar{g}\bar{h}}^{p-2}\bar{g}^{\alpha\beta}D_\beta u^j\bar{h}_{ij} - |Du|_{gh}^{p-2}g^{\alpha\beta}D_\beta u^j h_{ij}\right)\left(D_\alpha v^j - D_\alpha u^j\right)dz$$
$$- \int_{Q_{\Lambda(2R)^2,2R}} |Du|_{gh}^{p-2}g^{\alpha\beta}D_\beta u^j\left(h_{ij} - \bar{h}_{ij}\right)\left(D_\alpha v^j - D_\alpha u^j\right)dz. \tag{2.27}$$

Let $s > p$ be the same positive number as in (2.2), which is stipulated later, and recall $q \ge p > 1$. We use Hölder's and Young's inequality to make the second term in the right hand side bound by

$$\int_{Q_{\Lambda(2R)^2,2R}} \left|gh - \bar{g}\bar{h}\right|\,|Du|^{p-1}|Du - Dv|dz$$
$$\le \delta \int_{Q_{\Lambda(2R)^2,2R}} |Du - Dv|^p dz$$
$$+C(\delta^{-1})\left(\int_{Q_{\Lambda(2R)^2,2R}} \left|gh - \bar{g}\bar{h}\right|^{\frac{ps}{(p-1)(s-p)}} dz\right)^{\frac{s-p}{s}} \left(\int_{Q_{\Lambda(2R)^2,2R}} |Du|^s dz\right)^{\frac{p}{s}}$$

$$\le \delta \int_{Q_{\Lambda(2R)^2,2R}} |Du - Dv|^p dz \tag{2.28}$$

$$+C(\delta^{-1}) \left|Q_{\Lambda(2R)^2,2R}\right|^{\frac{s-p}{s}} (|g|_{*,hd} + |h|_{*,hd})^{\frac{s-p}{s}} \left(\int_{Q_{\Lambda(2R)^2,2R}} |Du|^s dz\right)^{\frac{p}{s}},$$

where, by the boundedness (1.2) of the coefficients, we have the bound for the mean oscillation of the coefficients

$$\frac{1}{|Q_{\Lambda(2R)^2,2R}|} \int_{Q_{\Lambda(2R)^2,2R}} \left|gh - \bar{g}\bar{h}\right|^{\frac{ps}{(p-1)(s-p)}} dz \le C\left(|g|_{*,hd} + |h|_{*,hd}\right), \tag{2.29}$$

where note that, since we choose a positive number d to be so small that $hd \le 1$ and $2R = 2h\rho_0 \le hd \le 1$ and the notation $|f|_{*,hd}$ is an abbreviation for (1.5). Similarly as (2.28) and (2.29), the third term is estimated by

$$\delta \int_{Q_{\Lambda(2R)^2,2R}} |Dv - Du|^p dz \tag{2.30}$$

$$+C(\delta^{-1}) \left|Q_{\Lambda(2R)^2,2R}\right|^{\frac{s-p}{s}} |h|_{*,hd}^{\frac{s-p}{s}} \left(\int_{Q_{\Lambda(2R)^2,2R}} |Du|^s dz\right)^{\frac{p}{s}}.$$

The first term are bounded by

$$C \int_{Q_{\Lambda(2R)^2,2R}} \left(|Dv - Du|^p + \left||F|^{p-2}F - \overline{|F|^{p-2}F}\right|^{\frac{p}{p-1}}\right) dz. \tag{2.31}$$

Combining (2.28), (2.30) and (2.31) with (2.27), using (2.26) and choosing a positive number δ to be small, we choose the positive constant C depends only on p, λ and Λ to arrived at the desired estimate (2.25).

