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The Hopf - Lax solution for state dependent Hamilton - Jacobi equations

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1 Introduction

Consider the Hamilton - Jacobi equation

$$u_t(x,t)+H(x,Du(x,t))=0 \text{ in } \mathbb{R}^N\times(0,+\infty) \tag{1.1}$$

with the initial condition

$$u(x,0)=g(x) \text{ in } \mathbb{R}^N. \tag{1.2}$$

It is well - known that if $H(x,p)=H(p)$ is convex, then the solution of the Cauchy problem (1.1), (1.2) is given by the function

$$u(x,t)=\inf_{y\in\mathbb{R}^N}\left[g(y)+tH^*\left(\frac{x-y}{t}\right)\right] \tag{1.3}$$

where H^* is the Legendre - Fenchel transform of H. The first result of this type goes back to E. Hopf [12] who proved that if H is convex and superlinear at infinity and g is Lipschitz continuous, then u satisfies (1.1) almost everywhere and achieves the initial condition. This result has been generalized in several directions, mainly in the framework of the theory of viscosity solutions, see [3], [11], [7], [2].

The Hopf - Lax formula (1.3) can be understood as a simplified expression of the classical representation of the solution of (1.1), (1.2) as the value function of the Bolza problem associated by duality with the Cauchy problem, namely

$$V(x,t) := \inf \int_0^t H^*(\dot{y}(s))ds + g(y(0)) \tag{1.4}$$

where the infimum is taken over all smooth curves y with $y(t) = x$. Indeed, if H does not depend on x then $u \equiv V$, the proof of the equivalence relying in an essential way on the fact that any pair of points x, y in $\mathbb{R}^N$ can be connected in a given time $t > 0$ by a curve of constant velocity, namely the straight line $y(s) = x + \frac{s}{t}(y - x)$, see [11]. When H depends on x as well the situation becomes more complicated and the simple, useful representation (1.3) of the solution of (1.1), (1.2) as the value function of an unconstrained finite dimensional minimization problem, parametrized by (x,t), is not available anymore.

This Note is dedicated to the presentation of a recent result due to H. Ishii and the author concerning the validity of an Hopf - Lax type representation formula for the solution of (1.1), (1.2) for a class of x - dependent Hamiltonians. In Section 2 we describe the main results contained in the forthcoming paper [10], give a brief sketch of their proofs and indicate some relevant examples.
Further comments on the Hopf - Lax formula in connection with large deviations problems and the related Maslov's approach to Hamilton - Jacobi equations are outlined in Section 3.

2 Results

From now on we assume that H is of the form

$$H(x,p) = \Phi\left(H_0(x,p)\right) \tag{2.1}$$

where H_0 is a continuous real valued function on $\mathbb{R}^{2N}$ satisfying the following conditions

$$p \mapsto H_0(x,p) \text{ is convex }, \; H_0(x,\lambda p) = \lambda H_0(x,p) \tag{2.2}$$

$$H_0(x,p) \geq 0 \;, \; |H_0(x,p) - H_0(y,p)| \leq \omega(|x-y|(1+|p|)) \tag{2.3}$$

for all x, y, p, for all $\lambda > 0$ and for some continuous, non decreasing function $\omega : [0,+\infty) \to [0,+\infty)$ such that $\omega(0) = 0$.

We assume also that H_0 is degenerate coercive in the sense that, for some $\epsilon > 0$, the conditions (2.4), (2.5) below hold:

$$\sigma(x)\left([-\epsilon,\epsilon]^M\right) \subseteq \partial H_0(x,0) \tag{2.4}$$

Here, $\partial H_0(x,0)$ is the subdifferential of the convex function $p \rightarrow H(x,p)$ at $p=0$, $[-\epsilon,\epsilon]^M$ is the cube of side ϵ in $\mathbb{R}^M$ and $\sigma(x)$ is an $N \times M$ matrix, $M \leq N$, depending smoothly on x, satisfying the Chow - Hörmander rank condition

$$\operatorname{rank} \mathcal{L}(\Sigma_1,\ldots,\Sigma_M)(x) = N \text{ for every } x \in \mathbb{R}^N \tag{2.5}$$

where $\mathcal{L}(\Sigma_1,\ldots,\Sigma_M)$ is the Lie algebra generated by the columns $\Sigma_1(x),\ldots,\Sigma_M(x)$ of the matrix $\sigma(x)$, see for example [8]. Concerning function Φ we assume

$$\Phi : [0,+\infty) \rightarrow [0,+\infty) \text{ is convex, non decreasing }, \ \Phi(0) = 0 \tag{2.6}$$

Under the assumptions made on H_0, the stationary equation

$$H_0(x, Dd(x)) = 1 \text{ in } \mathbb{R}^N \setminus \{y\} \tag{2.7}$$

is of eikonal type and, consequently, it is natural to expect that (2.7) has distance type solutions $d = d(x,y)$ and, on the basis of the analysis in [13], that the solution of

$$u_t(x,t) + \Phi\left(H_0(x, Du(x,t))\right) = 0 \text{ in } \mathbb{R}^N \times (0,+\infty) \tag{2.8}$$

$$u(x,0) = g(x) \text{ in } \mathbb{R}^N \tag{2.9}$$

can be expressed in terms of these distances.

