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Some formulae related to Hamilton–Jacobi equations of eikonal type

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Introduction

The purpose of this note is to present some formulae related to the Hamilton–Jacobi equation

$$H(x, Du) = 0 \tag{0.1}$$

in the framework of viscosity solutions theory.

We start in section 1 assuming the continuity of H in both variables as well as a convexity condition with respect to the second one. This is the most classical case, the formulae given here go back to the P.L. Lions' book . They are exploited to obtain existence results and to represent viscosity solutions of (0.1) coupled with suitable boundary conditions. Moreover under additional hypotheses they are interpreted from a metric point of view.

Two variations of the previous formulae are presented in sections 2 and 3 for the case where continuity but no convexity properties are assumed on H and where H is just measurable with respect to the state variable and verifies some convexity in the second one.

In the nonconvex case a suitable penalty term is introduced under the integral and a some game theory is used to get an inf–sup integral formula.

In the measurable case the formula of section 1 is recovered because of the convexity assumption but the set of admissible curves is modified imposing a transversality condition with respect to some sets of vanishing Lebesgue measure.

In the analysis a crucial role is played by the 0–sublevel sets of the Hamiltonian

$$Z(x) = \{P : H(x,p) \leq 0\} \quad \text{for } x \in \mathbb{R}^N \tag{0.2}$$

In the paper the term (sub–super) solution signifies viscosity (sub–super) solution.

The materials of sections 1,2 can be mainly found in [6],[7],[8] and those of section 3 in [2]. We refer the reader to the bibliographies of the above papers for further information.

1 A classical formula

Here we assume the Hamiltonian H to be continuous in both variables and verifying a quasiconvexity property with respect to the second one.

More precisely we require that the 0–sublevel sets of H defined in (0.2) are nonempty and convex. In addition we assume the coercivity condition

$$\liminf_{|p|\to+\infty} H(x,p) > 0 \quad \text{for any } x \tag{1.1}$$

and the relation

$$\partial Z(x) = \{H(x,p) = 0\} \tag{1.2}$$

These hypotheses guarantee that the set–valued map $x \mapsto Z(x)$ is continuous with respect to Hansdorff metric and compact convex valued.

Thanks to (1.2) the analysis of (0.1) as well as the representation formulae for solutions will depend only on Z and not on H. Therefore two Hamiltonians with the some 0–sublevel sets give rise to equations which cannot be distinguished from a viscosity solutions viewpoint.

We set for $(x,q) \in \mathbb{R}^N \times \mathbb{R}^N$

$$\sigma(x,q) = \max_{p\in Z(x)} qp$$

the support function of $Z(x)$ at q, and for any Lipschitz–continuous curve defined in [0, 1]

$$I(\xi) = \int_0^1 \sigma(\xi,\dot{\xi})dt \tag{1.3}$$

To choose interval [0, 1] as domain is just a matter of convenience being the value of the integral invariant for reparametrization of the curve.

For any couple of points x, y we denote by $\mathcal{A}_{y,x}$ the set of Lipschitz–continuous curves defined in [0, 1] and joining y to x. The quantity (1.3) is first used to get an existence result.

Proposition 1.1 *The following three conditions are equivalent*

i. *for any closed curve ξ , $I(\xi)$ is nonnegative*

ii. *equation (0.1) has a subsolution*

iii. *equation (0.1) has a solution*

i. is in fact equivalent to

$$L(y,x) := \inf\{I(\xi) : \xi \in \mathcal{A}_{y,x}\} > -\infty \tag{1.4}$$

for any y, x. Therefore for any fixed y_0

$$u := L(y_0, \cdot)$$

is a subsolution of (0.1) and also a solution in $\mathbb{R}^N \setminus \{y_0\}$. By the coercivity assumption (1.1) every subsolution is locally Lipschitz continuous. Actually in the present setting the notions of (viscosity) subsolution and locally Lipschitz–continuous a.e. subsolution are equivalent and they are characterized by the inequality

$$H(x,p) \leq 0$$

for any x and p in the (Clarke) generalized gradient of the function at x. The main contribution of viscosity solutions theory is in giving a supersolution condition. The (viscosity) solutions are particular Lipschitz–continuous a.e. solutions.

To show ii. one considers a sequence

$$u_n := L(y_n, \cdot)$$

with $|y_n| \to \infty$. It is is locally equiLipschitz–continuous being every u_n subsolution and equibounded up to addition of suitable constants. Then a subsequence converges locally uniformly to a function u which is a solution of (0.1) for the stability properties of viscosity solutions and because the "bad" points y_n have disappeared at infinity.