3. **PROOf Of THEOREM** Let η be a positive number determined later and λ be a positive number such that $\eta\lambda \ge \lambda_0$. Then, in the exactly same way as in (2.8) and (2.9), we can choose a "good" parabolic cylinder $Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}(z_0)$ for almost every $z_0 \in \{|Du| > \overline{C}\left|\tilde{Q}\right|^{\frac{1}{s}} \mu\}$, where $\mu = \eta\lambda\left(\min_{z_0 \in Q_i} |Q_i|\right)^{-\frac{1}{s-p+2}}$. Thus (2.12) and (2.19) hold for λ replaced by $\eta\lambda$ and $Q_{\Lambda(\rho_0)^2,\rho_0}$ replaced by $Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}$. For brevity, we use the notation for the integral average

$$(f)_r = \frac{1}{|Q^{\eta\lambda}_{\Lambda r^2,r}|} \int_{Q^{\eta\lambda}_{\Lambda r^2,r}} f dz. \tag{2.32}$$

Using the elementary inequality

$$||P|^p - |Q|^p| \le C\delta^{-p}|P - Q|^p + \delta|Q|^p, \tag{2.33}$$

which holds for any positive number δ and all $P=(P^i_\alpha), Q=(Q^i_\alpha)\in R^{mn}$, we have, for any positive number δ,

$$\begin{aligned}&\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}} \left||Du|^p-(|Du|^p)_{\rho_0}\right| dz\\ &\le C(\delta^{-1})\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}} |Dv-(Dv)_{\rho_0}|\,dz+\delta\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}} |Dv|^p dz\\ &\quad +C(\delta^{-1})\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}} |Dv-Du|^p dz. \qquad (2.34)\end{aligned}$$

Replacing $Q_{\Lambda(\rho_0)^2,\rho_0}$ by $Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}$ and substituting (2.12), (2.19) and (2.25) into (2.34), we have

$$\begin{aligned}&\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}} \left||Du|^p-(|Du|^p)_{\rho_0}\right| dz\\ &\le C\left(C\delta+C(\delta^{-1})\,h^{-p\alpha}\right)\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}} |Du|^p dz\\ &\quad+\left|Q^{\eta\lambda}_{\Lambda(2R)^2,2R}\right|^{\frac{s-p}{s}}\left(|g|^{\frac{s-p}{s}}_{*,hd}+|h|^{\frac{s-p}{s}}_{*,hd}\right)\left(\int_{Q^{\eta\lambda}_{\Lambda(2R)^2,2R}} |Du|^s dz\right)^{\frac{p}{s}}\\ &\quad+\int_{Q^{\eta\lambda}_{\Lambda(2R)^2,2R}} \left||F|^{p-2}F-\left(|F|^{p-2}F\right)_{2R}\right|^{\frac{p}{p-1}} dz. \qquad (2.35)\end{aligned}$$

We proceed to the estimation for the right hand side in (2.35).

$$\begin{aligned}&\overline{C}^p\left|\tilde{Q}\right|^{\frac{p}{s}}\mu^p\left|Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\right|=C\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}} |Du|^p dz\\ &=\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\left\{|Du|\le\eta\overline{C}|\tilde{Q}|^{\frac{1}{s}}\mu\right\}} |Du|^p dz+\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\left\{|Du|>\eta\overline{C}|\tilde{Q}|^{\frac{1}{s}}\mu\right\}} |Du|^p dz\\ &\le C\,\eta^p\overline{C}^p\left|\tilde{Q}\right|^{\frac{p}{s}}\mu^p\left|Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\right|+\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\left\{|Du|>\eta\overline{C}|\tilde{Q}|^{\frac{1}{s}}\mu\right\}} |Du|^p dz. \qquad (2.36)\end{aligned}$$

where we used (2.8). We choose a positive number η to be so small that $\eta^p<1$ and, then

$$\overline{C}^p\left|\tilde{Q}\right|^{\frac{p}{s}}\mu^p\left|Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\right|=C\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}} |Du|^p dz\le C\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\left\{|Du|>\eta\overline{C}|\tilde{Q}|^{\frac{1}{s}}\mu\right\}} |Du|^p dz. \qquad (2.37)$$

