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Image restoration through microlocal analysis with smooth tight wavelet frames*

大阪教育大学・数理科学 芦野 隆一 (Ryuichi Ashino)
Division of Mathematical Sciences, Osaka Kyoiku University
Steven J. Desjardins
Department of Mathematics and Statistics, University of Ottawa
Christopher Heil
School of Mathematics, Georgia Institute of Technology
大阪大学・理学研究科 長瀬 道弘 (Michihiro Nagase)
Graduate School of Sciences, Osaka University
Rémi Vaillancourt
Department of Mathematics and Statistics, University of Ottawa

Abstract

Results on general and tight wavelet frames and microlocal analysis in $\mathbb{R}^n$ are summarized. To perform microlocal analysis of tempered distributions in $\mathbb{R}^n$, tight frame wavelets, whose Fourier transforms consist of smoothed characteristic functions of cubes in $\mathbb{R}^n$, are constructed. Singularities in smooth images are localized in position and direction by means of frame coefficients computed in the Fourier domain. The numerical process of image restoration based on microlocal analysis with smooth tight wavelet frames is presented and two natural images are restored by this process.

1 Introduction

In previous work [1], [2], [3], the concept of microlocal analysis was described, with the goal of numerically studying the singularities of distributions in $\mathbb{R}^n$.

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The micro-analytic content of a distribution was localized by means of the coefficients of a multiwavelet expansion. The Fourier transforms of the multiwavelets were characteristic functions of boxes or squares that completely covered the Fourier domain. In order to have better resolution in the x domain, smooth tight wavelet frames were constructed in [4] by using several types of smoothings of the box functions. A numerically convenient smoothing was obtained in [5] by tapering the characteristic functions in $\mathbb{R}^2$. Image analysis in the Fourier domain allows the localization of the micro-analytic content and singularities in position and orientation. It also allows some denoising and compression of natural and geometric images.

This paper is a continuation of the previous work. We expand the discussion on general frame theory for the convenience of the reader and present, as a new application, a procedure for image restoration based on microlocal analysis and smooth frame expansion. In particular, singularities that have been localized by the above methods are removed from the image. The theoretical results on the construction and application of wavelet frames have to be adapted to finite images. Two simple examples are presented, where a singularity in the form of a short straight segment or a straight line is added to a natural image, thus producing a scarred image. The discrete Fourier transform of the scarred image is filtered by means of one or two wavelet frames with support in the high frequency part of the transformed image, at right angle to the scar, in order to pick up the singularity. This high pass filter cuts off the low frequencies which come mainly from the Fourier transform of the original image (which is rather smooth). The frame coefficients of the filtered image are computed in the Fourier domain. By means of the Plancherel theorem, the coefficients with larger absolute values localize the scar in the x domain. The scar is reconstructed by means of its wavelet frame expansion in the x domain. Because of its finite size, the one-pixel-thick scar is returned to the x domain as a few-pixel-thick segment or line after a direct and an inverse discrete Fourier transform. To remove small perturbations in the returned image, the values at each pixel are rounded, thus setting the small values to zero. Then the thickness of the line or segment is found and reduced by setting to zero the pixels off the center line. In some cases, further tuning may be needed. Then, subtracting the image of the scar from the initial scarred image restores the original image.

The paper is organized as follows. In section 2, frame theory is briefly reviewed. Tight wavelet frames in $\mathbb{R}^n$ are described in section 3. Sections 4 to 7 present the notions of frame multiresolution analysis, microlocal analysis, one- and multi-dimensional orthornormal and frame microlocal filtering. Section 9 describes the image restoration process based on the above theory. Two scarred natural images are numerically restored.

2 Frames

We briefly review frame theory in this section, referring to [6], [7] and [8] for detailed information. Frame theory was originally developed by Duffin and Schaeffer [9] to reconstruct band-limited signals f from irregularly spaced samples $\{f(t_n)\}_{n\in\mathbb{Z}}$. A function f is said to be band-limited if its Fourier transform is supported in a finite interval $[-\pi/T, \pi/T]$, Duffin and Schaeffer were motivated by the classical sampling theorem (see, for example, [8], p. 44), which asserts that a band-limited function $f(t)$ can be recovered from regularly spaced samples $f(nT)$.

Theorem 1 (Sampling Theorem) *If the support of f is included in the interval $[-\pi/T, \pi/T]$, then*

$$f(x) = \sum_{n=-\infty}^{\infty} f(nT)h_T(t - nT),$$

with

$$h_T(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

It is natural to consider general conditions under which one can recover a vector f in a separable Hilbert space $\mathcal{H}$ from inner products $\langle f, \phi_n\rangle$ with a family of vectors $\{\phi_n\}_{n\in\mathbb{J}}$, where the index set $\mathbb{J}$ might be finite or infinite.

