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Harmonic Relations between Green's Functions and Green's Matrices for Boundary Value Problems II

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1 Introduction

In previous papers [3], [4], we remarked that there is a harmonic relation between the Green functions $G(x,\xi)$ for

$$\begin{cases} -\frac{d}{dx}(p(x)\frac{du}{dx}) = f(x), & a < x < b \\ u(a) = u(b) = 0, & p(x) > 0 \text{ in } [a,b] \end{cases} \tag{1.1}$$

and the Green matrix $A_0^{-1} = (g_{ij})$ for the discretized system

$$\begin{cases} a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b, \ x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1}) & (1.2) \\ -\dfrac{p_{i+\frac{1}{2}}\frac{U_{i+1}-U_i}{h_{i+1}} - p_{i-\frac{1}{2}}\frac{U_i - U_{i-1}}{h_i}}{\frac{h_{i+1}+h_i}{2}} = f_i, \ h_i = x_i - x_{i-1}, \ i = 1,2,\cdots,n & (1.3) \\ U_0 = U_{n+1} = 0 \end{cases}$$

or

$$HA_0U = f$$

with

$$H = \begin{pmatrix} \frac{2}{h_1+h_2} & & \\ & \ddots & \\ & & \frac{2}{h_n+h_{n+1}} \end{pmatrix},$$

$$A_0 = \begin{pmatrix} a_1 + a_2 & -a_2 & & & \\ -a_2 & a_2 + a_3 & -a_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & -a_n & a_n + a_{n+1} \end{pmatrix}, \quad a_i = \frac{1}{h_i} p_{i-\frac{1}{2}} \tag{1.4}$$

$$U = (U_1, \cdots, U_n)^t, f = (f_1, \cdots, f_n)^t$$

In fact, we have

$$G(x_i, x_j) = \begin{cases} \left(\int_a^b \frac{ds}{p(s)}\right)^{-1} \int_a^x \frac{ds}{p(s)} \int_\xi^b \frac{ds}{p(s)} & (x \le \xi) \\ \\ \left(\int_a^b \frac{ds}{p(s)}\right)^{-1} \int_a^\xi \frac{ds}{p(s)} \int_x^b \frac{ds}{p(s)} & (x \ge \xi) \end{cases}$$

and

$$g_{ij} = \begin{cases} \left(\sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right)^{-1} \left(\sum_{k=1}^{i} \frac{h_k}{p_{k-\frac{1}{2}}}\right) \left(\sum_{k=j+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right) & i \le j \\ \\ \left(\sum_{k=1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right)^{-1} \left(\sum_{k=1}^{j} \frac{h_k}{p_{k-\frac{1}{2}}}\right) \left(\sum_{k=i+1}^{n+1} \frac{h_k}{p_{k-\frac{1}{2}}}\right) & i \ge j. \end{cases}$$

Hence,

$$G(x_i, x_j) = g_{ij} + O(h^2) \quad \forall i, j, \; h = \max_i h_i,$$

if $p \in C^{1,1}[a, b]$.

On the other hand, the finite element approximation $v_n = \sum_{i=1}^{n} \hat{U}_i \phi_i$ with piecewise linear polynomials is determined by solving

$$\sum_{j=1}^{n} \left(\int_a^b p(x)\phi_i'\phi_j' dx\right)\hat{U}_j = \int_a^b f(x)\phi_i(x)dx, \quad i = 1, 2, \cdots, n \tag{1.5}$$

with respect to $\hat{U}_j$, where $\phi_i, i = 1, 2, \cdots, n$ are piecewise linear polynomials satisfying $\phi_i(x_j) = \delta_{ij}$. The equations(1.5) can be written in the matrix-vector form

$$\hat{A}\hat{U} = \hat{f},$$

where $\hat{A} = \hat{A}_0$ is obtained by replacing a_i in (1.4) by

$$\hat{a}_i = \frac{1}{h_i}\rho_i, \qquad \rho_i = \frac{1}{h_i}\int_{x_{i-1}}^{x_i} p(x)dx,$$

$$\hat{f} = (\hat{f}_1, \cdots, \hat{f}_n), \qquad \hat{f}_i = \int_{x_{i-1}}^{x_{i+1}} f(x)\phi_i(x)dx.$$

