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On minimal vertical singular diffusion preventing overturning

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1 Introduction

This is a preliminary version of our work to a continuation of recent works [9], [10] of the second author.

In [9] we introduce the notion of proper viscosity solutions for a class of equations whose solutions may develop jump discontinuities. The class contains (scalar) conservation laws as special examples and a proper viscosity solution is essentially equivalent to an entropy solution for conservation laws. In [10] we propose to interpret this evolution as a result of the vertical singular diffusion. By a formal argument we have noted in [10] that there is a threshold value of the strength of the vertical diffusion such that it prevents overturning of the solution.

In this paper we give a rigorous proof for the fact that a solution develops overturning if the strength M of the vertical diffusion is smaller than the critical value by studying the Riemann problem for the Burgers equation:

$$u_t + uu_x = 0, \tag{1.1}$$

$$u(x,0) = (\mathrm{sgn} x)d/2. \tag{1.2}$$

If one views the graph of u as a level set of auxiliarly function $\psi(x,y,t)$, ψ must satisfy

$$\psi_t + y\psi_x = 0. \tag{1.3}$$

If we consider (1.3) in $\mathbf{R}^2 \times (0,T)$, each level set of ψ moves by (1.1) if it is represented by the graph of a function $u = u(x,t)$. This formulation is successful to track discontinuous solutions for

$$u_t + H(u, u_x) = 0$$

if $r \mapsto H(r,p)$ is nondecreasing so that solution does not develop discontinuities if the initial data is continuous [12]. However, for (1.1) the zero level set of the solution of (1.3) certainly overturn if initially

$$\{(x,y); \psi(x,y,0) > 0\} = \{(x,y); y < -d/2\} \cup \{(x,y); x < 0, -d/2 \le y < d/2\}; \quad (1.4)$$

in fact, the zero level set $\psi = 0$ for $t > 0$ cannot be viewed as the graph of a single valued function in any sense.

In [10] we propose to add the vertical diffusion term

$$\psi_t + y\psi_x = M|\nabla\psi|\partial_y(\psi_y/|\psi_y|). \quad (1.5)$$

A formal argument [10, Theorem 2.1] reflecting [3] says that if M is large so that

$$V_I \ge V - 2M \quad \text{on} \quad I = (-d/2, d/2), \quad (1.6)$$

then the zero level set of ψ with initial condition (1.4) does not overturn and equals the graph of the entropy solution of (1.1), (1.2). Here $V(\eta) = -\eta^2/2$ which is the primitive of $-y$ and V_I denotes its convex hull in I. An elementary calculation shows that the minimum value M_0 of M satisfying (1.6) is $d^2/16$. In the numerical simulation [16] we also observe that the overturning occurs if and only if $M < M_0 = d^2/16$. (There I is replaced by (a,b) but the value of M_0 equals $(b-a)^2/16$.)

In this paper we show analytically that M_0 is optimal in the sense that if $M < M_0$, the overturning is not prevented. It is also possible to prove that the overturning does not occur $M \ge M_0$ for more general equations but we shall discuss this problem in one of forthcoming papers.

Although the level set method (see e.g. [8]) allowing the singular diffusivity is well-studied by [4], [5], [6], the equation handled there is spatially homogeneous and excludes (1.5). Instead of developing a general theory for (1.5) we rather study its approximation. In fact, we shall prove that there is a sequence of level set equations

$$\psi_t + y\psi_x = M|\nabla\psi|\,\mathrm{div}\,(\nabla\gamma_\varepsilon(-\nabla\psi)) \quad (1.7)$$

approximating (1.5) such that the limit of zero level set of $\psi = \psi^\varepsilon$ develops 'overturning' if $M < M_0$. Here $\gamma_\varepsilon \in C^2(\mathbf{R}^2 \setminus \{0\})$ is convex and positively homogeneous of degree one.

