数理解析研究所講究録 1286 巻 2002 年 4-8

テスト関数の空間の完備性について

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The space $\mathcal{D}(\mathbf{R})$ of all infinitely differentiable functions $f : \mathbf{R} \to \mathbf{R}$ with compact support together with a locally convex structure defined by the seminorms

$$p_{\alpha,\beta}(f) := \sup_{n} \max_{l \leq \beta(n)} \sup_{|x| \geq n} 2^{\alpha(n)} |f^{(l)}(x)| \quad (\alpha, \beta \in \mathbf{N} \to \mathbf{N})$$

is an important example of a locally convex space. Classically the space $\mathcal{D}(\mathbf{R})$ - the space of test functions - is complete, but it is difficult to show that it is complete within the framework of Bishop's constructive mathematics. This leads us a difficulty in developing the theory of distributions in Bishop's constructive mathematics; see [1, Appendix A] and [2, Chapter 7, Notes].

Our aim of the paper is to find a principle which is necessary and sufficient to establish the completeness of $\mathcal{D}(\mathbf{R})$. Although it is formulated in the setting of informal Bishop-style constructive mathematics, the proofs could easily be formalized relative to a system based on intuitionistic finite-type arithmetics $\mathbf{HA}^{\omega}$ [8, Chapter 1], [9, Chapter 9]; see also [5].

A subset A of $\mathbf{N}$ is said to be *pseudobounded* if for each sequence $\{a_n\}_n$ in A,

$$\lim_{n \to \infty} \frac{a_n}{n} = 0.$$

A bounded subset of $\mathbf{N}$ is pseudobounded. The converse holds in classical mathematics, intuitionistic mathematics and constructive recursive mathematics of Markov's school; see [6]. However, the following principle is independent of Heyting arithmetic [4].

BD-N: Every countable pseudobounded subset of $\mathbf{N}$ is bounded.

BD-N has been proved to be equivalent to the following theorems [6, 7, 4]; Banach's inverse mapping theorem; the open mapping theorem; the closed graph theorem; the Banach-Steinhaus theorem; the Hellinger-Toeplitz theorem; every sequentially continuous mapping of a separable metric space into a metric space is pointwise continuous; every uniformly sequentially continuous mapping of a separable metric space into a metric space is uniformly continuous. In this paper, we will show that it is also equivalent to the completeness of $\mathcal{D}(\mathbf{R})$.

In the rest of the paper, we assume familiarity with the constructive calculus, as found in [1, Chapter 2], [3, Appendix], [2, Chapter 2], or [9, Chapter 6].

Before showing our main result, we shall show that the test function

$$\hat{\varphi}(x) := \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

is well-defined in Bishop's constructive mathematics.

A function $f : (a, b) \to \mathbf{R}$ is said to *vanish at end points* if for each k there exists m such that for all $x \in (a, b)$,

$$x < a + 2^{-m} \vee b - 2^{-m} < x \Rightarrow |f(x)| < 2^{-k}.$$

Proposition 1 *Let $f : (a, b) \to \mathbf{R}$ be a function which vanishes at end points and is uniformly continuous on each compact subinterval of (a, b). Then there exists a uniformly continuous function $\hat{f} : \mathbf{R} \to \mathbf{R}$ such that $\hat{f} = f$ on (a, b) and $\hat{f} = 0$ on $(-\infty, a) \cup (b, \infty)$.*

A function f from a subset X of $\mathbf{R}$ into $\mathbf{R}$ is *uniformly differentiable* on X, with a derivative f', if for each k, there exists n such that for all $x, y \in X$,

$$|x - y| < 2^{-n} \Rightarrow |f'(x)(x - y) - (f(x) - f(y))| < 2^{-k}.$$

We shall use the familiar notation for iterated derivatives: $f^{(0)} := f$, $f^{(l+1)} := (f^{(l)})'$.

Let $f, f' : (a,b) \to \mathbf{R}$ be functions which vanish at end points, and suppose that f is uniformly differentiable on each compact subinterval of (a,b) with a derivative f'. Then by [3, A.1], f and f' are uniformly continuous on each compact subinterval of (a,b), and hence they have the uniformly continuous extensions $\hat{f}$ and $\widehat{f'}$.

Proposition 2 *Let $f, f' : (a,b) \to \mathbf{R}$ be functions which vanish at end points, and suppose that f is uniformly differentiable on each compact subinterval of (a,b) with a derivative f'. Then $\hat{f}$ is uniformly differentiable on $\mathbf{R}$ with a derivative $\widehat{f'}$.*

The function

$$\varphi(x) := \exp\left(-\frac{1}{1-x^2}\right)$$

from $(-1,1)$ to $\mathbf{R}$ is infinitely differentiable on each compact subinterval of $(-1,1)$, and whose l-th derivative is

$$\varphi^{(l)}(x) = \frac{P_l(x)}{(1-x^2)^{2l}} \exp\left(-\frac{1}{1-x^2}\right)$$

for some polynomial P_l. Since for each m and k there exists n such that

$$t > 2^n \Rightarrow \frac{t^m}{\exp(t)} < 2^{-k} \quad (t \in \mathbf{R}),$$

each $\varphi^{(l)}$ vanishes at end points. Hence $\hat{\varphi} = \widehat{\varphi^{(0)}}$ is infinitely differentiable on $\mathbf{R}$, and whose l-th derivative $\hat{\varphi}^{(l)}$ is $\widehat{\varphi^{(l)}}$.

We shall show our main result with the completeness of the space $\mathcal{K}(\mathbf{R})$, which is another important example of a locally convex space, of all uniformly continuous functions $f : \mathbf{R} \to \mathbf{R}$ with compact support together with the seminorms

$$q_\alpha(f) := \sup_n \sup_{|x| \geq n} 2^{\alpha(n)} |f(x)| \quad (\alpha \in \mathbf{N} \to \mathbf{N}).$$

Note that since differentiable functions on a compact interval are uniformly continuous on the interval, functions in $\mathcal{D}(\mathbf{R})$ belong to $\mathcal{K}(\mathbf{R})$.

Lemma 3 *A subset A of $\mathbf{N}$ is pseudobounded if and only if for each sequence $\{a_n\}$ in A, $a_n < n$ for all sufficiently large n.*

Theorem 4 *The following are equivalent.*

1. $\mathcal{K}(\mathbf{R})$ *is complete.*

2. $\mathcal{D}(\mathbf{R})$ *is complete.*

3. *BD*-**N**.

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