Then it can also be shown that $\hat{A}^{-1} = (\hat{g}_{ij})$ satisfies

$$\hat{g}_{ij} = \begin{cases} \left(\sum_{k=1}^{n+1} \frac{h_k}{\rho_k}\right)^{-1} \left(\sum_{k=1}^{i} \frac{h_k}{\rho_k}\right) \left(\sum_{k=j+1}^{n+1} \frac{h_k}{\rho_k}\right) & i \leq j \\ \left(\sum_{k=1}^{n+1} \frac{h_k}{\rho_k}\right)^{-1} \left(\sum_{k=1}^{j} \frac{h_k}{\rho_k}\right) \left(\sum_{k=i+1}^{n+1} \frac{h_k}{\rho_k}\right) & i \geq j, \end{cases}$$

which indicates a similar harmony between the Green function $G(x, \xi)$ and the corresponding discrete Green function:

$$G(x_i, x_j) = \hat{g}_{ij} + O(h^2) \quad \forall i, j,$$

if $p \in C^{1,1}[a, b]$.

The purpose of this paper is to establish a similar relation for the Green function $G(x, \xi)$ for

$$Lu \equiv -\frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)\frac{du}{dx} + r(x)u = f(x), a < x < b \tag{1.6}$$

$$u(a) = u(b) = 0$$

and the discrete Green function $G_h(x_i, x_j)$ (Green matrix) for the discrétized system

$$\begin{cases} L_h U \equiv -\dfrac{p_{i+\frac{1}{2}} \frac{U_{i+1}-U_i}{h_{i+1}} - p_{i-\frac{1}{2}} \frac{U_i - U_{i-1}}{h_i}}{\frac{h_{i+1}+h_i}{2}} + q_i \dfrac{U_{i+1} - U_{i-1}}{h_{i+1} + h_i} + r_i U_i = f_i, \quad i = 1, 2, \cdots, n \\ U_0 = U_{n+1} = 0, \end{cases} \tag{1.7}$$

provided that $p(x) \in C^{3,1}$, $q(x), r(x) \in C^{1,1}[a, b]$, $p(x) > 0, r(x) \geq 0$ in $[a, b]$

2 Results

The discrete Green function $G_h(x_i, x_j)$ for the operator L_h is defined as the solution of the linear system

$$\begin{cases} L_h G_h(x_i, x_j) = \frac{2}{h_{j+1}+h_j}\delta_{ij}, & i, j = 1, 2, \cdots, n \\ G_h(x_i, x_j) = 0, i = 0, n+1, & 1 \leq j \leq n, \end{cases}$$

where δ_{ij} stands for the Kronecker symbol. This means that the $n \times n$ matrix$(G_h(x_i, x_j))$ is the inverse of the matrix $A = A_0 + Q + D$, where A_0 is defined by (1,3),

$$Q = \begin{pmatrix} 0 & & 0 & & \\ -\frac{q_2}{2} & \ddots & \frac{q_2}{2} & & \\ & & \ddots & & \\ & & & \ddots & \frac{q_{n-1}}{2} \\ & & -\frac{q_n}{2} & & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \frac{r_1(h_1+h_2)}{2} & & & & \\ & \frac{r_2(h_2+h_3)}{2} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \frac{r_n(h_n+h_{n+1})}{2} \end{pmatrix}$$

We first prove the following lemma:

Lemma 2.1. *Given positive integers N_a and N_b, we have*

$$\sum_{j=1}^{n} G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} = \begin{cases} O(h) & \text{if } i \le N_a \text{ or } i \ge n+1-N_b, \\ O(1) & \text{otherwise.} \end{cases}$$

Proof. Let $\phi(x) \in C^2[a,b]$ be the solution of the problem $Lu = 1$ in (a,b) and $u(a) = u(b) = 0$. Then we have

$$\sum_{j=1}^{n} G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} \le 2\phi(x_i) \quad \forall i$$

(cf.Matsunaga-Yamamoto[2]), which proves Lemma 2.1. □

Then we have the following result.