From the existence of a solution it is simple to recover the property i. of the previous statement using the Lipschitz–continuity of it.

If a subsolution of (0.1) exists, L can be used to give representation formulae for solutions of the equation coupled with suitable boundary conditions as well as to express admissibility conditions for boundary data.

If K is a compact subset of $\mathbb{R}^N$ and g a continuous function defined on ∂K then

$$w(x) := \min\{L(y,x) + g(y) : y \in \partial K\} \tag{1.5}$$

is a solution of (0.1) in $\mathbb{R}^N \setminus K$ and $u \leq g$ in ∂K, if in addition the condition

$$g(y_1) - g(y_2) \leq Ly_2, y_1) \quad \text{for } y_1, y_2 \in \partial K \tag{1.6}$$

holds true then $u = g$ in ∂K.

To obtain comparison principles and uniqueness results it must be assumed the existence of a strict subsolution.

In fact using a technique introduced by Ishii, it can be proved the relation

$$\max_{\Omega} u - v = \max_{\partial\Omega} u - v \tag{1.7}$$

for any open bounded set Ω, u subsolution of (0.1) in Ω upper semicontinuous in $\operatorname{cl}\Omega$, v supersolution lower semicontinuous in $\operatorname{cl}\Omega$. For any given continuous g defined on $\partial\Omega$ and verifying (1.6) in $\partial\Omega$, the function w defined as in (1.5) with $\partial\Omega$ in place of ∂K is the unique solution of (0.1) in Ω verifying

$$u = g \quad \text{in } \partial\Omega$$

It is also the maximal element in the class of Lipschitz–continuous a.e. subsolution of the equation in Ω verifying

$$u \leq g \quad \text{in } \partial\Omega$$

To give a certain geometric flavor to our construction, we proceed to assume that the null function is a strict subsolution of (0.1).

Actually we strengthen a bit this condition requiring

$$H(x,p) < 0 \quad \text{for any } x \,,\ |p| \le \frac{a}{|x|+b} \tag{1.8}$$

with a, b suitable positive constants.

In this case L defined as in (1.4) is a (nonsymmetric) distance in $\mathbb{R}^N$ wich is complete thanks to (1.8) and locally equivalent to the Euclidean metric.

More precisely it is a Finsler metric. It can be viewed as a generalization of a Riemannian one having convex compact sets containing 0 in its interior, as tangential balls instead of ellipsoids. Then $Z(x)$ is the closed cotangential ball of L at x.

We can state the following uniqueness result:

Proposition 1.2 *L is the unique complete metric on $\mathbb{R}^N$ such that $L(y_0, \cdot)$ is a solution of (0.1) in $\mathbb{R}^N \setminus \{y_0\}$*

This can be proved observing that the completeness is equivalent to

$$\lim_{|x| \to +\infty} L(y_0, x) = +\infty$$

for any y_0 and making use of the Kruzkov transform.

A converse construction is also possible, namely starting from a complete continuous Finsler metric it can be defined an Hamiltonian having the closed unit cotangential ball of it as 0–sublevel sets. The relation between the associated Hamilton–Jacobi equation and the metric is as in the statement of Proposition 1.2

2 First variation

In this section we remove any convexity condition on H and on Z. We still require continuity on the Hamiltonian as well as (1.2) and (1.8). In addition we need a locally uniform version of (1.1) to get the continuity of Z, namely we assume that for any compact set K there exists $R > 0$ verifying

$$\inf\{H(x,p) : \quad x \in K, \quad |p| > R\} > 0 \tag{2.1}$$

In this setting the set–valued maps $x \mapsto Z(x)$, $x \mapsto \partial Z(x)$ are continuous compact valued.

The distance L defined in (1.4) is not any more related to (0.1) as in the convex case. In fact since the support function of any set coincides with that of its convex hull, it is clear that L is related to a convexified form of the equation with Hamiltonian having $coZ(x)$ as 0–sublevel set, for any x.

We modify the formula of L to give a distance adapted to the non-convex setting. We will get an inf–sup integral formula and make use of some game–theory devices.