A sequence $\{\phi_n\}_{n\in\mathbb{J}}$ is called a *frame* for $\mathcal{H}$ if there exist constants $A > 0$ and $B > 0$ such that for any $f \in \mathcal{H}$,

$$A\,\|f\|^2 \le \sum_{n\in\mathbb{J}} |\langle f, \phi_n\rangle|^2 \le B\,\|f\|^2.$$

The constants A and B are called *frame bounds.* A frame is said to be *tight* if $A = B$. The operator $L : \mathcal{H} \mapsto \mathcal{H}$ defined by

$$Lf = \sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle\, \phi_n, \qquad \forall f \in \mathcal{H},$$

is called the *frame operator*, and is a positive, continuous mapping of H onto itself with continuous inverse. Define

$$\ell^2(\mathbb{J}) := \left\{x \;:\; \|x\|^2_{\ell^2(\mathbb{J})} := \sum_{n\in\mathbb{J}} |x[n]|^2 < +\infty\right\}$$

and define the *analysis operator* $U : \mathcal{H} \mapsto \ell^2(\mathbb{J})$ by

$$Uf[n] = \langle f, \phi_n\rangle, \qquad \forall n \in \mathbb{J}.$$

The *synthesis operator* is the adjoint U^* of U, and is given by

$$U^*x = \sum_{n\in\mathbb{J}} x[n]\,\phi_n, \qquad x \in \ell^2(\mathbb{J}).$$

Then the frame operator L factorizes as $L = U^*U$. The system $\{\widetilde{\phi}_n\}_{n\in\mathbb{J}}$ defined by

$$\widetilde{\phi}_n = L^{-1}\phi_n = (U^*U)^{-1}\phi_n$$

is a frame for $\mathcal{H}$ with frame bounds $1/B$, $1/A$, and is called the *dual frame* of $\{\phi_n\}_{n\in\mathbb{J}}$. If the frame is tight (i.e., $A = B$), then $\widetilde{\phi}_n = A^{-1}\,\phi_n$.

Let $\operatorname{ran} U$ denote the range of U, that is, the space of all Uf where $f \in \mathcal{H}$. If $\{\phi_n\}_{n\in\mathbb{J}}$ is a frame which is not a basis for $\mathcal{H}$, then $\operatorname{ran} U$ is strictly included in $\ell^2(\mathbb{J})$ and U admits an infinite number of left inverses $\bar{U}^{-1}$:

$$\bar{U}^{-1}Uf = f, \qquad \forall f \in \mathcal{H}.$$

The left inverse that is zero on $\operatorname{ran} U^{\perp}$ is called the *pseudo-inverse* of U and is denoted by $\widetilde{U}^{-1}$:

$$\widetilde{U}^{-1}x = 0, \qquad \forall x \in \operatorname{ran} U^{\perp}.$$

In infinite-dimensional spaces, the pseudo-inverse $\widetilde{U}^{-1}$ of an injective operator is not necessarily bounded. This induces numerical instabilities when trying to reconstruct f from Uf. The pseudo-inverse can be expressed in the form

$$\widetilde{U}^{-1} = (U^*U)^{-1}U^*,$$

and

$$f = \widetilde{U}^{-1}Uf = \sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle\, \widetilde{\phi}_n = \sum_{n\in\mathbb{J}} \langle f, \widetilde{\phi}_n\rangle\, \phi_n = L(L^{-1}f).$$

When the frame is tight (i.e., $A = B$), then $\widetilde{\phi}_n = A^{-1}\,\phi_n$, so in this case

$$f = \widetilde{U}^{-1}Uf = \frac{1}{A}\sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle\, \phi_n.$$

Hence, for a tight frame, by replacing ϕ_n by $\phi_n/\sqrt{A}$ we may without loss of generality always assume that the frame bound is $A = 1$.

Let us describe some numerical algorithms to recover a signal f from its frame coefficients $Uf[n] = \langle f, \phi_n\rangle$. When the dual frame vectors $\widetilde{\phi}_n = (U^*U)^{-1}\phi_n$ are precomputable, we can recover each f from the frame expansion

$$f = L^{-1}Lf = \sum_{n\in\mathbb{J}} \langle f, \phi_n\rangle\, \widetilde{\phi}_n.$$

But in some applications, the dual frame vectors $\widetilde{\phi}_n$ cannot be computed in advance. In this case, another approach is to apply the pseudo-inverse to Uf in the form:

$$f = \widetilde{U}^{-1}Uf = (U^*U)^{-1}(U^*U)f = L^{-1}Lf.$$

Whether we precompute the dual frame vectors or apply the pseudo inverse on the frame data, both approaches require an efficient way to compute $f = L^{-1}g$ for some $g \in \mathcal{H}$. One way is to use the following *Richardson's extrapolation* scheme when the frame bounds A and B are known.

Lemma 1 (Richardson's Extrapolation) *Let $g \in \mathcal{H}$. To compute $f = L^{-1}g$, initialize $f_0 = 0$. Let $\gamma > 0$ be a relaxation parameter. For any $n > 0$, define*

$$f_n = f_{n-1} + \gamma\,(g - Lf_{n-1}).$$

If

$$\delta = \max\{|1-\gamma A|, |1-\gamma B|\} < 1,$$

then

$$\|f - f_n\| \leq \delta^n\,\|f\|, \tag{1}$$

and hence $\lim\limits_{n\to+\infty} f_n = f$.

This algorithm for frame inversion appears in [9] and is commonly referred to as the *frame algorithm.* The convergence rate is maximized when δ is minimum:

$$\delta = \frac{B-A}{B+A} = \frac{1-A/B}{1+A/B},$$

which corresponds to the relaxation parameter

$$\gamma = \frac{2}{A+B}.$$

The algorithm converges quickly if A/B is close to 1. If A/B is small then

$$\delta \approx 1 - 2\frac{A}{B}. \tag{2}$$

Inequality (1) proves that we obtain an error smaller than ϵ for a number n of iterations, such that the following inequality holds,

$$\frac{\|f-f_n\|}{\|f\|} \leq \delta^n = \epsilon.$$

Inserting (2) in this inequality gives

$$n \approx \frac{\log \epsilon}{\log(1 - 2A/B)} \approx \frac{-B}{2A} \log \epsilon.$$

Thus, the number of iterations is directly proportional to the frame bound ratio B/A.