We obtain from (2.9) and (2.37)

$$\begin{aligned}\int_{Q^{\eta\lambda}_{\Lambda(2R)^2,2R}} |Du|^s dz &\le \overline{C}^s\left|\tilde{Q}\right|\mu^s\left|Q^{\eta\lambda}_{\Lambda(2R)^2,2R}\right| \qquad (2.38)\\ &\le C\,(2h)^{m+2}\left(\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\left\{|Du|>\eta\overline{C}|\tilde{Q}|^{\frac{1}{s}}\mu\right\}} |Du|^p dz\right)^{\frac{s}{p}}\left|Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\right|^{1-\frac{s}{p}}.\end{aligned}$$

Noting that $\Lambda = \mu^{2-p}$ and $\mu = (\eta\lambda)|Q_{i_0}|^{-\frac{1}{s-p+2}}$ and using (2.37), we have the boundedness for the third term of (2.35) by

$$
\begin{aligned}
&(2h)^{m+2}\left|Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\right|\left([|F|^{p-2}F]_{*,\frac{p}{p-1}}\right)^{\frac{p}{p-1}} &(2.39)\\
&\le (2h)^{m+2}\eta^{-p}\lambda^{-p}\overline{C}^{-p}\left|\tilde{Q}\right|^{-\frac{p}{s}}\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\left\{|Du|>\eta\overline{C}|\tilde{Q}|^{\frac{1}{s}}\mu\right\}}|Du|^p dz\left([|F|^{p-2}F]_{*,\frac{p}{p-1}}\right)^{\frac{p}{p-1}},
\end{aligned}
$$

where we use that $|Q_{i_0}| \le 1$ and $[|F|^{p-2}F]_{*,\frac{p}{p-1},\tilde{Q}}$ is abbreviated to $[|F|^{p-2}F]_{*,\frac{p}{p-1}}$. Since, by the definition of "good" parabolic cylinders,

$$
\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}}|Du|^p dz = \overline{C}^p\mu^p\left|\tilde{Q}\right|^{\frac{p}{s}}\left|Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\right|,\quad \mu = (\eta\lambda)|Q_{i_0}|^{-\frac{1}{s-p+2}},
$$

we put $\overline{\mu} = \lambda|Q_{i_0}|^{-\frac{1}{s-p+2}}$, and note that $\mu = \eta\overline{\mu}$ to have the estimation

$$
\begin{aligned}
&\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}}\left||Du|^p - (|Du|^p)_{\rho_0}\right| dz\\
&\ge \int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\left\{|Du|>\overline{C}|\tilde{Q}|^{\frac{1}{s}}\overline{\mu}\right\}}\left||Du|^p - (|Du|^p)_{\rho_0}\right| dz\\
&\ge (1-\eta^p)\overline{C}^p\left|\tilde{Q}\right|^{\frac{p}{s}}\overline{\mu}^p\left|Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\left\{|Du| > \overline{C}\left|\tilde{Q}\right|^{\frac{1}{s}}\overline{\mu}\right\}\right|\\
&\ge \tfrac{1}{2}\overline{C}^p\left|\tilde{Q}\right|^{\frac{p}{s}}\overline{\mu}^p\left|Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\left\{|Du| > \overline{C}\left|\tilde{Q}\right|^{\frac{1}{s}}\overline{\mu}\right\}\right|, &(2.40)
\end{aligned}
$$

where, in the second inequality, we use (2.38) and, in the last inequality, we choose a positive number η to be small such that $2\eta^p \le 1$. Combine (2.38), (2.39), (2.40) with (2.35) and divide the resulting inequality by $\overline{C}^p\left|\tilde{Q}\right|^{\frac{p}{s}}|Q_{i_0}|^{-\frac{p}{s-p+2}}$ to have

$$
\begin{aligned}
&\tfrac{1}{2}\lambda^p\left|Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\{|\mathcal{G}| > \lambda\}\right| &(2.41)\\
&\le C\left(\delta + h^{-\alpha}\right)\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\{|\mathcal{G}|>\eta\lambda\}}|\mathcal{G}|^p dz\\
&+C\left(2h\right)^{m+2}\left(|g|^{\frac{s-p}{s}}_{*,hd} + |h|^{\frac{s-p}{s}}_{*,hd}\right)\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\{|\mathcal{G}|>\eta\lambda\}}|\mathcal{G}|^p dz\\
&\quad+\overline{C}^{-p}\left|\tilde{Q}\right|^{-\frac{p}{s}}(2h)^{m+2}\eta^{-p}\lambda^{-p}\left([|F|^{p-2}F]_{*,\frac{p}{p-1}}\right)^{\frac{p}{p-1}}\int_{Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}\cap\{|\mathcal{G}|^p>\eta^p\lambda^p\}}|\mathcal{G}|^p dz,
\end{aligned}
$$