Roughly speaking the modifications can be described as follows:

starting point

$$\inf_{\xi} \int_0^1 \sigma(\xi(t), \dot{\xi}(t)dt = \inf_{\xi} \int_0^1 \sup_{\eta(t) \in Z(\xi(t))} \eta(t)\dot{\xi}(t)dt$$

1^{st} *step*

"commute" sup and $\int$ to get

$$\inf_{\xi} \sup_{\eta \in Z(\xi)} \int_0^1 \eta(t)\dot{\xi}(t)dt$$

2^{nd} *step*

introduce a penalty term to eliminate the constraint in the sup and obtain

$$\inf_{\xi} \sup_{\eta} \int_0^1 \eta(t)\dot{\xi}(t) - |\dot{\xi}(t)| d^*(\eta(t), Z(\xi(t))dt$$

where d^* represents the signed Euclidean distance, this term is indeed positive in the interior of $Z(\xi(t))$ and negative outside it.

3^{rd} *step*

relate η and $\dot{\xi}$ using the notion of nonanticipative strategy to get the final formula.

We denote, for any T, by B^T the space of measurable essentially bounded functions defined in $]0, T[$ with values in $\mathbb{R}^N$ and set for any couple y, x of points

$$B^T_{y,x} = \{\zeta \in B^T : y + \int_0^T \zeta dt = x\}$$

we write $\Gamma^T, \Gamma^T_{y,x}$ for the set on nonanticipative strategies from B^T to B^T and from B^T to $B^T_{y,x}$, respectively.

For $\eta \in B^T$, $\gamma \in \Gamma^T$, we denote by $\xi(\eta,\gamma,y,\cdot)$ the integral curve of $\gamma[\eta]$ which equals y at 0.

For $\eta \in B^T$, $\gamma \in \Gamma^T$ we finally define

$$\mathcal{I}_y^T(\eta,\gamma) = \int_0^T \gamma[\eta]\eta - |\gamma[\eta]| d^*(\eta, Z(\xi(\eta,\gamma,y,\cdot))dt$$

We set for any y , x

$$S(y,x) = \inf_{\gamma \in \Gamma_{y,x}} \sup_{\eta \in B^1} \mathcal{I}_y^1(\eta,\gamma) \tag{2.2}$$

S satisfies the following dynamical programming principle:

Proposition 2.1 *For any y, x, $T > 0$*

$$S(y,x) = \inf_{\gamma \in \Gamma^T} \sup_{\eta \in B^T} \left\{ \mathcal{I}_y^T(\eta,\gamma) + S((\xi(\eta,\gamma,y,T),x) \right\}$$

Exploiting it, one can prove that S is a complete distance on $\mathbb{R}^N$ locally equivalent to the Euclidean one with $S \leq L$. Moreover it can be related it to the equation (0.1) as follows:

Proposition 2.2 *For any y_0 $u = S(y_0,\cdot)$ is solution of (0.1) in $\mathbb{R}^N \setminus \{y_0\}$ and subsolution in $\mathbb{R}^N$.*

Observe that by the coercivity condition (2.1) every subsolution is a locally Lipschitz continuous a.e. subsolution. However in contrast to the convex case the notions of (viscosity) subsolution and locally Lipschitz continuous a.e. subsolution are not any more equivalent.

There are no uniqueness results unless Z is assumed to have values strictly star–shaped with respect to 0. If this is the case then the techniques of section 1 can be used to recover the Proposition 1.2 with S in place of L.

We now address the question of examining the relations between the distances S and L and the metric counterpart of the lack of convexity in H .

To do that we first need some definitions.

Given a general (possibly nonsymmetric) distance D on $\mathbb{R}^N$, we define for any continuous curve ξ defined in $[0,T]$ for a certain $T > 0$, the intrinsic length $l_D(\xi)$ as the total variation of the curve with respect to the distance. Namely:

$$l_D(\xi) = \sup \sum_i D(\xi(t_{i-1}), \xi(t_i))$$

where the supremum is taken with respect to all finite increasing sequences $\{t_1, \ldots, t_n\}$ with $t_1 = 0$ and $t_n = T$.

A metric can be then defined via the formula

$$D_l(y, x) = \inf\{l_D(\xi) : \xi \text{ continuous curve joining } y \text{ to } x\}$$

for $y, x \in \mathbb{R}^N$. It is apparent that

$$D \leq D_l$$

We term D a path metric, see [5], if equality holds in the previous formula. The passage from D to D_l can be viewed as a sort of metric convexification. If indeed a distance is complete then the property of being a path metric, convex in Menger's sense, and having any couple of points joined by a curve whose length realizes the distance, are equivalent; moreover $(D_l)_l = D_l$.

