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APPROXIMATION SCHEME OF THE MEAN CURVATURE FLOW BY THE BENCE-MERRIMAN-OSHER ALGORITHM

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1. Introduction

Let $\{\Gamma(t)\}_{t\geq 0}$ be a smooth hyper surface embedded into $\mathbb{R}^{n+1}$ and $V(t,x)$ denote the normal velocity of the surface $\Gamma(t)$ at x. We consider the mean curvature evolution equation;

$$\begin{cases} V=\kappa\nu, & \text{in } (0,\infty)\times\mathbb{R}^n, \\ \Gamma(0)=\partial C_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\nu=\nu(x)$ is the outward unit normal to the interface and $\kappa=\kappa(x)$ is the mean curvature at x, respectively. We denote a region that is enclosed by the interface at t as $C(t)$. The equation (1.1) has been widely studied by many authors. Among others, Evans-Spruck [9] and Chen-Giga-Goto [5] considered a weak solution for (1.1) by a notion of the viscosity solutions. It is now standard to introduce a level set function to describe the equation (1.1). For some smooth (continuous) function $u:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ such that

$$\Gamma(t)=\{(t,x)|u(t,x)=0\},$$

it follows from (1.1) that

$$\begin{cases} \partial_t u-\Delta u+\dfrac{\nabla u\otimes\nabla u}{|\nabla u|^2}\nabla^2 u=0, \\ u(0,x)=u_0(x), \qquad \partial C_0=\{x;u_0(x)=0\}. \end{cases} \tag{1.2}$$

In this note, we are particularly concerned with an algorithm of numerical computation to (1.1): a scheme introduced by Bence-Merriman-Osher [3] which compute the motion by mean curvature by a simple procedure using a linear heat equation.

Let $\Gamma(t)$ be a smooth surface given by the 0 - level set of a level set function u as $\Gamma(t) = \{x; u(t,x) = 0\}$. We also suppose that $\Gamma(t)$ is given by the boundary of an inner region; $C(t)$. We set $\chi_C(x)$ as a characteristic function of C , i.e.,

$$\chi_C(x) = \begin{cases} 1, & x \in C, \\ 0, & x \notin C. \end{cases}$$

For the initial data $u_0(x) = \chi_C(x) - \chi_{C^c}(x)$, we solve the initial boundary value problem of the heat equation:

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = 0, & \text{in } (0,+\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x) = \begin{cases} 1, & x \in C, \\ -1, & x \in C^c, \end{cases} & \text{in } \mathbb{R}^n, \\ u(t,x) \to -1, & (|x| \to \infty) \end{cases} \tag{1.3}$$

up to $t = h$, where $h > 0$ is the width of the time discretization. By the solution of (1.3) at $t = h$, i.e., $u_0(h,x)$, we define a new set;

$$C_1 = \{x \in \mathbb{R}^n | u_0(h,x) \geq 0\}.$$

Again we solve the heat equation (1.3) with initial data u_1 instead of u_0. Repeating these procedure, we construct a sequence of sets $\{C_k\}_{k=0,1,\cdots}$ and a solution $u_h(t,x)$ of

$$\begin{cases} \partial_t u_k(t,x) - \Delta u_k(t,x) = 0, & \text{in } (kh,(k+1)h] \times \mathbb{R}^n, \\ u_k(0,x) = \chi_{C_k}(x) - \chi_{C_k^c}(x), & \text{in } \mathbb{R}^n, \\ u(t,x) \to -1 & (|x| \to \infty), \end{cases} \tag{1.4}$$

where

$$C_{k+1} = \{x \in \mathbb{R}^n | u_k((k+1)h, x) \geq 0\}.$$

Then setting

$$\Gamma_h(t) = \partial C_k, \qquad kh \leq t < (k+1)h \quad k = 0,1,2,\cdots$$

and keeping $T = kh > 0$ as a constant and we let the time step $h \searrow 0$: Then we see the approximation interface $\Gamma_h(t)$ converges to the real interface $\Gamma(t)$ which is governed by (1.1):

$$\Gamma_h(t) \longrightarrow \Gamma(t) \quad (0 \leq t \leq T).$$

Mathematical proof of the convergence of this approximation scheme has been done by several authors. See L.C. Evans [7], G. Barles-G. Georglin [2], H. Ishii [12], H. Ishii-G.E. Pires-P.E. Souganidis [14], L. Vivier [22] and F. Leoni [17] (Japanese surveys can be found in S. Goto [10] and K. Ishii [15]).