As Gröchenig has shown, much faster algorithms for frame inversion can be derived by making use of ideas from conjugate gradient methods [10]. In particular, those methods do not require knowledge of the frame bounds, and can be fast even when B/A is not close to 1.

3 Tight Wavelet Frames

Since the dual of a tight frame is a constant multiple of the frame itself, recovering functions from their frame coefficients does not require the computation of the dual frame. Hereafter, we shall focus on tight wavelet frames.

Given $f \in L^2(\mathbb{R}^n)$, let f_{jk} denote the scaled and shifted function

$$f_{jk}(x) = 2^{nj/2} f(2^j x - k), \qquad j \in \mathbb{Z}, \quad k \in \mathbb{Z}^n. \tag{3}$$

Let $\mathbb{L}$ be a finite index set. A system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L}, j\in\mathbb{Z}, k\in\mathbb{Z}^n} \subset L^2(\mathbb{R}^n)$ is called a *tight wavelet frame* with frame bound A if

$$f = \frac{1}{A} \sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \langle f, \psi^\ell_{jk}\rangle \, \psi^\ell_{jk}, \qquad \forall f \in L^2(\mathbb{R}^n). \tag{4}$$

A system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L}, j\in\mathbb{Z}, k\in\mathbb{Z}^n} \subset L^2(\mathbb{R}^n)$ is called an *orthonormal wavelet basis* if it is an orthonormal basis for $L^2(\mathbb{R}^n)$. This is equivalent to saying that the system $\{\psi^\ell_{jk}\}_{\ell\in\mathbb{L}, j\in\mathbb{Z}, k\in\mathbb{Z}^n}$ is a tight wavelet frame with frame bound 1 and $\|\psi^\ell\|_{L^2(\mathbb{R}^n)} = 1$ for $\ell \in \mathbb{L}$.

The following general theorem, which is essentially Theorem 1 as stated and proved in [11] for $\mathbb{R}^n$, gives necessary and sufficient conditions to have a tight wavelet frame in $\mathbb{R}^n$ with frame bound 1.

Theorem 2 *Suppose $\psi^\ell \subset L^2(\mathbb{R}^n)$ for $\ell \in \mathbb{L}$. Then*

$$\|f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} |\langle f, \psi^\ell_{j,k}\rangle|^2 \tag{5}$$

for all $f \in L^2(\mathbb{R}^n)$ *if and only if the set of functions* $\{\psi^\ell\}_{\ell\in\mathbb{L}}$ *satisfies the following two equalities:*

$$\sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}}} \left|\widehat{\psi^\ell}(2^j\xi)\right|^2 = 1, \qquad a.e.\, \xi \in \mathbb{R}^n, \tag{6}$$

$$\sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}_+}} \widehat{\psi^\ell}(2^j\xi)\,\overline{\widehat{\psi^\ell}(2^j(\xi+q))} = 0, \qquad a.e.\, \xi \in \mathbb{R}^n,\ \forall q \in \mathbb{Z}^n\backslash(2\mathbb{Z})^n, \tag{7}$$

where $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ *and* $q \in \mathbb{Z}^n\backslash(2\mathbb{Z})^n$ *means that at least one component* q_j *is odd.*

Corollary 1 *Under the hypotheses of Theorem 2, any function* $f \in L^2(\mathbb{R}^n)$ *admits the tight wavelet frame expansion*

$$f = \sum_{\substack{\ell\in\mathbb{L}\\ j\in\mathbb{Z}\\ k\in\mathbb{Z}^n}} \langle f, \psi^\ell_{jk}\rangle\, \psi^\ell_{jk}. \tag{8}$$

By using the localization property of the frame wavelet in the Fourier domain, one can study the directions of growth of $\widehat{f}(\xi)$ by looking at the size of the frame coefficients

$$\langle f, \psi^\ell_{jk}\rangle = (2\pi)^{-n}\langle \widehat{f}, \widehat{\psi}^\ell_{jk}\rangle, \tag{9}$$

where the Fourier transform of f is defined by

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi}\, f(x)\, dx$$

and the inverse Fourier transform of g is defined by

$$\mathcal{F}^{-1}[g](x) := (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi}\, g(\xi)\, d\xi.$$

Moreover, by using the localization property of the frame wavelets in x-space, one can localize the singular support of $f(x)$ by varying ℓ, j and k in (9).

4 Frame Multiresolution Analysis

The notion of frame multiresolution analysis was introduced by Benedetto and Li [12]. Let us recall that an *orthonormal multiresolution analysis* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^n)$ satisfying

(i) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$;

(ii) $f(\cdot) \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$, for all $j \in \mathbb{Z}$;

(iii) $\bigcap_{j\in\mathbf{Z}} V_j = \{0\}$;

(iv) $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbb{R}^n)$;

(v) There exists a function $\phi \in V_0$ such that $\{\phi(\cdot - k)\}_{k\in\mathbf{Z}^n}$ is an orthonormal basis for V_0.

The function $\phi \in L^2(\mathbb{R}^n)$ whose existence is asserted in condition (v) is called an *orthonormal scaling function* for the given orthonormal multiresolution analysis.