While L, being Finsler, is a path metric, this property is in general not true for S due to the lack of convexity. A natural issue is then to determine S_l and a candidate for it is of course L.

The equality $S_l = L$ should establish a connection between two different type of convexification, namely the convexification of $Z(x)$ leading to a quasiconvex Hamiltonian and the metric convexification of S leading to S_l.

We are able to prove the following:

Proposition 2.3 *If the values of Z are star–shaped with respect to 0 then $S_l = L$*

The star–shaped condition is exploited to show the existence for any x_0, $p_0 \in \text{int } Z(x_0)$ of a set–valued continuous convex compact valued map Z_0 verifying

$$Z_0(x) \subset Z(x) \quad \text{for any } x$$

$$p_0 \in Z_0(x_0)$$

along with other suitable conditions. From this and Proposition 2.2 it can be proved the relation

$$\lim_{\substack{y,x\to x_0\\ x\neq y}} \frac{S(x,y)}{\sigma(x_0, y-x)} = 1 \tag{2.3}$$

which implies the equality

$$l_S(\xi) = l_L(\xi) \tag{2.4}$$

for any Lipschitz-continuous curve ξ. Knowing that and using the local equivalence of S and the Euclidean metric, Proposition 2.3 is finally proved.

It is worth noticing that there is an inversion in the structure of the analysis in presence or lack of convexity.

In the convex case in fact the metric properties of the Finsler distance are used to relate it to (0.1), while in the nonconvex setting, from Proposition 2.2 it is obtained (2.3), (2.4) and eventually the property of S stated in Proposition 2.3.

3 Second variation

Here we treat the case where $H(x,p)$ is measurable in x for any p and continuous in x for a.e. p.

We assume the following coercivity condition:

for any compact subset K there is $R > 0$ with

$$\text{ess inf}\{H(x,p) : |p| > R, \quad x \in k\} > 0 \tag{3.1}$$

Moreover we require the convexity of $Z(x)$, (1.2), (1.8) for a. e. x.

The first problem is to adapt the notion of viscosity solution to equations with measurable Hamiltonianians. We suitably modify L and examine how the quantity given by the modified formula is related to the equation.

These relations will be taken as basis for the definition of solution. The difficulty is that L is the infimum of integrals on curves and these are negligible with respect to the Lebesgue measure and so difficult to handle under measurability assumptions on H.

In modifying L we try to recover the maximality property of $L(y_0, \cdot)$ between the locally Lipschitz–continuous a.e. subsolution of the equation vanishing at y_0.

It holds for any y_0 in the continuous case, but it is not valid any more even if H is upper semicontinuous. To see this, consider the equation

$$|Du| = f \quad \text{in } \mathbb{R}^2 \tag{3.2}$$

with f equal to $1/2$ on the line $x_2 = 0$ and to 1 on the complement. Then $u(x) = |x|$ is the maximal Lipschitz–continuous a. e. subsolution of (3.2) vanishing at 0, while the strict inequality

$$L(0, x) < |x|$$

holds for x with $|x_2|$ suitably small.

To justify the formula (3.7) given later, let us argue heuristically still assuming the upper semicontinuity of H in x and consequently the lower semicontinuity of the set–valued map Z.

We consider a locally Lipschitz continuous a.e. subsolution v of (0.1) vanishing at a certain point y_0 and $\xi \in \mathcal{A}_{y_0,x}$ for $x \in R^N$. It results

$$v(x) = \int_0^1 \frac{d}{dt} v(\xi(t))dt \tag{3.3}$$

and if t_0 is a differentiability point of ξ then

$$\frac{d}{dt} v(\xi(t_0)) = p(t_0)\dot{\xi}(t_0) \tag{3.4}$$

for a suitable element $p(t_0)$ of the generalized gradient of $\partial v(\xi(t_0))$.

If Z is continuous or even upper–semi continuous, it is apparent by the very definition of generalized gradient and the convex character of Z that

$$\partial v(x) \subset Z(x) \quad \text{for any } x$$

It yields by (3.3), (3.4)

$$v(x) \leq \int_0^1 \sigma(\xi, \dot{\xi})dt \tag{3.5}$$

and

$$v(x) \leq L(y_0, x)$$

The situation is different if just lower semicontinuouos are assumed in Z or, as in the general setting, measurability. However the inequality (3.5) is still valid if ξ is required to verify

$$\mathcal{L}^1(\{t : \xi(t) \in E\}) = 0 \tag{3.6}$$

with E denoting the null set (i.e. set of vanishing N–dimensional Lesbegue measure) where u is not differentiable and $\mathcal{L}^1$ the one–dimensional Lebesgue measure.