In this note, we briefly explain our proof in Goto-Ishii-Ogawa [11] of the convergence of the Bence-Merriman-Osher (hereafter we abbreviate it either by B-M-O or BMO) algorithm by a different way that is inspired by a result due to H.M.Soner [21] where he derived an interface equation from the Allen-Cahn equations. In view for showing the convergence, the most of previous results were proven by the level set approach and therefore the convergence of the B-M-O scheme is more or less in-direct way. Our aim is to show the convergence of the B-M-O scheme in more directly. To this end, we employed the method of the signed distance function. A function $d(t,x)$ to a set $C(t) = \{(t,x); \text{surrounded by } \Gamma(t)\}$ is defined as

$$d(t,x) = \begin{cases} d(x,\Gamma(t)) & x \in C(t), \\ -d(x,\Gamma(t)) & x \in C(t)^c. \end{cases}$$

It is well-known that the distance function is the Lipschitz continuous and satisfies the Eikonal equation in the sense of viscosity solution. Moreover once we know that the distance function satisfies the heat equation, then the corresponding motion of the interface is governed by (1.1) (c.f. [20]). From the B-M-O scheme, we introduce an approximation signed distance function; $z_h(t,x)$ (defined in (3.14) blow) and we show that $z_h(t,x) \to d(t,x)$ for any (t,x) as $h \to 0$. In fact, this approximation distance z_h satisfies the following type of the semi-linear heat equation:

$$\begin{cases} \partial_t z_h - \Delta z_h + \dfrac{z_h}{2t_k}(|\nabla z_h|^2 - 1) = 0, & t > 0, x \in \mathbb{R}^n, t_k = t - kh, \\ z_h(kh,x) = d_k(x), \quad t = kh, x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

where $d_k(x)$ is the signed distance function to the approximation interface ∂C_k. Then it is proved that the limit distance function $d(t,x)$ solves a heat equation in the sense of viscosity solution;

$$\begin{cases} \partial_t d - \Delta d = 0, \\ -|\nabla d|^2 + 1 = 0. \end{cases} \tag{1.6}$$

This shows the approximation scheme converges to the original motion by mean curvature. Furthermore, under a slightly stronger assumption, we prove that the derived surface is in fact continuous in space and time variable.

Hereafter, we use the following notations. $UC(\Omega)$ is the set of uniformly continuous functions. $\|f(x)\|_\infty = \text{ess.}\sup_x |f(x)|$. For a function $f(t,x)$, the lower and upper semi-continuous envelop are defined by $f_*(t,x) = \liminf_{(s,y)\to(t,x)} f(s,y)$, $f^*(t,x) = \limsup_{(s,y)\to(t,x)} f(s,y)$, respectively. For a set valued function $C(t,x)$ on $\mathbb{R}_+ \times \mathbb{R}^n$, $C(t)$ denotes its t-section $C(t,\cdot)$.

2. The signed-distance functions and the viscosity solutions

First, we introduce the "signed-distance function" to describe the motion of the inter-

Definition. Assume that $\Gamma(t)$ is a hypersurface in $\mathbb{R}^n$ and the boundary of a fixed open and bounded subset $C(t) \subset \mathbb{R}^n$. Then we define *the signed-distance function* to $\partial C(t)(=\Gamma(t))$; $d(t,x)$; $[0,\infty) \times \mathbb{R} \to \mathbb{R}$ as next

$$d(t,x) = \begin{cases} \operatorname{dist}(x, \partial C(t)), & x \in C(t), \\ -\operatorname{dist}(x, \partial C(t)), & x \notin C(t). \end{cases}$$

REMARK. The distance function to $\partial C(t)$ is the Lipschitz continuous only in x and satisfies $\|\nabla d\|_\infty = 1$.