Theorem 2.2. *If $p \in C^{3,1}[a,b]$, $q(x), r(x) \in C^{1,1}[a,b]$, $p(x) > 0$, $r(x) \ge 0$ in $[a,b]$, then*

$$G_h(x_i, x_j) - G(x_i, x_j) = \begin{cases} O(h^3) & (i \le N_a \text{ or } i \ge n+1-N_b), \\ O(h^2) & (\text{otherwise}). \end{cases}$$

Proof. Let $\{V_i\}$ be any mesh function defined on $\mathrm{I} = \{x_0, x_1, \cdots, x_n, x_{n+1}\}$ such that $V_0 = V_{n+1} = 0$. Then it is easy to see

$$V_i = \sum_{j=1}^{n} G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} L_h V_j, \quad i = 1, 2, \cdots, n$$

Hence

$$G(x_i, x_j) = \sum_{k=1}^{n} G_h(x_i, x_k) \frac{h_k + h_{k+1}}{2} L_h G(x_k, x_j), \quad i, j = 1, 2, \cdots, n \tag{2.1}$$

Furthermore, a careful computation leads to

$$L_hG(x_k,x_j)=\begin{cases}\frac{2}{h_k+h_{k+1}}[(h_k^2-h_{k+1}^2)\phi_{kj}+\mathrm{O}(h_k^3+h_{k+1}^3)] & (k\neq j)\\ \frac{2}{h_j+h_{j+1}}[1+(h_{j+1}^2\phi_j^+-h_j^2\phi_j^-)+\mathrm{O}(h_j^3+h_{j+1}^3)] & (k=j),\end{cases}$$

where

$$\begin{aligned}\phi_{kj}&=\frac{1}{6}p_k\frac{\partial^3G(x_k,x_j)}{\partial x^3}+\frac{1}{4}(p_k'-q_k)\frac{\partial^2G(x_k,x_j)}{\partial x^2}+\frac{1}{8}p_k''\frac{\partial G(x_k,x_j)}{\partial x},\\ \phi_j^+&=\frac{1}{4}q_j\frac{\partial^2G(x_j+0,x_j)}{\partial x^2}-\frac{1}{8}p_j''\frac{\partial G(x_j+0,x_j)}{\partial x}-\frac{1}{4}p_j'\frac{\partial^2G(x_j+0,x_j)}{\partial x^2}\\ &\quad-\frac{1}{6}p_j\frac{\partial^3G(x_j+0,x_j)}{\partial x^3},\\ \phi_j^-&=\frac{1}{4}q_j\frac{\partial^2G(x_j-0,x_j)}{\partial x^2}-\frac{1}{8}p_j''\frac{\partial G(x_j-0,x_j)}{\partial x}-\frac{1}{4}p_j'\frac{\partial^2G(x_j-0,x_j)}{\partial x^2}\\ &\quad-\frac{1}{6}p_j\frac{\partial^3G(x_j-0,x_j)}{\partial x^3}.\end{aligned}$$

Substituting this relation into (2.1) yields

$$\begin{aligned}G(x_i,x_j)=&\sum_{\substack{k=1\\k\neq j}}^{n}G_h(x_i,x_k)\{(h_k^2-h_{k+1}^2)\phi_{kj}+O(h_{k+1}^3+h_k^3)\}\\ &+G_h(x_i,x_j)\{1+(\phi_j^+h_{j+1}^2-\phi_j^-h_j^2)+O(h_{j+1}^3+h_j^3)\}\end{aligned}$$

or

$$\begin{aligned}G(x_i,x_j)-G_h(x_i,x_j)=&\sum_{k\neq j}G_h(x_i,x_j)\{(h_k^2-h_{k+1}^2)\phi_{kj}+O(h_{k+1}^3+h_k^3)\}\\ &+G_h(x_i,x_j)\{(\phi_j^+h_{j+1}^2-\phi_j^-h_j^2)+O(h_{j+1}^3+h_j^3)\}\end{aligned}\tag{2.2}$$

Hence there exists a constant $C_1>0$ such that

$$|G(x_i,x_j)-G_h(x_i,x_j)|\leq C_1h\sum_{\substack{k\neq j\\k=1}}^{n}G_h(x_i,x_k)(h_k+h_{k+1})+O(h^2),$$