where we put

$$
\begin{aligned}
&\mathcal{G}(z) = \overline{C}^{-1}\left|\tilde{Q}\right|^{-\frac{1}{s}}\left(\min_{z\in Q_i}|Q_i|^{\frac{1}{s-p+2}}\right)|Du(z)|,\\
&\mathcal{F}(z) = \overline{C}^{-1}\left|\tilde{Q}\right|^{-\frac{1}{s}}\left(\min_{z\in Q_i}|Q_i|^{\frac{1}{s-p+2}}\right)|F(z)|\quad\text{for all } z \in Q_d &(2.42)
\end{aligned}
$$

and we use that $|Q_{i_0}| \le 1$ and we see from the way of dividing Q_d into the Whitney type cylinders Q_i, $i = 1, 2, \cdots$, that, for all $z \in Q^{\eta\lambda}_{\Lambda(2R)^2,2R}$, the ratio of $\min_{z\in Q_i} |Q_i|$ for $\min_{0\in Q_i} |Q_i|$ is bounded above and below by the absolute constant.

Note that the set $Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}(z_0)$ is selected for any $z_0 \in \{|\mathcal{G}| > \eta\lambda\}$ in the way as in (2.8) and (2.9) and thus, we can choose the set $Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}(z_0)$ for each $z_0 \in \{|\mathcal{G}| > \lambda\}$. Apply the Vitali type covering lemma to obtain the family of cylinders $Q_i = Q^{\eta\lambda}_{\Lambda(\rho_i)^2,\rho_i}(z_i)$, $z_i \in \{|\mathcal{G}| > \lambda\}$ $i = 1, 2, \cdots$, such that

$$\begin{aligned} &Q_i \cap Q_j = \emptyset, \quad i \neq j, \\ &\{|\mathcal{G}| > \lambda\} \subset \bigcup_{i=1}^{\infty} Q_i' \subset Q_d \quad \text{almost everywhere}, \end{aligned} \tag{2.43}$$

where $Q_i' = Q^{\eta\lambda}_{\Lambda(5\rho_i)^2,5\rho_i}(z_i)$, $i = 1, 2, \cdots,$. Then (2.41) with $Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}(z_0)$ replaced by Q_i, $i = 1, 2, \cdots$, hold. Multiply the both side of (2.41) with $Q^{\eta\lambda}_{\Lambda(\rho_0)^2,\rho_0}(z_0)$ replaced by Q_i, $i = 1, 2, \cdots$, by λ^{q-p-1} and sum up the resulting inequality over the coverings Q_i, $i = 1, 2, \cdots$, to have

$$\begin{aligned} &\lambda^{q-1} |Q_d \cap \{|\mathcal{G}| > \lambda\}| \\ &\le C\, 5^{m+2} \Big(\delta + h^{-\alpha}) + (2h)^{m+2} \Big(|g|^{\frac{s-p}{s}}_{*,R} + |h|^{\frac{s-p}{s}}_{*,R}\Big) \\ &\quad + \big|\tilde{Q}\big|^{-\frac{p}{s}} (2h)^{m+2} \eta^{-p} \Big([|F|^{p-2}F]_{*,\frac{p}{p-1}}\Big)^{\frac{p}{p-1}} \lambda^{-p}\Big) \lambda^{q-p-1} \int_{Q_d \cap \{|\mathcal{G}| > \eta\lambda\}} |\mathcal{G}|^p dz, \end{aligned} \tag{2.44}$$