The main idea of the proof is to convert the problem of evolution of $\{\psi = 0\}$ to the evolution of $x = v(y,t)$ starting with $v(y,0) = 0$. (For this purpose we assume that $\nabla^2\gamma(0,1) = 0$ so that the line segment on the line $y = \pm d/2$ does not move.) We study the equation for v derived from (1.7) and prove that it converges to a function which has strictly monotone increasing part in y if $M < M_0$. This means that 'overturning' occurs. Unfortunately, the boundary condition for v at $y = \pm d/2$ is not conventional. It is formally equals the Neumann condition

$$v_y(\pm d/2, t) = -\infty.$$

This is hard to handle so we estimate from above and below by solutions of a homogeneous Neumann problem and on inhomogeneous Dirichlet problem. We prove that solutions of the latter two problems converges to the same function having desired property.

2 Explicit solutions for some inhomogeneous very singular diffusion equations

We consider a singular degenerate parabolic equation for $v = v(\eta, t)$ of the form

$$v_t = M(\mathrm{sgn} v_\eta)_\eta + \eta \quad \text{in} \quad I \times (0, \infty), \tag{2.1}$$

$$v = 0 \quad \text{on} \quad \partial I \times (0, \infty), \tag{2.2}$$

$$v|_{t=0} = 0 \tag{2.3}$$

with $I = (-d/2, d/2)$, where $M > 0$ is a parameter. Since $(\mathrm{sgn} v_\eta)_\eta$ formally equals $\delta(v_\eta)v_{\eta\eta}$, the diffusion is degenerate for $v_\eta \neq 0$ and is very strong for $v_\eta = 0$. Naively, the meaning of a 'solution' is not clear. Fortunately, the theory of nonlinear semigroups [15] or subdifferential equations provides a suitable notion of a solution. We shall briefly review its notion and give an explicit representation formula of the solution.

We first give a subdifferential interpretation of the problem (2.1)-(2.3). For $v \in H = L^2(I)$ we associate the energy $E(v)$ defined by

$$E(v) := \int_{\mathbf{R}} \{M|\tilde{v}_\eta(\eta)| - \eta\tilde{v}(\eta)\} \mathrm{d}\eta \quad \text{if} \quad v \in BV(I)$$

and $E(v) := \infty$ if $v \notin BV(I)$. Here $BV(I)$ denotes the space of functions with bounded variation in I and $\tilde{v}$ denotes the extention of v to $\mathbf{R}$ such that $\tilde{v} = 0$ outside I. The integral $\int_{\mathbf{R}} |\nabla\tilde{v}(n)| d\eta$ denotes the total variation of $\tilde{v}$ in $\mathbf{R}$. Then as in [7, the first lemma in §2] the functional E is convex, lower semicontinuous in the Hilbest space H equipped with the standard inner product $(f, g) = \int_I fg d\eta$. Note that (2.1) is formally a gradient flow of E. Thus we formulate the problem (2.1)-(2.3) as

$$\frac{dv}{dt} \in -\partial E(v), \tag{2.4}$$

$$v(0) = 0, \tag{2.5}$$

where ∂E denotes the subdifferential of E in H. A general theory [15], [1] yields that there is a unique solution v of (2.4) and (2.5) in the sense that

(i) $v \in C([0, \infty), H)$ i.e., v is continuous from the time interval $[0, \infty)$ to H. Moreover, v satisfies (2.5).

(ii) v is absolutely continuous with values in H on each compact set in $(0,\infty)$ and solves (2.4) for almost all $t \geq 0$.

As well-known (e.g. [1], see also [7, §2]) the solution $v(t)$ is right-differentiable at all $t > 0$ with values in H and its right derivative d^+v/dt satisfies

$$\frac{d^+v}{dt} = -\partial^0 E(v) \quad \text{for all} \quad t > 0. \tag{2.6}$$

where $\partial^0 E(v)$ is the canonical restriction (or minimal section) of $\partial E(v)$, i.e., $\partial^0 E(v)$ is the unique element of the closed convex set $\partial E(v)$ which is closest to the origin of H. Moreover, we have another definition of solution equivalent to (i) (ii). Namely, v is the solution of (2.4) and (2.5) if and only if v fulfills (i) and

(ii)' v is absolutely continuous with values in H on each compact set in $(0,\infty)$ and solves (2.6) for all $t > 0$.

Here and hereafter by solution of (2.1)-(2.3) we mean that v satisfies (i) and (ii)'. Fortunately, the solution can be represented in an explicit formula.