Definition. Let $d^*(t,x)$ and $d_*(t,x)$ be the upper and lower semi-continuous envelope of $d(t,x)$.

Next, we recall the notion of the "viscosity solution" , which is a weak solutions in a point-wise sense. We consider the level set formulation of (1.1).

$$\begin{cases} \partial_t u - \Delta u + \dfrac{\nabla u \otimes \nabla u}{|\nabla u|^2} \nabla^2 u = 0, & \text{in } [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = \chi_{C_0}(x) - \chi_{C_0^c}(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{2.7}$$

Definition. (Definition of the viscosity solution of Mean curvature evolution equation)
(i) A bounded and continuous function $u \in C([0,\infty) \times \mathbb{R}^n) \cap L^\infty([0,\infty) \times \mathbb{R}^n)$ is a viscosity sub (super)-solution of (2.7), if it holds

$$\begin{cases} \partial_t \phi - \left(\Delta \phi - \dfrac{\nabla \phi \otimes \nabla \phi}{|\nabla \phi|^2} \nabla^2 \phi \right) \le (\ge) 0, & \nabla \phi(t_0, x_0) \neq 0, \\ \partial_t \phi - (\Delta \phi - \eta_i \eta_j)\, \phi \le (\ge) 0, & \nabla \phi(t_0, x_0) = 0 \text{ and } |\eta| \neq 1. \end{cases} \tag{2.8}$$

whenever the test function $\phi \in C^\infty(\mathbb{R}^{n+1})$ such that $u - \phi$ attains its maximum (minimum) at a point $(t_0, x_0) \in ([0,\infty) \times \mathbb{R}^n)$.

(ii). We say $u \in C([0,\infty) \times \mathbb{R}^n) \cap L^\infty([0,\infty) \times \mathbb{R}^n)$ is a viscosity solution of (2.7) if u is both a viscosity sub- and a viscosity super-solution.

We refer to Crandall-Ishii-Lions [6] for the details of the theory of the viscosity solution,

3. Approximated distance function

We describe our main idea to show the convergence of the approximated motion by B-M-O scheme. Let $u_h(t,x); \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be the solution of B-M-O scheme that satisfies for $h > 0$ and $k \in \mathbb{N}$,

$$\begin{cases} \partial_t u_h - \Delta u_h = 0, & t \in [kh, (k+1)h) \times \mathbb{R}^n, \\ u_h(kh, x) = \chi_{C_k}(x) - \chi_{C^c}(x), & \\ C_k = \left\{ (t,x);\ \lim\limits_{t \to kh-} u_h(t,x) \ge 0 \right\}. & \end{cases} \tag{3.9}$$

Note that by the maximal principle, u_h satisfies $\|u_h(t,\cdot)\|_\infty < 1, \quad t \neq kh$.

Our main idea to show the convergence of the BMO algorithm is to consider the approximation distance function $z_h(t,x)$ defined from the BMO scheme (3.9). To introduce this function, we call the result of the singular limit process to the Allen-Cahn equations (c.f. [4], [21]). It is well understood that that the motion of the normal direction of interface is much smaller than the tangential direction. Therefore the motion of approximated interface is described by the one dimensional heat equation: Let the solution of heat equation;

$$u(t,r,\eta) = u(t,r).$$

And we initialize the process by setting

$$u(0,x) = \chi_{C_0}(x) - \chi_{C_0{}^c}(x) \quad \text{in } \mathbb{R}^n.$$

Then we denote $u(t,x)$ is the solution of the one dimensional heat equation , and the solution $u(t,r)$ would be given by:

$$\begin{aligned} u(t,r) =& \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \exp(-\frac{|r-y|^2}{4t})\{\chi_{C_0}(y) - \chi_{C_0^c}(y)\}dy \\ =& \frac{1}{\sqrt{4\pi t}} \left(\int_{-r}^{r} \exp(-\frac{z^2}{4t})dz \right). \end{aligned}$$

Hence we introduce an auxiliary function

$$U_0(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^\zeta e^{-\eta^2} d\eta. \tag{3.10}$$