A *frame multiresolution analysis* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ of $L^2(\mathbb{R}^n)$ satisfying (i), (ii), (iii), (iv) and

(v-1) There exists a function $\phi \in V_0$ such that $\{\phi(\cdot - k)\}_{k\in\mathbf{Z}^n}$ is a frame for V_0.

The function $\phi \in L^2(\mathbb{R}^n)$ whose existence is asserted in condition (v-1) is called a *frame scaling function* for the given frame multiresolution analysis.

Let D be a finite index set. An *orthonormal multiwavelet multiresolution analysis* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ of $L^2(\mathbb{R}^n)$ satisfying (i), (ii), (iii), (iv) and

(v-2) There exists a system of functions $\{\phi_\delta\}_{\delta\in D} \subset V_0$ such that $\{\phi_\delta(\cdot - k)\}_{\delta\in D,\ k\in\mathbf{Z}^n}$ is an orthonormal basis for V_0.

The set of functions $\{\phi_\delta\}_{\delta\in D}$ whose existence is asserted in condition (v-2) is called a set of *orthonormal multiscaling functions.*

A *frame multiwavelet multiresolution analysis* consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ of $L^2(\mathbb{R}^n)$ satisfying (i), (ii), (iii), (iv) and

(v-3) There exists a system of functions $\{\phi_\delta\}_{\delta\in D} \subset V_0$ such that $\{\phi_\delta(\cdot - k)\}_{\delta\in D,\ k\in\mathbf{Z}^n}$ is a frame for V_0.

The set of functions $\{\phi_\delta\}_{\delta\in D}$ whose existence is asserted in condition (v-3) is called a set of *frame multiscaling functions.*

5 Microlocal Analysis

Our approach to microlocal analysis is based on the theory of hyperfunctions ([13], [14], [15]). Hyperfunctions are powerful tools in several applications; for example, vortex sheets in two-dimensional fluid dynamics are a realization of hyperfunctions of one variable. Microlocal analysis deals with the direction along which a hyperfunction can be extended analytically. In other words, it decomposes the "singularity" into microlocal directions. Microlocal analysis plays an important role in the theory of hyperfunctions, partial differential operators, and other areas. In this theory, for example, one can consider the product of hyperfunctions and discuss the partial regularity of hyperfunctions with respect to any independent variable.

Here, we give only a rough sketch. A more complete treatment of microlocal filtering can be found in [1] (see also [4]). The important point is to find directions in which a hyperfunction can be continued analytically. Let $\Omega \subset \mathbb{R}^n$ be an open set, and $\Gamma \subset \mathbb{R}^n$ be a convex open cone with vertex at 0. From now on, every cone is assumed to have vertex at 0. The set $\Omega + i\Gamma \subset \mathbb{C}^n$ is called a *wedge.* An *infinitesimal wedge* $\Omega + i\Gamma 0$ is an open set $U \subset \Omega + i\Gamma$ which approaches asymptotically to Γ as the imaginary part of U tends to 0 (see Figure 1).

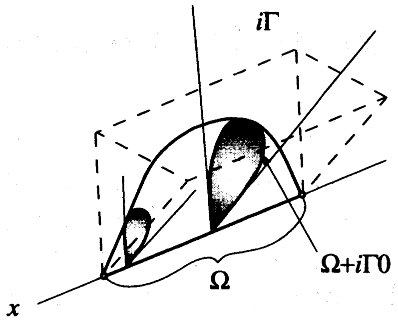

Figure 1: An infinitesimal wedge $\Omega + i\Gamma 0$.

A *hyperfunction* $f(x)$ can be defined as a sum

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0), \qquad x \in \Omega,$$

of formal boundary values

$$F_j(x + i\Gamma_j 0) = \lim_{\substack{y \to 0 \\ x+iy \in \Omega + i\Gamma_j 0}} F_j(x + iy)$$

of holomorphic functions $F_j(z)$ in the infinitesimal wedges $\Omega + i\Gamma_j 0$.

A hyperfunction is said to be *micro-analytic* at $x_0 \in \mathbb{R}^n$ in the direction $\xi_0 \in \mathbb{S}^{n-1}$ or, in short, at (x_0, ξ_0), if there exists a neighborhood Ω of x_0 and holomorphic functions F_j in infinitesimal wedges $\Omega + i\Gamma_j 0$ such that $f = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0)$ and

$$\Gamma_j \cap \{y \in \mathbb{R}^n \ : \ y \cdot \xi_0 < 0\} \neq \emptyset$$

for all j.

A simple aspect of the relation between micro-analyticity and the Fourier transform is given as follows.