where we use the disjointness of Q_i, $i = 1, 2, \cdots$. For a moment, we assume that $\mathcal{G}$ is L^q–integrable and proceed to our arguments. Integrate the both side of (2.44) on λ in $\left(\frac{\lambda_0}{\eta}, \infty\right)$ to have

$$\begin{aligned} &\int_{\frac{\lambda_0}{\eta}}^{\infty} \lambda^{q-1} |Q_d \cap \{|\mathcal{G}| > \lambda\}|\, d\lambda \\ &\le C \Big(\delta + h^{-\alpha} + (2h)^{m+2} \Big(|g|^{\frac{s-p}{s}}_{*,R} + |h|^{\frac{s-p}{s}}_{*,R}\Big) \\ &\quad + \big|\tilde{Q}\big|^{-\frac{p}{s}} (2h)^{m+2} \eta^{-p} \Big([|F|^{p-2}F]_{*,\frac{p}{p-1}}\Big)^{\frac{p}{p-1}} \lambda^{-p}\Big) \int_{\frac{\lambda_0}{\eta}}^{\infty} \lambda^{q-p-1} \Big(\int_{Q_d \cap \{|\mathcal{G}| > \eta\lambda\}} |\mathcal{G}|^p dz\Big)\, d\lambda. \end{aligned} \tag{2.45}$$

By changing variables and Fubini's theorem, we make calculation of the integral in the both side of (2.45)

$$\begin{aligned} &\int_{\frac{\lambda_0}{\eta}}^{\infty} \lambda^{q-1} |Q_d \cap \{|\mathcal{G}| > \lambda\}|\, d\lambda = \tfrac{1}{q} \int_{Q_d \cap \{|\mathcal{G}| > \frac{\lambda_0}{\eta}\}} |\mathcal{G}|^q dz, \\ &\int_{\frac{\lambda_0}{\eta}}^{\infty} \lambda^{q-p-1} \Big(\int_{Q_d \cap \{|\mathcal{G}| > \eta\lambda\}} |\mathcal{G}|^p dz\Big)\, d\lambda \\ &= \tfrac{\eta^{-q+p}}{q-p} \Big(\tfrac{p}{q} \int_{Q_d \cap \{|\mathcal{G}| > \lambda_0\}} |\mathcal{G}|^q dz - (\lambda_0)^{q-p} \int_{Q_d \cap \{|\mathcal{G}| > \lambda_0\}} |\mathcal{G}|^p dz\Big). \end{aligned} \tag{2.46}$$

Combine (2.45) with (2.46) to have

$$\begin{aligned}\int_{Q_d} |\mathcal{G}|^q dz &\le (\lambda_0)^{q-p} \int_{Q_d\cap\{|\mathcal{G}|\le\lambda_0\}} |\mathcal{G}|^p dz + \int_{Q_d\cap\{|\mathcal{G}|>\lambda_0\}} |\mathcal{G}|^q dz \\ &\le (\lambda_0)^{q-p} \int_{Q_d} |\mathcal{G}|^p dz + C\left(\delta + h^{-\alpha} + (2h)^{m+2}\left(|gh|_{*,hd}^{\frac{s-p}{s}} + |h|_{*,hd}^{\frac{s-p}{s}}\right)\right. \\ &\quad \left. + \left|\tilde{Q}\right|^{-\frac{p}{s}} (2h)^{m+2}\eta^{-p}\left([|F|^{p-2}F]_{*,\frac{p}{p-1}}\right)^{\frac{p}{p-1}} \lambda^{-p}\right) \frac{p\eta^{-q+p}}{q(q-p)} \int_{Q_d} |\mathcal{G}|^q dz.\end{aligned} \tag{2.47}$$

We choose positive numbers δ and h small and large enough, respectively, to have

$$C\left(\delta + h^{-\alpha}\right) \le \tfrac{1}{6}\tfrac{q(q-p)\eta^{q-p}}{p}.$$

Then we let a positive number d to be so small that

$$C\,(2h)^{m+2}\left(|g|_{*,hd}^{\frac{s-p}{s}} + |h|_{*,hd}^{\frac{s-p}{s}}\right) \le \tfrac{1}{6}\tfrac{q(q-p)\eta^{q-p}}{p}.$$