We should remark that U_0 is a strictly increasing function and satisfies

$$\begin{cases} U_0''(\zeta) + 2\zeta U_0'(\zeta) = 0, \\ U_0(+\infty) = +1, \quad U_0(-\infty) = -1, \quad U_0(0) = 1. \end{cases} \tag{3.11}$$

<u>*Definition*</u> (Approximated distance function).
Let $u_h(t,x)$ be the solution of (3.9). Then for $k \in \mathbb{N} \cup \{0\}$, we define $z_h(t,x); \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$u_h(t,x) = U_0\left(\frac{z_h(t,x)}{2\sqrt{t-kh}}\right) \text{ in } (kh,(k+1)h] \times \mathbb{R}^n, \tag{3.12}$$

where

$$U_0(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^\zeta e^{-\eta^2} d\eta. \tag{3.13}$$

Therefore z_h is defined by

$$z_h(t,x) = 2\sqrt{t-kh} \times U_0^{-1}(u_h(t,x)) \qquad (t \neq kh). \tag{3.14}$$

REMARK. It should be mentioned that Leoni [17] considered the B-M-O scheme deduced from a semi-linear heat equation. Her method is based on the Perron-Ishii argument of the viscosity solution. She used some upper and lower barrier that is closely related to the above function we defined.

We now introduce the approximation surface and limiting regions N and P which are separated by the limiting surface, where each of them represents the out-side and inside of the surface, respectively.

Definition. Let functions z^* and z_* defined by

$$z^*(t,x) = \limsup_{h\to 0\ (s,y)\to(t,x)} z_h(s,y)$$
$$z_*(t,x) = \liminf_{h\to 0\ (s,y)\to(t,x)} z_h(s,y).$$

Definition. (Limiting interface and separated regions)

$$\begin{aligned} &N = \{(t,x) | z^*(t,x) < 0\} \qquad \text{(outside of interface)},\\ &P = \{(t,x) | z_*(t,x) > 0\} \qquad \text{(inside of interface)},\\ &\Gamma = (N \cup P)^c \qquad \text{(interface)}. \end{aligned} \tag{3.15}$$

¿From above definition , clearly N and P are open set. Therefore Γ is closed set.

4. Main Result

By the motion by mean curvature, the interface occasionally shrink into one point. Namely the interface may disappear within a finite time. Therefore we show the convergence of the algorithm while the interface survives. To be specific, we define the extinction time of the interface and shall only handle the interface until the extinction time.

Definition (the extension time of interface).

$$T_{ex} = \sup\{T \in [0,+\infty) \mid |z^*(t,x)|,\ |z_*(t,x)| < +\infty, \text{for all}(t,x) \in [0,T] \times \mathbb{R}^n\}.$$

In previous subsection we define z_h as a function that denote the distance from the interface (that denote the motion of the interface). To show the convergence of the BMO scheme, we claim that

$$z_h(t,x) \longrightarrow d(t,x) \qquad (h \to 0)$$

where $d(t,x)$ is signed-distance function to $\Gamma(t)$, and $d(t,x)$ satisfies the mean curvature evolution equation.

In general, there is a case that interface at some time has interior point, although the first interface do not have any. This phenomenon is called as "fattening of the interface" and large part of this cases still remains open. If $\Gamma(t)$ has an interior point, the approximation from the inside is not equivalent to one from the outside. Here we only consider the case when the fattening do not occur. In order to assure this case, we assume that for all $t > 0$,

$$\text{int } \Gamma(t) = \emptyset.$$

Theorem 4.1. (Main Theorem)[11] *We let* $\Gamma = \mathbb{R}^n \backslash (P \cup N)$ *and* $\text{int}\,\Gamma_t = \emptyset$. *Then for* $t \in [0, T_{ex})$, *the approximation distance function* $z_h(t,x)$ *by the B-M-O scheme (3.9) through (3.14) converges to the limiting distance function* $d(t,x) = dist((t,x), \Gamma_t)$ *to* $\Gamma(t)$. *Namely*

$$z_h(t,x) \to d(t,x) \quad (t,x) \in [0, T_ex) \times \mathbb{R}^n. \tag{4.16}$$