Lemma 2 *Let $\Gamma \subset \mathbb{R}^n$ be a closed cone and let $x_0 \in \mathbb{R}^n$. For a tempered distribution f, if there exists a tempered distribution g such that* $\operatorname{supp}\widehat{g} \subset \Gamma$ *and $f - g$ is analytic in a neighborhood of x_0, then f is micro-analytic at (x_0, ξ) for every $\xi \in \Gamma^c \cap \mathbb{S}^{n-1}$, where Γ^c denotes the complement of Γ.*

We shall construct orthonormal multiwavelet bases or tight frames which enable us to obtain information on the microlocal content of signals or functions. Since this separation of microlocal contents can be considered as a filtering operation, we call it *microlocal filtering.*

6 One-dimensional Orthonormal Microlocal Filtering

Our aim is to construct wavelets $\{\phi_\delta\}_{\delta \in D}$ having good localization both in the base space $\mathbb{R}$ and in the direction space $\mathbb{S}^0 = \{\pm 1\}$ within the limits of the uncertainty principle. Here "good localization" at a point $(x_0, \xi_0) \in \mathbb{R} \times \mathbb{S}^0$ means that ϕ_δ is essentially concentrated in a neighborhood of a point $x_0 \in \mathbb{R}$ and $\widehat{\phi_\delta}$ is essentially concentrated in a conic neighborhood of a point $\xi_0 \in \mathbb{S}^0$. We call this "*good microlocalization.*"

Define the classical Hardy spaces $H^2(\mathbb{R}_\pm)$ by

$$H^2(\mathbb{R}_\pm) = \{f \in L^2(\mathbb{R}) : \ \widehat{f}(\xi) = 0 \quad \text{a.e. } \xi \leq (\geq)\, 0\}.$$

Each function of $H^2(\mathbb{R}_\pm)$ has good localization in the direction space $\mathbb{S}^0 = \{\pm 1\}$. Hence if we construct wavelets in $H^2(\mathbb{R}_\pm)$ with good localization in the base space, those wavelets have good microlocalization.

In these cases, an orthonormal wavelet function ψ_+ and an orthonormal scaling function ϕ_+ for orthonormal wavelets of $H^2(\mathbb{R}_+)$ are defined by

$$\widehat{\psi_+} = \chi_{[2\pi, 4\pi]}, \qquad \widehat{\phi_+} = \chi_{[0, 2\pi]}.$$

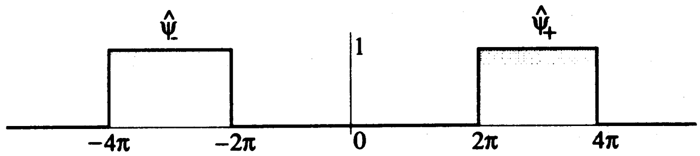

Figure 2: The Fourier transform of the orthonormal wavelet functions ψ_+ and ψ_-.

From the two-scale relation

$$2\widehat{\phi}_+(2\xi) = m_0(\xi)\,\widehat{\phi}_+(\xi)$$

it is found that the corresponding lowpass filter is

$$m_0(\xi) = 2\chi_{[0,\pi]}(\xi) = 2\widehat{\phi}_+(2\xi)$$

on $[0, 2\pi)$, and extended 2π-periodically to the line. From the two-scale relation

$$2\widehat{\psi}_+(2\xi) = e^{i\xi}\,\overline{m_0(\xi+\pi)}\,\widehat{\phi}_+(\xi) = m_1(\xi)\,\widehat{\phi}_+(\xi)$$

it is found that the corresponding highpass filter is

$$m_1(\xi) = e^{i\xi}\,\overline{m_0(\xi+\pi)} = 2\widehat{\psi}_+(2\xi)$$

on $[0, 2\pi)$, and extended 2π-periodically to the line.

By the same argument, we have an orthonormal wavelet function ψ_- and an orthonormal scaling function ϕ_- for orthonormal wavelets of $H^2(\mathbb{R}_-)$. Since

$$L^2(\mathbb{R}) = H^2(\mathbb{R}_+) \oplus H^2(\mathbb{R}_-),$$

$\{\psi_+, \psi_-\}$ and $\{\phi_+, \phi_-\}$ can be regarded as sets of orthonormal multiwavelet functions and orthonormal multiscaling functions, respectively, of $L^2(\mathbb{R})$. This decomposition of $L^2(\mathbb{R})$ into the orthogonal sum of the classical Hardy spaces $H^2(\mathbb{R}_\pm)$ corresponds to the intuitive definition of hyperfunction:

$$f(x) = F_+(x+i0) - F_-(x-i0),$$

where $F_+(z)$ and $F_-(z)$ are holomorphic in the upper half plane and in the lower half plane, respectively.

Auscher [16] essentially proved that there is no smooth orthonormal wavelet ψ in the classical Hardy space $H^2(\mathbb{R}_+)$, that is, there is no orthonormal wavelet ψ whose Fourier transform $\widehat{\psi}$ is continuous on $\mathbb{R}$ and satisfies the regularity condition:

$$\exists\alpha > 0; \qquad |\widehat{\psi}(\xi)| = O\big((1+|\xi|)^{-\alpha-1/2}\big) \quad \text{at } \infty.$$

The decay of a function at infinity in x space corresponds to the smoothness of its Fourier transform in ξ space. Hence the non-existence of smooth wavelets implies that it is impossible to have any smooth orthonormal wavelet having good microlocalization. Thus our aim is to construct smooth tight frame wavelets with good microlocalization properties.

7 Multi-dimensional Orthonormal Microlocal Filtering

The following notation will be used.

- $\eta = (\eta_1, \ldots, \eta_n) \in H := \{\pm 1\}^n$.
- $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E := \{0,1\}^n \setminus \{0\}, \quad j \in \mathbb{Z}_+$.
- $Q_\eta := \prod_{k=1}^n [0, \eta_k], \quad \varepsilon.{*}\eta := (\varepsilon_1\eta_1, \ldots, \varepsilon_n\eta_n)$.
- $\mathcal{Q}_{j,\varepsilon,\eta} := \Big\{\prod_{k=1}^n [\eta_k(\ell_k - 1), \eta_k\ell_k] + 2^j(\varepsilon.{*}\eta) :$ $1 \le \ell_1, \ldots, \ell_n \le 2^j, \ell_1, \ldots, \ell_n \in \mathbb{N}\Big\}$.
- $\mathbb{Z}_+^{E\times H}$ is the set of all functions from $E \times H$ to $\mathbb{Z}_+$.