and, by Lemma 2.1, we have

$$G_h(x_i,x_j)=G(x_i,x_j)+O(h).\tag{2.3}$$

Substituting this into (2.2) we have

$$
\begin{aligned}
G(x_i,x_j) - G_h(x_i,x_j) &= \sum_{k\neq j} G(x_i,x_k)(h_k^2 - h_{k+1}^2)\phi_{kj} + G(x_i,x_j)(\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h^2) \\
&= h_1^2 G(x_i,x_1)\phi_{1j} + \sum_{k=1}^{j-2} h_{k+1}^2 [G(x_i,x_{k+1})\phi_{k+1j} - G(x_i,x_k)\phi_{kj}] \\
&\quad - h_j^2[G(x_i,x_{j-1})\phi_{j-1j} + G(x_i,x_j)\phi_j^-] \\
&\quad + h_{j+1}^2[G(x_i,x_j)\phi_j^+ + G(x_i,x_{j+1})\phi_{j+1j}] \\
&\quad + \sum_{k=j+1}^{n-1} h_{k+1}^2[G(x_i,x_{k+1})\phi_{k+1j} - G(x_i,x_k)\phi_{kj}] \\
&\quad - h_{n+1}^2[G(x_i,x_n)\phi_{nj}] + O(h^2) \\
&= O(h^2) \quad \text{(an improvement of (2.3))}
\end{aligned}
$$

since

$$
\begin{aligned}
&G(x_i,x_{k+1})\phi_{k+1j} - G(x_i,x_k)\phi_{kj} \\
&= [G(x_i,x_{k+1}) - G(x_i,x_k)]\phi_{k+1j} + G(x_i,x_k)[\phi_{k+1j} - \phi_{kj}] \\
&= \mathrm{O}(h_{k+1})\phi_{k+1j} + G(x_i,x_k)h_{k+1} = \mathrm{O}(h_{k+1}), \quad \textit{etc.}
\end{aligned}
$$

Replacing $O(h)$ in (2.3) by $O(h^2)$ and repeating similar argument as above, we obtain for $i \leq N_a$ or $i \geq n+1-N_b$

$$G(x_i,x_j) - G_h(x_i,x_j) = O(h^3).$$

This proves Theorem 2.2. □

We can apply Theorem 2.2 to derive the superconvergence of the Shortley-Weller approximation applied to the semilinear problem

$$
\begin{cases}
-\dfrac{d}{dx}(p(x)\dfrac{du}{dx}) + q(x)\dfrac{du}{dx} + f(x,u) = 0, \quad a < x < b & (2.4) \\
u(a) = \alpha, \quad u(b) = \beta & (2.5)
\end{cases}
$$

with any nodes (1.2):

Theorem 2.3. *In addition to the assumptions of Theorem 2.2, assume that f is continuous on $\mathcal{R} : a \leq x \leq b,\ -\infty < u < +\infty$. Furthermore, assume that f is continuously differentiable with respect to u on $\mathcal{R}$ and $f_u \geq 0$. Then the finite difference method*

$$
\begin{cases}
-\dfrac{p_{i+\frac{1}{2}}\frac{U_{i+1}-U_i}{h_{i+1}} - p_{i-\frac{1}{2}}\frac{U_i-U_{i-1}}{h_i}}{\frac{h_{i+1}+h_i}{2}} + q_i\dfrac{U_{i+1}-U_{i-1}}{h_{i+1}+h_i} + f(x_i,U_i) = 0, \quad i = 1,2,\cdots,n, & (2.6) \\
U_0 = \alpha, \quad U_{n+1} = \beta
\end{cases}
$$

for solving (2.4)–(2.5) is superconvergent with any nodes (1.2):

$$u_i - U_i = \begin{cases} O(h^3), & i \in \Gamma = \{1, 2, \cdots, N_a, n - N_b + 1, n - Nb + 2, \cdots, n\} \\ O(h^2), & i \notin \Gamma \end{cases}$$

as $h \to 0$, where N_a and N_b are arbitrary given positive integers.

Remark. If the boundary conditions (2.5) are replaced by

$$\alpha_1 u(a) + \alpha_2 u'(a) = \alpha \quad \text{and} \quad \beta_1 u(b) + \beta_2 u'(b) = \beta,$$

where $\alpha_2\beta_2 \neq 0$, $\alpha_1\alpha_2 \geq 0$ and $\beta_1\beta_2 \geq 0$, then it can be shown that the corresponding Shortley-Weller approximation (2.6) is quadratic convergent with any nodes (1.2):

$$u_i - U_i = O(h^2) \quad \forall i$$

as $h \to 0$. However, superconvergence can not be expected in general.

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