Lemma 2.1. *Let v be the solution of (2.1)-(2.3). Then v is represented by*

$$v(\eta, t) = tv_1(\eta), \ t \geq 0 \tag{2.7}$$

with v_1 satisfying

$$v_1(\eta) = \min(\eta, (\frac{d}{2} - 2M^{1/2})_+) \qquad \text{for} \quad \eta \in [0, \frac{d}{2})$$
$$v_1(\eta) = -v_1(-\eta) \qquad \text{for } \eta \in (-\frac{d}{2}, 0],$$

where $\alpha_+ = \max\,(\alpha, 0)$. In particular, $v_1 \equiv 0$ if and only if $M \geq d^2/16$ and otherwise v_1 has a strictly increasing part.

Remark 2.2. (i) If we replace the homogeneous Dirichlet condition (2.2) by the homogeneous Neumann condition

$$v_\eta = 0 \quad \text{on} \quad \partial I \times (0, T), \tag{2.2$'$}$$

the solution of (2.1) with (2.2$'$), (2.3) is the same as in (2.7). Here we should replace the definition of E by

$$E_N(v) := \int_I \{M|v_\eta| - \eta v\} d\eta \quad \text{if} \quad v \in BV(I) \tag{2.8}$$

and $E_N(v) := \infty$ if $v \notin BV(I)$ so that (2.1), (2.2$'$) (2.3) is formulated by (2.4), (2.5) with E replaced by E_N.

(ii) We may replace the homogeneous Dirichlet condition (2.2) by inhomogeneous Dirichlet condition

$$v = \mp R \quad \text{at} \quad \eta = \pm d/2. \tag{2.2''}$$

The solution of (2.1) with (2.2″), (2.3) is the same as in (2.7) for $R > 0$. Here we should replace E by

$$E_R(v) := \int_{\mathbf{R}} \{M|\bar{v}_\eta| - \eta\bar{v}\} d\eta \quad \text{if} \quad v \in BV(I) \tag{2.9}$$

and $E_R(v) := \infty$ if $v \notin BV(I)$. The extention $\bar{v}$ of v equals $-R$ for $\eta \geq d/2$ and R for $\eta \leq -d/2$. The equation (2.1), (2.2″), (2.3) is now formulated by (2.4), (2.5) with E replaced by E_R.

To show these statements it suffices to verify (2.6) as in [3].

3 Neumann problems for some non-uniform parabolic equations

To study solutions of problems approximating (2.1)-(2.3) we consider the Neumann problem:

$$v_t = a(v_\eta)v_{\eta\eta} + \eta \quad \text{in} \quad I \times (0,\infty), \tag{3.1}$$

$$v_\eta = -\alpha \quad \text{on} \quad \partial I \times (0,\infty), \tag{3.2}$$

$$v|_{t=0} = 0. \tag{3.3}$$

Here $a \in C^1(\mathbf{R})$ is assumed to be positive and α is a non-negative constant. Since v_η of (3.1) solves

$$v_{\eta t} = (a(v_\eta)v_{\eta\eta})_\eta + 1, \tag{3.4}$$

by the maximum principle we have an a priori bound $|v_\eta(n,t)| \leq \max(t,\alpha)$ for v_η. So in $I \times (0,T)$ with $T > 0$ we may assume that equation is uniformly parabolic by restricting a on $[-\max(T,\alpha),\ \max(T,\alpha)]$. A general theory of parabolic equations [14] yields an unique global classical solution $v \in C^{2,1}(I \times [0,\infty)) \cap C^{2,1}(\bar{I} \times (0,\infty))$ of (3.1)-(3.3).

Our main goal in this section is to prove several properties of the solution of (3.1)-(3.3).

Theorem 3.1. *Let v^α be the solution of (3.1)-(3.3) with $\alpha \geq 0$.*

(i) (Symmetry). $v^\alpha(\eta,t) = -v^\alpha(-\eta,t)$ for $\eta \in I$, $t \geq 0$. In particular, $v^\alpha(0,t) = 0$ for $t > 0$.