Theorem 3 *Fix $j \in \mathbb{Z}_+$, $\varepsilon \in E$, $\eta \in H$. For a cube $Q \in \mathcal{Q}_{j,\varepsilon,\eta}$, define ψ_Q by*

$$\widehat{\psi}_Q = \chi_{2\pi Q},$$

where $\chi_{2\pi Q}$ is the characteristic function of the cube $2\pi Q$. For $\rho \in \mathbb{Z}_+^{E\times H}$, let

$$\mathcal{Q}_\rho := \bigcup_{(\varepsilon,\eta)\in E\times H} \mathcal{Q}_{\rho(\varepsilon,\eta),\varepsilon,\eta}.$$

Then $\Psi := \{\psi_Q\}_{Q\in\mathcal{Q}_\rho}$ is a set of orthonormal wavelets. Define ϕ_η by

$$\widehat{\phi}_\eta := \chi_{2\pi Q_\eta}.$$

Then $\{\phi_\eta\}_{\eta\in H}$ is a set of frame scaling functions for these wavelets.
In particular, when $\rho(\varepsilon, \eta)$ is constant, Ψ is a set of multiwavelets.

Figure 3 illustrates the 2-D multiwavelets constructed in Theorem 3. Multiwavelets are masks in Fourier space — they are characteristic functions of cubes $2\pi Q$. The left part of Fig. 3 shows 12 multiwavelet functions. For finer resolution in Fourier space, we need a greater number of multiwavelets. The right part of Fig. 3 shows 27 multiwavelet functions.

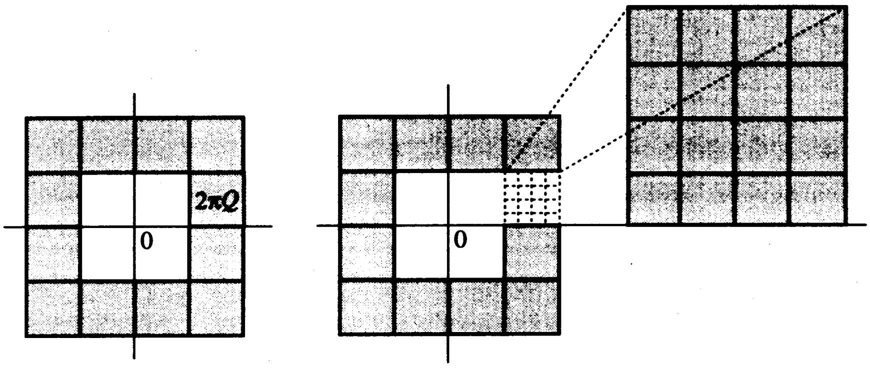

Figure 3: 2-D orthonormal multiwavelet functions in Fourier space.

8 Multi-dimensional Frame Microlocal Filtering

Smooth tight multiwavelet frames are obtained by convolving characteristic functions of cubes πQ so that the support of the smoothed functions have support inside cubes $2\pi Q$. This is achieved by considering the next inside annulus of cubes πQ in the left part of Figure 3.

Let $\vartheta(t)$ be a $C_0^\infty(\mathbb{R})$-function of one variable satisfying

$$\vartheta(t) \geq 0, \qquad \vartheta(t) = \vartheta(-t), \qquad \int_{\mathbb{R}} \vartheta(t)\, dt = 1, \qquad \vartheta(t) = \begin{cases} 1, & |t| \leq \frac{1}{3}; \\ 0, & |t| \geq \frac{2}{3}. \end{cases}$$

For $\alpha > 0$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, let

$$\vartheta_\alpha(\xi) = \frac{1}{\alpha^n} \prod_{j=1}^{n} \vartheta\Big(\frac{\xi_j}{\alpha}\Big).$$

We have the following theorem.

Theorem 4 *Fix $j \in \mathbb{Z}_+$, $\varepsilon \in E$, $\eta \in H$, and $\alpha \in (0, 1/2)$. Define*

$$\lambda_Q(\xi) := (\vartheta_\alpha * \chi_{\pi Q})(\xi) = \int_{\mathbb{R}^n} \vartheta_\alpha(\xi - \zeta)\, \chi_{\pi Q}(\zeta)\, d\zeta, \qquad Q \in \mathcal{Q}_{j,\varepsilon,\eta},$$

where $\chi_{\pi Q}$ is the characteristic function of the cube πQ. For $\rho \in \mathbb{Z}_+^{E\times H}$, let

$$\tau_\rho(\xi) := \sum_{j\in\mathbb{Z}, Q\in\mathcal{Q}_\rho} |\lambda_Q(2^j\xi)|^2,$$

and, for $Q \in \mathcal{Q}_\rho$, define $\psi_Q(x)$ by

$$\widehat{\psi}_Q(\xi) := \tau_\rho(\xi)^{-1/2}\,\lambda_Q(\xi).$$

Then $\Psi := \{\psi_Q\}_{Q\in\mathcal{Q}_\rho}$ is a set of tight frame wavelets.