If (3.6) holds in fact

$$\frac{d}{dt}v(\xi(t)) = Dv(\xi(t))\dot{\xi}(t) \quad \text{for a.e. } t$$

and so the limiting procedure we have applied before is not any more necessary.

The relation given in (3.6) between a curve and a null set will play a central role in the analysis and it will be expressed saying that ξ is transversal to E, in symbols $\xi \pitchfork E$.

The previous discussion leads to the following formal inequality

$$v(x) \leq \sup_{|E|=0} \inf\{I(\xi) : \xi \in \mathcal{A}_{y_0,x}, \quad \xi \pitchfork E\} \tag{3.7}$$

The notion of transversality as well as sup–inf formulae similar to (3.7) have been introduced in [3], [4] in the framework of the study of the so–called Lip–manifold and of some class of metrics defined in it.

We proceed to show that the sup–inf formula in (3.7) is indeed the modification of L we were looking for.

Our assumptions imply that Z is measurable as a map from $\mathbb{R}^N$ with the Lebesgue measure to the space of compact subset of $\mathbb{R}^N$ endowed with the Hausdorff topology.

Therefore $x \mapsto \sigma(x, q)$ is measurable for any q. We give the conventional value $+\infty$ to $I(\xi)$ whenever $t \mapsto \sigma(\xi(t), \dot{\xi}(t))$ is not measurable. It can be proved using Fubini's theorem that for any null set E, the subset of $\mathcal{A}_{y,x}$ of curves transversal to E and suche that $I(\xi)$ is well defined and finite, is nonempty. This implies that the sup – inf is finite.

This quantity is invariant with respect to the following equivalence relation between Hamiltonians:

$$H \sim H' \quad \text{if } H(x,p) = H'(x,p)$$

for a certain null set E and $(x,p) \in (\mathbb{R}^N \setminus E) \times \mathbb{R}^N$. We set for any y, x

$$T(y,x) = \sup_{|E|=0} \inf\{I(\xi) : \xi \in \mathcal{A}_{y,x},\ \xi \pitchfork E\} \tag{3.8}$$

If we renounce to the invariance with respect to the previously defined equivalence relation and fix a representative H (or equivalently Z) then we can select a null set E_0 such that for any y, x

$$T(y,x) = \inf\{I(\xi) : \xi \in \mathcal{A}_{y,x},\ \xi \pitchfork E_0\} \tag{3.9}$$

Taking into account (3.8), we see that (3.9) still holds if we replace E_0 by any null set containing it. This remark is frequently exploited in the analysis.

From the previous heuristic discussion it can be understood that $T = L$ if H is lower semicontinuous with respect to x and so Z upper semicontinuous.

We proceed to examine the relation between T and the equation (0.1). It comes from our assumptions that $T(y_0, \cdot)$ is locally Lipschitz–continuous for any y_0.

The first step is the following:

Proposition 3.1 *For a.e.* x_0

$$\limsup_{x \to x_0} \frac{T(x, x_0)}{\sigma(x, x_0 - x)} \leq 1 \tag{3.10}$$

The previous inequality holds, more precisely, if x_0 is an approximate continuity point of Z, i.e. a point verifying for any $\varepsilon > 0$

$$\lim_{r \to 0} \frac{|\{x : d_H(Z(x), Z(x_0)) < \varepsilon\} \cap B(x_0, r)|}{|B(x_0, r)|} = 1$$

where d_H denotes the Hausdorff metric and $|\cdot|$ the Lebesgue measure.

We recall that the usual local characterization of measurability holds for set–valued maps and so the complement of the set of approximate continuity points has vanishing measure.