(ii) (Concavity). $v^\alpha(\eta,t) \leq \eta t$, $v_t^\alpha(\eta,t) \leq \eta$ for $\eta \in I_+$, $t \geq 0$ with $I_+ = (0,d/2)$. In particular, $v_{\eta\eta}^\alpha \leq 0$ in $I_+ \times (0,\infty)$.

(iii) (Monotonicity). $v^\alpha \leq v^\beta$ *in* $I_+ \times (0,\infty)$ *if* $\alpha \geq \beta \geq 0$. *Moreover* $v^\alpha_\eta \leq v^\beta_\eta$ *in* $I_+ \times (0,\infty)$ *if* $\alpha \geq \beta \geq 0$.

(iv) (Lower bound). Assume that

$$c_0 := \int_{-\infty}^{0} a(\tau)\mathrm{d}\tau \leq \frac{d^2}{8} \tag{3.5}$$

and

$$c_1 := \int_{-\infty}^{0} |\tau| a(\tau)\mathrm{d}\tau < \infty. \tag{3.6}$$

Then $v^\alpha(\eta,t) \geq -c_0c_1$ *for* $\eta \in [0,d/2], t \geq 0$.

Proof. (i) Since $-v^\alpha(-\eta,t)$ solves (3.1)-(3.3), the uniqueness of a solution yields the symmetry.
(ii) Clearly ηt is a supersolution of (3.1)-(3.3) in $I_+ \times (0,\infty)$ with zero boundary condition at $\eta = 0$ so the comparison principle yields $v \leq \eta t$ in $I_+ \times (0,\infty)$. We differentiate (3.1), (3.2) in t to get

$$w_t = a(v^\alpha_\eta)w_{\eta\eta} + a'(v^\alpha_\eta)w_\eta v^\alpha_{\eta\eta} \quad \text{in} \quad I \times (0,\infty)$$
$$w_\eta(d/2,t) = 0, \;\; w(0,t) = 0 \text{ (by } (i))$$

for $w = v^\alpha_t$. Since $v^\alpha_t \leq \eta$ at $t = 0$ on I_+ by $v^\alpha \leq \eta t$, the maximum principle implies that $w \leq \eta$ in $[0,d/2] \times [0,\infty)$. The concavity follows from $v_t \leq \eta$ and the equation (3.1) since $a > 0$.
(iii) For $\beta \leq \alpha$ the solution v^β is a supersolution of (3.1)-(3.3) with $v = 0$ at $\eta = 0$ in $I_+ \times (0,\infty)$, the comparison principle yields $v^\alpha \leq v^\beta$ in $I_+ \times (0,\infty)$. Since $v^\alpha \leq v^\beta$ and $v^\alpha = v^\beta = 0$ at $\eta = 0$, we observe that $v^\alpha_\eta \leq v^\beta_\eta$ at $\eta = 0$. Since v^β_η solves (3.4) and $v^\alpha_\eta \leq v^\beta_\eta$ at $\eta = d/2$, the comparison principle yields $v^\alpha_\eta \leq v^\beta_\eta$ in $I_+ \times (0,\infty)$.
(iv) As in the next Lemma we shall construct a time independent subsolution $f = f_\alpha$ for (3.1)-(3.3) in $I_+ \times (0,\infty)$ with the zero-boundary condition at $\eta = 0$ such that $f_\alpha \geq -c_0c_1$. Once such a subsolution is constructed, the comparison principle yields the bound $v^\alpha \geq -c_0c_1$ for v^α.

Lemma 3.2. *Assume that (3.5) holds. Then there exists a unique* $\sigma \in I_+ = (0,d/2)$ *and a* C^1 *function* $f = f_\alpha$ *on* $\tilde{I}_+$ *such that*

$$-(A(f'(\eta)))' = \eta \quad \text{on} \quad I_+, \tag{3.7}$$

$$f'(d/2) = -\alpha, \;\; f'(\sigma) = f(\sigma) = 0, \tag{3.8}$$

where $A(q) = \int_0^q a(\tau)\mathrm{d}\tau$ *and* f' *denotes the derivative of* f. *If moreover* a *satisfies (3.6), then*

$$-c_0c_1 \leq \inf\{f_\alpha(\eta); \quad \eta \in [0,d/2], \alpha \geq 0\} = \inf\{f_\alpha(d/2); \alpha \geq 0\} \tag{3.9}$$

(The zero-exterision of f_α to $[0,\sigma]$ is still denoted by f_α).