Moreover d is a weak solution of the mean curvature flow (2.7) in the sense of the viscosity solution. Namely for $\underline{d}_*$*;its upper semi-continuous envelop,* $\bar{d}^*$, $\underline{d}^* = (\min d((t,x),0))^*$, $\bar{d}^* = (\max d((t,x),0))^*$. *Then* $\underline{d}^*$ *and,* $\bar{d}^*$ *satisfies that*

$$\begin{aligned} &\partial_t \underline{d}_* - \Delta \underline{d}_* + \frac{\nabla \underline{d}_* \otimes \nabla \underline{d}_*}{|\nabla \underline{d}_*|^2} D^2 \underline{d}_* \leq 0 \quad \textit{(viscosity sub-solution)}, \\ &\partial_t \bar{d}^* - \Delta \bar{d}^* + \frac{\nabla \bar{d}^* \otimes \nabla \bar{d}^*}{|\nabla \bar{d}^*|^2} D^2 \bar{d}^* \geq 0 \quad \textit{(viscosity super-solution)}. \end{aligned} \tag{4.17}$$

In particular , the motion of the limiting interface $\Gamma(t)$ *governed by the mean curvature flow in the weak sense.*

REMARK. The saturated distance function which represents the limiting function of $z_h(t,x)$ satisfies equation (2.7) only on $\Gamma(t)$. In this sense, the evolution is understood in a weak sense. The theory of the viscosity solution is necessary for this formulation.

Under the slightly stringent condition on the interfaces, it is possible to show the limiting distance function to the interface is continuous respect to (t,x) ([11]).

Theorem 4.2 ([11]). *If the interface satisfies*

$$\Gamma(t) = \partial P(t) = \partial N(t),$$

then

(1)$\Gamma(t)$ *is continuous in the sense of Housdorff distance .*
i.e.,

$$\lim_{t \to s} d_H(\Gamma(t), \Gamma(s)) = 0$$

(2) Distance function $d(t,x)$ *is continuous with respect to* (t,x).

(3)In addition, for $(t,x) \in [0, +\infty) \times \mathbb{R}^n$, *the approximated distance function* z_h *converges to the distance function d locally and uniformly as* $h \to 0$.

REMARK. Note that the condition

$$\Gamma(t) = \partial P(t) = \partial N(t)$$

implies int $\Gamma(t) = \emptyset$, while the reverse is not necessarily true.

5. Equation of the approximation distance

In this section, we derive several nature of the approximation distance function $z_h(t,x)$. Naturally from the definition, z_h solves some semi-linear heat equation:

Proposition 5.1. *Let z_h be the approximation distance function defined in (3.12). Then there holds*

$$\begin{cases} \partial_t z_h - \Delta z_h + \dfrac{z_h}{2(t-kh)}(|\nabla z_h|^2 - 1) = 0, & t \in (kh,\, (k+1)h],\ x \in \mathbb{R}^n, \\ z_h(0,x) = d_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{5.18}$$

Moreover, z_h is subject to the a priori estimate,

$$\|\nabla z_h(t,\cdot)\|_\infty \le 1 \quad \textit{for } t \in [0, T_{ex}).$$

Proof of Proposition 5.1. Since $u_h(t,x) = U_0\left(\frac{z_h(t,x)}{2\sqrt{t-kh}}\right)$ satisfies (3.9),

$$\begin{aligned} 0 =&(t-kh)^{-\frac{3}{2}} U_0{}'(\cdot) \times \left(\frac{(t-kh)\partial_t z_h(t)}{2} - \frac{z_h(t,x)}{4} \right) \\ &- 2(t-kh)^{-1} U_0{}''(\cdot) \times \left| \frac{\nabla z_h(t)}{2} \right|^2 - (t-kh)^{-1/2} U_0{}'(\cdot) \times \left(\frac{\Delta z_h(t)}{2} \right). \end{aligned}$$