From the previous proposition we get

Proposition 3.2 *For any* y_0*,* $L(y_0, \cdot)$ *is a.e. subsolution of (0.1). In addition for any* x_0 *and* φ C^1*-supertangent to* $L(y_0, \cdot)$ *at* x_0

$$D\varphi(x_0) \in \overline{Z}(x_0) \tag{3.11}$$

where $\overline{Z}(x_0)$ is defined via the formula

$$\overline{Z}(x_0) = \cap \left\{ C : \lim_{r\to 0} \frac{|\{x : Z(x) \supset C\} \cap B(x_0, r)|}{|B(x_0, r)|} = 0 \right\}$$

In any point of approximate continuity x it result $\overline{Z}(x) = Z(x)$. For the equation (3.1) the condition (3.11) reads

$$D\varphi(x_0) \le \text{ap} \limsup_{x\to x_0} f(x) = \inf\{a : \lim_{r\to 0} \frac{|\{x : f(x) > a\} \cap B(x_0, r)|}{|B(x_0, r)|} = 0\}$$

An important difference with respect to the convex continuous case is that $w = T(y_0, \cdot)$ is not in general a.e. solution of the equation, however it can be proved that the equality $H(x, Du(x)) = 0$ holds in a dense subset of $\mathbb{R}^N$.

The following example shows this phenomenon (see [3]), it is based on a dense set such that the complement has positive measure.

Let a_n be a sequence made up by all rational numbers, we set

$$A = \{a \in \mathbb{R} : |a - a_n| < \frac{1}{2^n} \text{ for some } n \in \mathbb{N}\}$$

$$B = (\mathbb{R} \times A) \cup (A \times \mathbb{R})$$

Consider the equation (3.2) in $\mathbb{R}^2$ with f equal to $1/2$ in B and 1 on the complement. Since B is dense in $\mathbb{R}^2$, it results for any fixed $y_0 = (y_0^1, y_0^2)$ and $x = (x_1, x_2)$

$$w(x) = L(y_0, x) = \frac{1}{2}|x_1 - y_1^0| + \frac{1}{2}|x_2 - y_2^0|$$

and so

$$|Dw(x)| < 1 = f(x) \quad \text{for any } x \in \mathbb{R}^2 \setminus B$$

The following proposition specifies the supersolution property verified by w.

Proposition 3.3 *For any $x_0 \neq y_0$ and any Lipschitz-continuous function ψ subtangent to w at x_0*

$$\text{ess} \limsup_{x\to x_0} \gamma(x, D\psi(x)) \ge 1 \tag{3.12}$$

In the statement γ is the gauge function defined for any x , p by

$$\gamma(x,p) = \inf\{\lambda > 0 : \frac{p}{\lambda} \in Z(x)\}$$

The equations (0.1) and $\gamma(x, Du) - 1 = 0$ are equivalent in the sense specified in section 1.

The essential lim sup is given by

$$\operatorname{ess}\limsup_{x\to x_0} = \lim_{r\to 0}\{\operatorname{ess}\sup_{B(x_0,r)} g\}$$

This notion has been used in the definition of viscosity solutions for second order measurable equations, see [1].

Proposition 3.3 can be equivalently stated requiring the non existence of locally Lipschitz continuous subtangents φ at x_0 verifying

$$H(x, D\varphi(x)) \leq -\varepsilon$$

for a.e. x in a certain neighborhood of x_0 and a certain $\varepsilon > 0$.

From the properties of w proved in Propositions 3.2, 3.3 we derive the definition of solution of (0.1) for H measurable.

We say that a continuous function u is a solution if (3.11) is verified for any C^1 supertangent φ and (3.12) holds for any Lipschitz–continuous subtangent ψ. The lack of symmetry in this definition is crucial for proving comparison and uniqueness results.

Note that the subsolution condition implies the local Lipschitz–continuity of the subsolutions and it is actually equivalent, as in the case of continuous H, to require such a regularity property and the inequality $H(x, Du) \leq 0$ a.e.

The formulae and the comparison results established in section 1 can be recovered here with T in place of L using minor modifications of the usual techniques.

It can be proved that for any y_0 $w = T(y_0, \cdot)$ is the maximal locally Lipschitz continuous subsolution of the equation vanishing at y_0.

The relation (1.7) is still valid for any bounded set Ω, u subsolution and v supersolution.

Formula (1.5) with T and Ω in place of L and K, respectively, represents the unique solution of (0.1) in Ω equaling g on $\partial\Omega$ if the compatibility condition (1.6) holds with T replacing L.

Finally Proposition 1.2 holds true for T. The metric T is not any more Finsler as in the continuous case, however it can be shown that it is a complete path metric locally equivalent to the Euclidean one.

Conversely if D is a metric of this type, it can be indicated a procedure starting from the derivatives $\lim_{t \to 0} \frac{D(x,x+tq)}{t}$ for x , $q \in \mathbb{R}^N$ which leads to an Hamilton–Jacobi equation related to D as (0.1) and L in Proposition 1.2.

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