Theorem 4 follows from Theorem 2.

9 Numerical Restoration of Images

In this section, we apply the above theory to the restoration of finite images represented by matrices. Since the Fourier transform of a finite region gives rise to oscillations of the type of cardinal sine, care must be taken in the restoration process.

The restoration process involves the following steps.

- The figure A to be restored is Fourier transformed into B.
- B is filtered by multiplication with a tapered characteristic function with support far from the origin and at right angle with the singularity to be localized. This produces C.
- In view of the Plancherel theorem, the wavelet coefficients of C, in (9),

$$\langle \widehat{f}, \widehat{\psi}^{\ell}_{jk}\rangle = (2\pi)^2\langle f, \psi^{\ell}_{jk}\rangle,$$

  are constructed in the Fourier domain and used in the x domain, to produce D which is the wavelet frame expansion (8) of Corollary 1.
- The extra width of D, caused by the side lobes in the support of ψ^{ℓ}_{jk}, is narrowed to eliminate oscillations due the cardinal sine effect when transforming functions with finite support.
- A tuned multiple of D is subtracted from A to restore the original image.

In Figure 4, the scarred woman image is restored. One notices in the top right part of the figure the wide width of the negative of the Fourier transform of the one-bit-wide short scar. The frame expansion of the inverse Fourier transform of the top right part produced a five-bit-wide segment. The width of this segment was reduced to one bit shown as a negative in the bottom left part of the figure. A multiple of the bottom left part of the figure, as a positive, was subtracted from the top left part to produce the restored woman figure shown in the bottom right part. In this case, only one frame wavelet was used as highpass filter in the top right part of the figure in the Fourier domain. Using a second filter in the lower left part of the Fourier domain does not seem to modify the final result.

In Figure 5, the boy image with a diagonal line is restored. One notices in the top right part of the figure the narrow width of the negative of the Fourier transform of the one-bit-wide long diagonal line. The frame expansion of the inverse Fourier transform of the top right part produced an eight-bit-wide segment. The width of this segment was reduced to one bit. Moreover, fine tuning required that the fourth root of this segment be taken. The result is shown as a negative in the bottom left part of the figure. A multiple of the bottom left part of the figure, as a positive, was subtracted from the top left part to produce the restored boy figure shown in the bottom right part. In this case, two frame wavelets were used as highpass filters in the top right and bottom left parts of the figure in the Fourier domain. Using only one filter in the upper right or lower left part in the Fourier domain does not seem to modify the final result.

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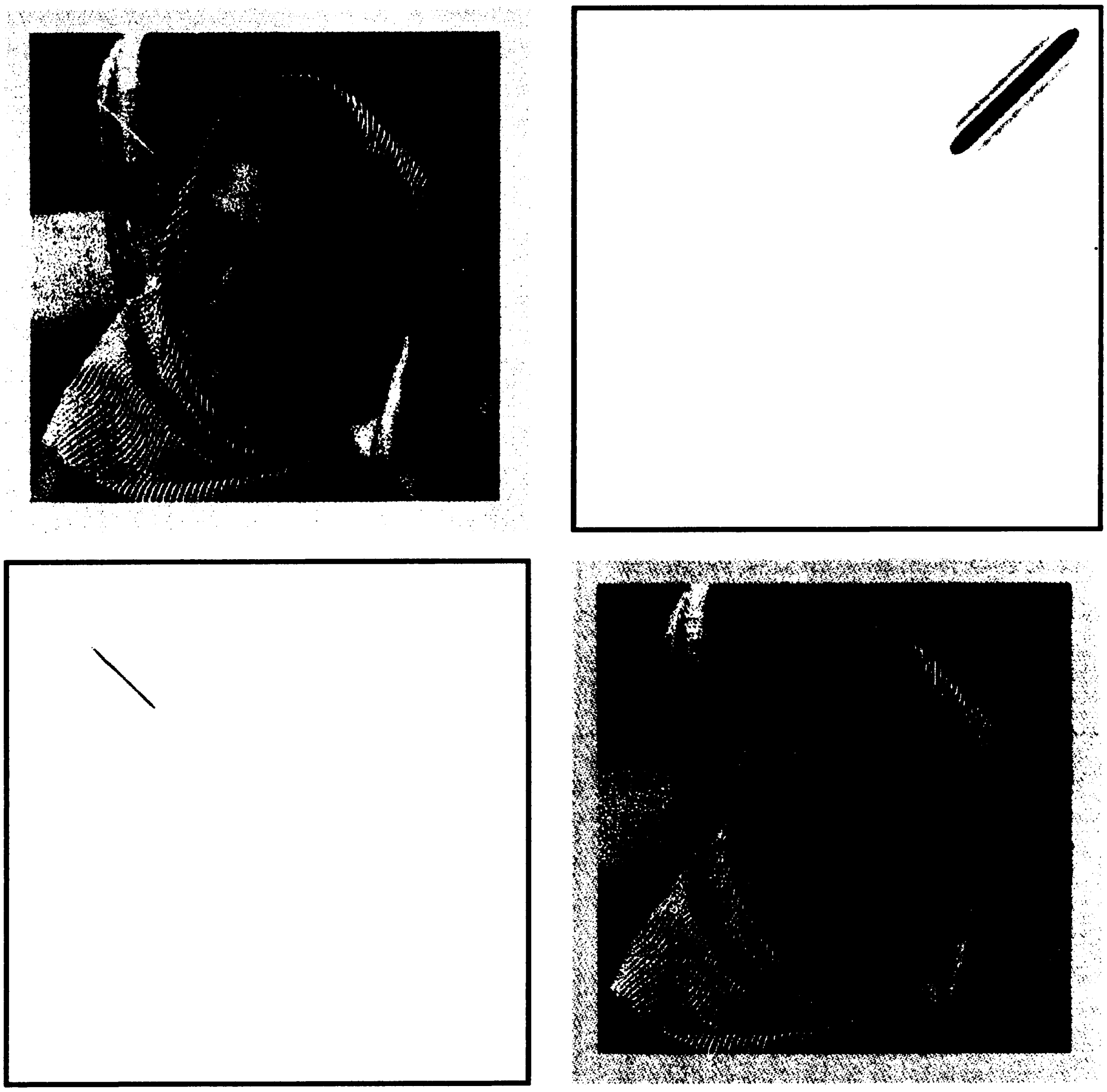