From $U_0{}''(\zeta) + 2\zeta U_0{}'(\zeta) = 0$ in (3.11), we have

$$\partial_t z_h - \Delta z_h + \frac{z_h}{2(t-kh)}(|\nabla z_h|^2 - 1) = 0.$$

By differentiating the both sides of (5.18) by x_i,

$$\partial_t(\nabla_i z_h) - \Delta \nabla_i z_h + \frac{\nabla_i z_h(|\nabla z_h|^2 - 1)}{2(t-kh)} + \frac{z_h}{t-kh}(\nabla z_h \cdot \nabla \nabla_i z_h) = 0.$$

Multiply the both sides of above equation by $\nabla_i z_h$, and make a summation over $i = 1, 2, \dots N$. Then letting $w = |\nabla z_h|^2$,

$$\begin{cases} \partial_t w - \Delta w + \dfrac{w}{t-kh}(w-1) + \dfrac{2z_h}{t-kh}(\nabla z_h \cdot \nabla w) = -|\nabla^2 z_h|^2 \le 0, \\ w(0,x) = 1. \end{cases} \tag{5.19}$$

Since $\overline{w}(t,x) \equiv 1$ satisfies

$$\begin{cases} \partial_t \bar{w} - \Delta \bar{w} + \dfrac{\bar{w}}{t-kh}(\bar{w}-1) + \dfrac{2z_h}{t-kh}(\nabla z_h \cdot \nabla \bar{w}) = 0, \\ \bar{w}(0,x) = 1, \end{cases} \tag{5.20}$$

applying the comparison principle, we conclude $w(t,x) \le 1$. Hence z_h satisfies $\|\nabla z_h(\cdot,\cdot)\|_\infty \le 1$. □

One may obtain that the limiting functions z^* and z_* are subject to the Eikonal equation in the sense of the viscosity solution.

Proposition 5.2. *z^* is viscosity sub-solution of the Eikonal equation $-|\nabla z|^2+1=0$ on N. i.e., z^* satisfies*

$$-|\nabla z^*|^2+1\le 0 \quad \mathit{in}\, N=\{(t,x)|z^*(t,x)<0\} \tag{5.21}$$

in the viscosity sense.
z_* is viscosity super-solution of $-|\nabla z|^2+1=0$ on P. i.e., z_* satisfies

$$|\nabla z_*|^2-1\ge 0 \quad \mathit{in}\, P=\{(t,x)|z_*(t,x)>0\} \tag{5.22}$$

in the viscosity sense.

6. Outline of the proof of Main Theorem

Proposition 6.1. *(1) $u_h(t,x)\to +1(h\to 0)$ locally uniformly on P.*
(2) $u_h(t,x)\to -1(h\to 0)$ locally uniformly on N.

Proof of Proposition 6.1. We shall only show (1), since (2) is similar. For any compact set $K\subset\subset P$, let $(t_0,x_0)\in K$ and define $z_*(t_0,x_0)\equiv\gamma>0$. Then for any $|t-t_0|<\delta$, $x\in B_\delta(x_0)$, there exists $\delta>0$ such that

$$z_*(t,x)\ge\frac{1}{2}z_*(t_0,x_0)=\frac{1}{2}\gamma>0.$$

Therefore suppose that there are $h_0>0,\delta_0>0$ satisfying

$$z_h(t,x)\ge\frac{1}{2}z_*(t_0,x_0)>0,\quad 0<h<h_0,\quad |t-t_0|<\delta_0,\quad x\in B_\delta(x_0).$$

Since $U_0(r)\le 1$ and U_0 is a monotone function ,

$$1\ge u_h(t,x)=U_0\left(\frac{z_h(t,x)}{2\sqrt{t-kh}}\right)\ge U_0\left(\frac{\gamma}{4\sqrt{t-kh}}\right).$$

When $kh<t<(k+1)h$ $(0<t-kh<h)$, by passing $h\to 0$, we deduce

$$\liminf_{h\to 0}U_0\left(\frac{\gamma}{4\sqrt{t-kh}}\right)\ge 1,$$

which shows

$$\lim_{h\to 0}u_h(t,x)=1 \qquad \mathrm{on} B_\delta(x_0)\times\{|t-t_0|<\delta\}$$

uniformly. Since K is arbitrary compact set, u_h converges 1 locally uniformly on P. ☐