Moreover, we choose a positive constant λ_0 such that

$$\tfrac{1}{6}(\lambda_0)^p \ge C\left|\tilde{Q}\right|^{-\frac{p}{s}} (2h)^{m+2}\frac{p\eta^{-q}}{q(q-p)}\left([|F|^{p-2}F]_{*,\frac{p}{p-1}}\right)^{\frac{p}{p-1}}, \tag{2.48}$$

where note that the positive number η is determined in (2.37) and thus, positive numbers h and η are depending only on γ, Γ, m and p. Therefore, we can absorb the second term in the right hand side into the left hand side in (2.47) to have

$$\begin{aligned}\int_{Q_d} |\mathcal{G}|^q dz &\le C\,(\lambda_0)^{q-p}\int_{Q_d} |\mathcal{G}|^p dz \\ &\le C\left(1 + \left(\tfrac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |Du|^s dz\right)^{\frac{q-p}{s-p+2}} + \left(\tfrac{1}{|\tilde{Q}|}\right)^{\frac{q-p}{s}}\left([|F|^{p-2}F]_{*,\frac{p}{p-1}}\right)^{\frac{q-p}{p-1}}\right)\int_{Q_d} |\mathcal{G}|^p dz.\end{aligned} \tag{2.49}$$

Here we need the Gehring inequality available for a solution of (1.1). The proof is referred to [11, 16]. Let α_0 and β_0 be positive numbers such that $\alpha_0 = p\left(\frac{m}{2}+1\right) - m$ and

$$\beta_0 = \begin{cases} 2, & \text{if } p > 2, \\ 4-p, & \text{if } \frac{2m}{m+2} < p < 2. \end{cases}$$

Lemma 5 *Let u be a small solution of (1.1) with $p > \frac{2m}{m+2}$. Then there exist positive constants ϵ and C depending only on m, p, a, γ, Γ and M such that*

$$\begin{aligned}\frac{1}{|Q_{\rho^p,\rho}|}\int_{Q_{\rho^p,\rho}(z_0)} |Du|^{p+\epsilon} dz &\le C\rho^{\epsilon\left(\frac{p}{\beta_0}-1\right)}\left(\frac{1}{|Q_{(2\rho)^p,2\rho}|}\int_{Q_{(2\rho)^p,2\rho}(z_0)} |Du|^p dz\right)^{1+\frac{\epsilon}{\beta_0}} \\ &\quad + C\rho^{\epsilon\left(\frac{p}{\alpha_0}-1\right)}\left(\frac{1}{|Q_{(2\rho)^p,2\rho}|}\int_{Q_{(2\rho)^p,2\rho}(z_0)} |Du|^p dz\right)^{1+\frac{\epsilon}{\alpha_0}} \\ &\quad + C\frac{\rho^{-p-\epsilon}}{|Q_{(2\rho)^p,2\rho}|}\int_{Q_{(2\rho)^p,2\rho}(z_0)} |F|^p dz.\end{aligned} \tag{2.50}$$

holds for all $Q_{\rho^p,\rho}(z_0) \subset Q_{(2\rho)^p,2\rho}(z_0) \subset Q$ with $\rho > 0$.

Now we note (2.42) to rewrite (2.49) for $|Du|$ and set $s = p + \epsilon$ in the resulting inequality, and then we apply (2.50) with $\rho = (r_0)^{\frac{2}{p}}$ to arrived at the desired estimation (1.6).
As a result, we have shown the validity of (1.6), provided $\mathcal{G}$ is L^q−integrable. Now we will remove the integrability assumption of $\mathcal{G}$. Let $L > \lambda_0$ be a positive number and put $\mathcal{G}_L = \min\{\mathcal{G}, L\}$. Then we see that (2.45)-(2.49) hold with $\mathcal{G}$ replaced by $\mathcal{G}_L$. Finally, we can take the limit as $L \to \infty$ in (2.49) with $\mathcal{G}$ replaced by $\mathcal{G}_L$ and use Fatou lemma to obtain (2.49).

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