Proof. Integrating (3.7) from σ to η yields

$$-A(f'(\eta)) = (\eta^2 - \sigma^2)/2 \tag{3.10}$$

since $f'(\sigma) = 0$. Since $A(p) \leq d^2/8$ for $p \leq 0$ by (3.5), there is unique $\sigma \in I_+$ such that

$$-A(-\alpha) = \frac{1}{2}\left(\frac{d}{2}\right)^2 - \frac{\sigma^2}{2}.$$

We fix such a σ and then taking the inverse A^{-1} of (3.10) to get

$$f'(\eta) = A^{-1}((\sigma^2 - \eta^2)/2), \ \eta \in [\sigma, d/2]. \tag{3.11}$$

Integrating this with $f(\sigma) = 0$ we obtain the solution f and $\sigma \in I_+$ satisfying (3.7), (3.8).

By (3.11) $f'(\eta) \leq 0$ in I_+ so $\inf_{I_+} f = f(d/2)$. Thus to prove (3.9) if suffices to prove that

$$\inf_\alpha f_\alpha(d/2) > -\infty. \tag{3.12}$$

Integrating (3.11) over $[\sigma, d/2]$ to get

$$\begin{aligned} - \ f_\alpha(d/2) &= -\int_\sigma^{d/2} A^{-1}((\sigma^2 - \eta^2)/2)\mathrm{d}\eta \\ &= -\int_{A(-\alpha)}^0 A^{-1}(\xi)\xi \mathrm{d}\xi \leq -A(-\infty)\int_{A(-\infty)}^0 A^{-1}(\xi)\mathrm{d}\xi. \end{aligned}$$

Since

$$-\int_{A(-\infty)}^0 A^{-1}(\tau)\mathrm{d}\tau = \int_{-\infty}^0 (A(p) - A(-\infty))\mathrm{d}p = \int_{-\infty}^0 |\tau| a(\tau)\mathrm{d}\tau = C_0$$

we now obtain that $-f_\alpha(d/2) \leq c_0 c_1$. □

4 Approximate problems

Let v^α be the solution of (3.1)-(3.3). We define v^∞ by

$$v^\infty(\eta, t) = \inf_{\alpha>0} v^\alpha(\eta, t), \eta \in I_+ = (0, d/2)$$

$$v^\infty(\eta, t) = -v^\infty(-\eta, t), \eta \in (-d/2, 0)$$

$$v^\infty(0, t) = 0.$$

By the monotone properties and bounds (Theorem 3.1) v^∞ is well-defined and $\eta \mapsto v^\infty(\eta, t)$ is C^1 and concave in I_+.

Our goal in this section is to prove the convergence of v^∞ to v in (2.7) when $f^q a$ approximates $M\mathrm{sgn}q$.

Theorem 4.1. *Assume that $a = a^\varepsilon \in C^1(\mathbf{R}), a^\varepsilon > 0$ satisfies (3.5) and (3.6). Assume that c_0^ε, c_1^ε defined by (3.5), (3.6) with $a = a^\varepsilon$ are bounded as $\varepsilon \to 0$. Assume that $A^\varepsilon(q) = \int_0^q a^\varepsilon(\tau)\mathrm{d}\tau$ converges to $M\mathrm{sgn}\eta + c$ with some constant c as $\varepsilon \to 0$ (in the sense of monotone graphs). Let v_ε^∞ be the solution of (3.1), (3.2), (3.3) with $a = a^\varepsilon$ and let $v_\varepsilon^\infty = \inf_{\alpha>0} v_\varepsilon^\alpha$. Let v be the function defined in (2.7). Then v_ε^∞ converges to v as $\varepsilon \to 0$ uniformly in every compact subset of $I \times [0, \infty)$.*

We shall prove this result by estimating v_ε^∞ from above by the solution of the homogeneous Neumann problem and from below by that of a nonhomogeneous Dirichlet problem.