Figure 4: Top left: positive scarred woman figure. Top right: framed negative filtered Fourier transform of top left figure. Bottom left: framed negative frame expansion of the inverse Fourier transform of top right figure. Bottom right: positive restored woman figure.

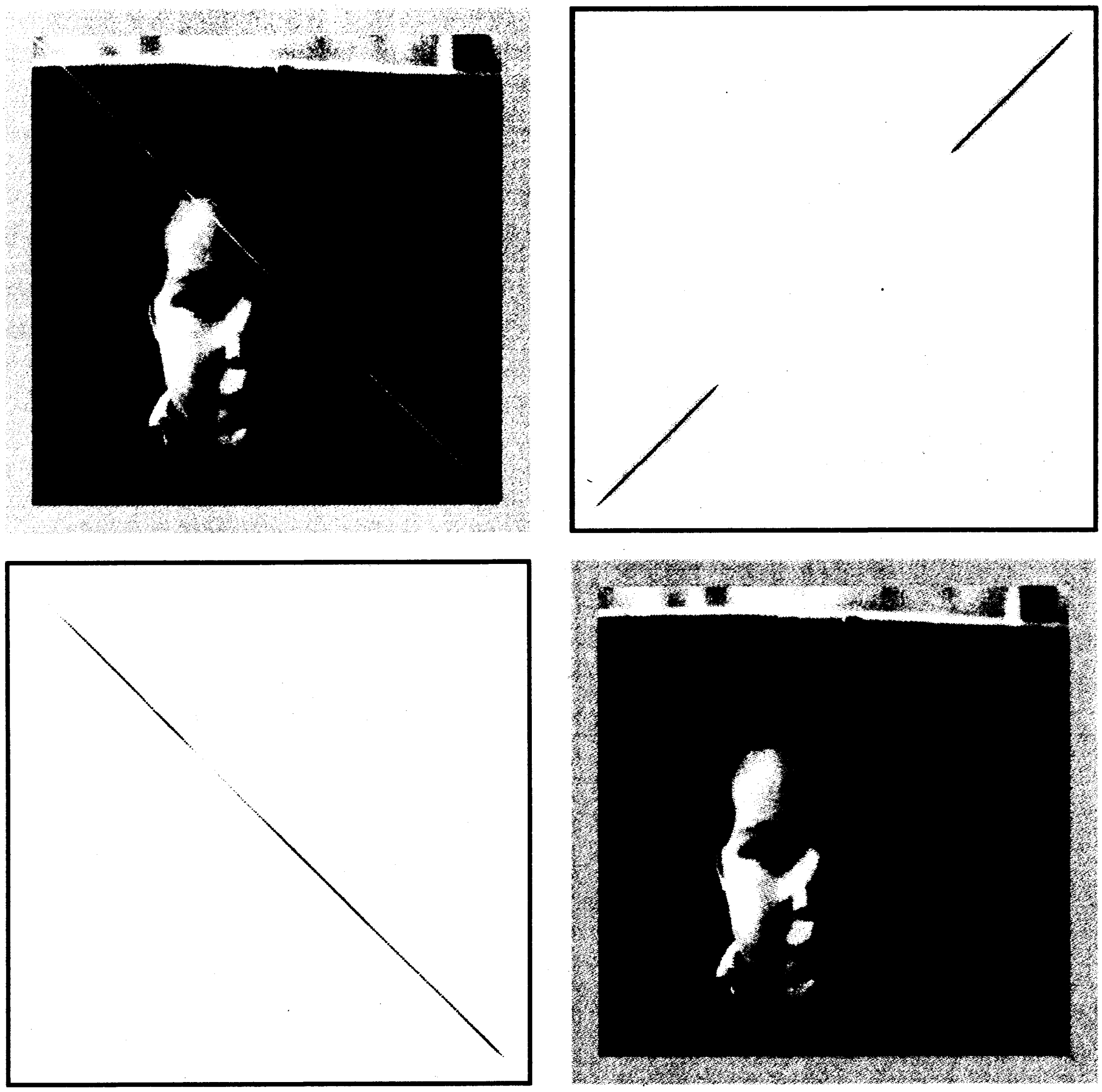

Figure 5: Top left: positive boy figure with diagonal line. Top right: framed negative filtered Fourier transform of top left figure. Bottom left: framed negative frame expansion of the inverse Fourier transform of top right figure. Bottom right: positive restored boy figure.

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