Proposition 6.2. *Assume that* $\operatorname{int}\Gamma(t)=\emptyset$ *and*

$$d(t,x)=\begin{cases}\mathit{dist}(x,\Gamma(t)), & x\in P(t),\\ -\mathit{dist}(x,\Gamma(t)), & x\in N(t),\\ 0, & x\in\Gamma(t).\end{cases}$$

Then for any $(t,x) \in \{[0,T_{ex}) \times \mathbb{R}^n\}$, *we have*

(6.23) $\qquad z_*(t,x) = d_*(t,x) = d(t,x), \qquad (t,x) \in P,$

(6.24) $\qquad z^*(t,x) = d^*(t,x) = d(t,x), \qquad (t,x) \in N,$

(6.25) $\qquad d^*(t,x) = d(t,x) = 0, \qquad (t,x) \in \Gamma \cap \{(t,x) | z^*(t,x) = 0\},$

(6.26) $\qquad d_*(t,x) = d(t,x) = 0, \qquad (t,x) \in \Gamma \cap \{(t,x) | z_*(t,x) = 0\},$

where d^* *and* d_* *are the upper and lower semi-continuous envelope of* d*, respectively.*

REMARK. Since Γ is not necessarily equal to ∂N (or ∂P), $d(t,x)$ is not necessary the signed distance function to the interface $\Gamma(t)$.

We invoke the following lemma due to Barles-Soner-Souganidis [2]. This assures that the signed-distance function satisfies the mean curvature evolution equation.

Lemma 6.3. [2] *If* z_* *is viscosity sub- (super-)solution of* $\partial_t z_* - \Delta z_* = 0$ *and viscosity solution of* $|\nabla z_*| = 1$, z_* *is viscosity sub-(super-)solution of the Mean curvature evolution equation ;*

$$\partial_t z_* - \left(\Delta z_* - \frac{\nabla z_* \otimes \nabla z_*}{|\nabla z_*|^2} \nabla^2 z_* \right) = 0.$$

We are now ready to show Theorem 4.1(Main Theorem).

Proof of Theorem 4.1 (Main Theorem). The convergence of the approximated distance function (4.16) has been shown by (6.23)-(6.26) in Proposition 6.2.

We recall that from Proposition 5.1, z_h satisfies

$$\partial_t z_h - \Delta z_h + \frac{z_h}{2(t-kh)}(|\nabla z_h|^2 - 1) = 0, \qquad (t,x) \in [0,T_{ex}) \times \mathbb{R}^n.$$

We show that z^* is the sub-solution of (2.7). The other case is quite similar.
If $(t_0,x_0) \in P$, $z_h(t_0,x_0) > 0$. Additionally by Proposition 5.1, $\|\nabla z_h\|_\infty \leq 1$ implies

$$\partial_t z_h - \Delta z_h \geq 0 \quad \text{for all } h \in (0,1)$$

on P. Let $\phi \in C^\infty$ be a smooth function and (t_0,x_0) be a strict local minimizer of $z_* - \phi$. Hence $z_*(t,x)$ is finite and and there exist a subsequence h_m and local minimizers (t_m,x_m) of the difference $z_h - \phi$ converging to (t,x) as $k \to \infty$. Hence on (t_m,x_m)

$$\partial_t z_{h_m} - \Delta z_{h_m} \geq 0.$$

In other hands, since $z_h(t_m,x_m) - \phi(t_m,x_m)$ attains the minimum (note that z_h is smooth),

$$\partial_t \phi(t_m,x_m) = \partial_t z_{h_m}(t_m,x_m), \quad \Delta z_{h_m}(t_m,x_m) \geq \Delta \phi(t_m,x_m).$$

Therefore

$$\partial_t \phi - \Delta \phi \geq 0 \quad \text{at } (t_m,x_m).$$

By passing $m \to \infty$ in above inequality, we see that z_* satisfies

$$\partial_t z_* - \Delta z_* \geq 0$$

in the sense of the viscosity solution.

Proposition 6.2 and Lemma 6.3 implies conclusion of theorem. $\square$

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