4.1 Convergence of the Neumann problem

Proposition 4.2. *Assume that $A^\varepsilon(q) = \int_0^q a^\varepsilon(\tau)\mathrm{d}\tau$ convergence to* Msgn *$\eta + c$ with some constant c as $\varepsilon \to 0$, where $a^\varepsilon \in C^1(\mathbf{R})$ and $a^\varepsilon > 0$. Let v_ε^0 be the solution of (3.1)-(3.3) with $\alpha = 0$. Then v_ε^0 converges to v (defined by (2.7)) as $\varepsilon \to 0$ uniformly in $\bar{I} \times [0, T]$ for any $T > 0$.*

Proof. We formulate the problem (3.1)-(3.3) by using a subdifferential equation $u_t \in -\partial E_N^\varepsilon(u)$, $u|_{t=0} = 0$. By a stability theorem of J. Watanabe [17] based on [2] the solution v_ε^0 converges to a solution u of $u_t \in -\partial E_N$ in $C([0,T], L^2(I))$ for any $T > 0$. Since the solution of $u_t \in -\partial E_N$ with $u|_{t=0} = 0$ equals v of (2.7) as in Remark 2.2, $v_\varepsilon^0 \to v$ in $C([0,T], L^2(I))$. By Theorem 3.1 $v_\varepsilon^0(\eta, t)$ is concave in $\eta \in I_+$ and $v_{\varepsilon\eta}^0 \le 1$ at $\eta = 0$. Since $v_{\varepsilon\eta}^0(d/2, t) = 0$, we see that $v_{\varepsilon_j}^{0j}(\cdot, t_j)$ always contains a uniform convergent subsequence on I as $j \to \infty$ if $\varepsilon_j \to 0$, $t_j \in [0,T]$. Since $v_\varepsilon^0 \to v$ in $C([0,T], L^2(I))$ this implies the uniform convergence of v_ε^0 in $\bar{I} \times [0,T]$ as stated in the next lemma whose proof is elementary.

Lemma 4.3. *Assume that $u^\varepsilon \to u$ in $C([0,T], L^2(\Omega))$ as $\varepsilon \to 0$, where Ω is an open set in $\mathbf{R}^d$. Assume that $\{u^{\varepsilon_j}(\cdot, t_j)\}$ has a uniform convergent subsequence in $\bar{\Omega}$ provident that $\varepsilon_j \to 0$, $t_j \in [0,T]$. Then $u^\varepsilon \to u$ uniformly in $[0,T] \times \bar{\Omega}$.*

4.2 Dirichlet problem

We consider the Dirichlet problem for (3.1), (3.3) with $a = a^\varepsilon$ with the boundary condition

$$v(\pm d/2, t) = \mp R, \tag{4.1}$$

where R is a positive constant. Let v_{R^ε} be the solution of (3.1), (3.3) with (4.1). The solution may not be satisfies (4.1). It can be understood as the limit of a uniformly parabolic problem which approximates (3.1), (3.3) and (4.1). Since we may assume that we conclude that $v_{R\varepsilon,\eta\eta} \le 0$ in $I_t \times (0, \infty)$.

Proposition 4.4 *Assume the same hypotheess of Proposition 4.2 concerning a^ε. Let v_{R^ε} be the solution of (3.1), (3.3) and (4.1). with $a = a^\varepsilon$. Then $v_{R^\varepsilon} \to v$ as $\varepsilon \to 0$ uniformly in each compact subset of $I \times [0,\infty)$, where v is defined by (2.7).*

Proof. As in the proof of Proposition 4.2 we observe that $v_{R^\varepsilon} \to v$ in $C([0,T], L^2(I))$. Again $v_{R\varepsilon}$ is concave in $\eta \in I_+$ and $v_{R\varepsilon,\eta}(0,t) \leq 1$. However, there is no control on $v_{R\varepsilon,\eta}(d/2,t)$. All we expect is that $v_{R\varepsilon}$ is bounded in $I_+ \times [0,T]$ and $v_{R\varepsilon}$ is concave in η. From these facts we are able to prove that $v_{R\varepsilon_v}(\cdot,\ t_j)$ has a uniform convergent subsequence in $[0, d/2-\delta]$ for each $\delta > 0$ if $t_j \in [0.T]$ and $\varepsilon_j \to 0$. By Lemma 4.3 we now conclude that $v_{R\varepsilon} \to v$ in each compact subset of $I \times [0,\infty)$

Proof of Theorem 4.1. By Theorem 3.1 (iii) we see that $v_\varepsilon^\infty \leq v_\varepsilon^0$ in $I_+ \times (0,\infty)$. We take $R \geq c_0^\varepsilon c_1^\varepsilon$ for small $\varepsilon > 0$. Then by the comparison for the Dirichlet problem

$$v_{R\varepsilon} \leq v_\varepsilon^\alpha \quad \text{in} \quad I_+ \times (0,\infty).$$

since $v_{R\varepsilon} = v_\varepsilon^\alpha = 0$ at $\eta = 0$. This implies

$$v_{R\varepsilon} \leq v_\varepsilon^\infty \quad \text{in} \quad I_+ \times (0,\infty).$$

The convergence results (Propositions 4.2, 4.4) yield the convergence $v_\varepsilon^\infty \to v$. □

5 Level set solutions

We consider the level set equation of the form

$$\psi_t + y\psi_x = M|\nabla\psi|\text{div}\{\nabla\gamma(-\nabla\psi/|\nabla\psi|)\} \quad \text{in} \quad \mathbf{R}^2 \times (0,\infty) \tag{5.1}$$

Here γ is a convex, positively homogeneous of degree one in $\mathbf{R}^2$. If $M = 0$, the set $\{\psi = 0\}$ formally represents the graph of a solution of the Burgers equation for $u = u(x,t)$:

$$u_t + uu_x = 0.$$

We shall use the convention that $\psi > 0$ below the graph of u. By a standard theory of the level set equation for each $\psi_0 \in \text{BUC}(\mathbf{R}^2)$ there is a unique viscosity solution $\psi \in \text{BUC}(\mathbf{R}^2 \times [0,T])$ for any $T > 0$ of (5.1) satisfying $\psi(x,y,t) = \psi_0(x,\eta)$ provided that $\gamma \in C^2(\mathbf{R} \setminus \{0\})$; see [11], [13]. We consider the initial data ψ_0 satisfying

$$\{\psi_0 > 0\} = \{(x,\eta); y < -d/2\} \cup \{(x,\eta); x > 0, y < d/2\} =: D_0.$$

and call the set $D = \{\psi > 0\}$ is the level set solution (of (5.1)) with the initial data D_0. The set D is independent of the choice of ψ_0 and is uniquely determined by D_0.

Our main goal is to show that if $M < d^2/16$, then for a large class of γ such that $\nabla\gamma(-\nabla\psi/|\nabla\psi|)$ approximating $\psi_y/|\psi_y|$, the limit of D develop 'overturning'.

Lemma 5.1. *Let $\gamma \in C^2(\mathbf{R}^2 \setminus \{0\})$ be convex and positively homogeneous of degree one. Then*

$$\nabla^2\gamma(0,1) = 0$$

if and only if $|q|^3 W''(q) \to 0$ as $q \to -\infty$ for $W(q) = \gamma(1,-q)$.

Proof. By definition

$$\gamma_2(1,-q) = -W'(q) \quad \text{and} \quad \gamma_{22}(1,-q) = W''(q),$$

where $\gamma_i = \partial\gamma/\partial p_i$, $\gamma_{ij} = \partial^2\gamma/\partial p_i\partial p_j$. Since γ_i is positively homogeneous of degree one, we have

$$\gamma_{12}(1,-q) - q\gamma_{22}(1,-q) = 0$$
$$\gamma_{11}(1,-q) - q\gamma_{12}(1,-q) = 0.$$

Thus

$$\gamma_{11}(1,-q) = q^2 W''(\gamma), \quad \gamma_{12}(1,-q) = gW'(q).$$

Since γ_{ij} is positively homogeneous of degree -1,

$$\gamma_{ij}(1/(1+q^2)^{1/2}, -q/(1+q^2)^{1/2}) = (1+q^2)^{1/2}\gamma_{ij}(1,-q) \to \gamma_{ij}(0,1)$$

as $q \to -\infty$. Thus $q^3 W''(q) \to 0$ as $q \to \infty$ is equivalent to $\gamma_{ij}(0,1) = 0$ for all $1 \le i,\ j \le 2$. □

The next lemma relates the level set solution D and a solution of (3.1), (3,3).

Lemma 5.2 *Let $\gamma \in C^2(\mathbf{R} \setminus \{0\})$ be convex and positively homogeneous of degree. Assume that $|q^3|W''(q) \to 0$ as $q \to -\infty$ for $W(q) = \gamma(1,-q)$. Assume that $W''(q) > 0$. For $a(q) = M(1+q^2)^{1/2}W''(q)$ let v^α the solution of (3.1)-(3.3) and $v^\infty = \inf_{\alpha>0} v^\alpha$. Let D be the level set solution with initial data D_0. Then*

$$D = \{(x,y,t); y < -d/2\} \cup \{(x,y,t); x < v^\infty(y,t),\ -d/2 \le y < d/2\}. \tag{5.2}$$

The proof is not short. We here indicate the idea of the proof.

Step1. The right hand side (denoted $\tilde{D}$) of (5.2) is a solution of (5.1) in the sense that the characteristic function of $\tilde{D}$ solves (5.1) in the viscosity sense. We use the fact that the straight part of $\partial\tilde{D} \subset \{y = \pm d/2\}$ does not move because of Lemma 5.1. We also note that $v^\infty_\eta(\eta,t) \to -\infty$ as $\eta \uparrow d/2$, This is important to prove that $\tilde{D}$ is the solution of (5.1). Note that if the boundary of $\tilde{D}$ is written as $x = v(y,t)$, then v satisfies (3.1).

Step.2 The set $\tilde{D}$ is the level set solution. This can be proved by showing that there is no fattening for $\tilde{D}$.

As an application of Theorem 4.1 we have a convergence result.

Theorem 5.3. *Let γ^ε fulfills the assumption of γ in Lemma 5.2 with $W^\varepsilon(q) = \gamma^\varepsilon(1, -q)$. Assume that $W^{\varepsilon'}(q) \to \operatorname{sgn} q + c$ with some constant c as $\varepsilon \to 0$ in the sense of monotone graphs. Let D^ε be the level set solution of (5.1) with $\gamma = \gamma^\varepsilon$ starting with D_0 Assume that there is $r > 0$ such that*

$$\int_{-\infty}^{0} (1+q^2)^{1/2}\, W^{\varepsilon''}(q)dq \le r \quad \text{for small} \quad \varepsilon$$

and

$$\sup_{0<\varepsilon<1} \int_{-\infty}^{0} |q|(1+q^2)^{1/2}\, W^{\varepsilon''}(q)dq < \infty.$$

Then $\bar{D}^\varepsilon$ converges to

$$E = \{(x,y,t); y < -d/2\} \cup \{(x,y,t); x < v(y,t), -d/2 \le y < d/2\}$$

in the sense of Hausdorff distance topology provided that $Mr \le d^2/8$.

Example. If $W^\varepsilon(q) = \int_0^q \tanh(\tau/\varepsilon)d\tau$, then

$$\int_{-\infty}^{0} (1+q^2)^{1/2}\, W^{\varepsilon''}(q)dq \to 1,$$

so for each $\delta > 0$, there is $\varepsilon_0 > 0$ such that

$$\int_{-\infty}^{0} (1+q^2)^{1/2}\, W^{\varepsilon''}(q)dq \le 1+\delta \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0).$$

The condition

$$\sup_{0<\varepsilon<1} \int_{-\infty}^{0} q(1+q^2)^{1/2}\, W^{\varepsilon''}(q)dq < \infty$$

is evidently fulfilled. Thus the convergence results holds for $M(1+\delta) \le d^2/8$. If $\delta > 0$ is taken small so that $(1+\delta)/16 < 8$, then we have a threshold value $M = d^2/16$ such that if $M < d^2/16$, then E experiences 'overturning' in the sense that there is a point (x_0, y_0, t_0) and (x_0, y_1, t_0) satisfying $y_1 < y_0$ such that

$$(x_0, y_0, t_0) \in E \quad \text{while} \quad (x_1, y_1, t_0) \notin E.$$

If $M \ge d^2/16$, $E = D_0 \times (0, \infty)$ so no overturn occurs.

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