We have indeed the following results, see [10]:

Theorem 2.1 *For each $y \in \mathbb{R}^N$, equation (2.7) has a unique viscosity solution $d = d(x,y)$ such that*

$$d(x,y) \geq 0 \text{ for all } x,y \ , \ d(y,y) = 0 \ . \tag{2.10}$$

Theorem 2.2 *Assume that*

$$g \textit{ lower semicontinuous }, \ g(x) \geq -C(1+|x|) \textit{ for some } C > 0 \ . \tag{2.11}$$

Then, the function

$$u(x,t) = \inf_{y \in \mathbb{R}^N} \left[g(y) + t\Phi^* \left(\frac{d(x,y)}{t} \right) \right] \tag{2.12}$$

is the unique lower semicontinuous viscosity solution of (2.8) *which is bounded below by a function of linear growth and such that*

$$\liminf_{(y,t)\to(x,0^+)} u(y,t) = g(x) \tag{2.13}$$

Theorem 2.1 extends to the present setting previous well - known results on the minimum time function for nonlinear control systems, see [4], [5].

The proof of the theorem starts from the construction by optimal control methods of the candidate solution d. Consider the set - valued mapping $x \to \partial H_0(x,0)$ and the differential inclusion

$$\dot{X}(t) \in \partial H_0(X(t),0) \tag{2.14}$$

By standard results on differential inclusions, for all $x \in \mathbb{R}^N$ there exists a global solution of (2.14) such that $X(0) = x$, see [1]. Assumptions (2.4), (2.5) imply that the set $F_{x,y}$ of all trajectories $X(\cdot)$ of (2.14) such that

$$X(0) = x, \ X(T) = y$$

for some $T = T(X(\cdot)) > 0$ is non empty for any $x, y \in \mathbb{R}^N$ since it contains all trajectories of the symmetric control system

$$\dot{X}(t) = \sigma(X(t))\,\epsilon(t)\ ,\ X(0) = x \tag{2.15}$$

where the control ϵ is any measurable function of $t \in [0,+\infty)$ taking values in $[-\epsilon,\epsilon]^M$. Indeed, thanks to assumption (2.5), the Chow's Connectivity Theorem implies that any pair of points x, y can be connected in finite time by a trajectory of (2.15) and so, a fortiori, by a trajectory of (2.14), see for example [8].
Define then

$$d(x,y) = \inf_{X(\cdot)\in F_{x,y}} T(X(\cdot)) < +\infty\,. \tag{2.16}$$

It is easy to check that

$$d(x,y) \geq 0, \ \ d(x,x) = 0, \ \ d(x,z) \leq d(x,y) + d(y,z)$$

for all x, y, z. Therefore, d is a sub - Riemannian distance of Carnot - Carathéodory type on $\mathbb{R}^N$, non symmetric in general.
Moreover, if Ω is a bounded open set and $k \in \mathbb{N}$ is the minimum length of commutators needed to guarantee (2.5) in Ω, then there exists $C = C(\Omega)$ such that

$$\frac{1}{C}|x-y| \leq d(x,y) \leq C|x-y|^{\frac{1}{k}}$$

for all $x, y \in \Omega$, see [17]. Hence,

$$d(x,y) - d(z,y) \leq \max(d(x,z), d(z,x)) \leq 2C|x-z|^{\frac{1}{k}}$$

which shows that $x \rightarrow d(x;y)$ is $\frac{1}{k}$ Hölder - continuous.
Also, it is not hard to check that function d satisfies the following form of the Dynamic Programming Principle

$$d(x,y) = \inf_{X(\cdot) \in F_{x,y}} [t + d(X(t), y)] \tag{2.17}$$

for all x, y and $0 \leq t \leq d(x,y)$, from which one formally argues that d is a candidate to be the required solution of the eikonal equation.
Due to the lack of regularity of the mapping $x \rightarrow \partial H_0(x,0)$ which is, in general, only upper semicontinuous and whose values may have empty interior, some technical refinements to the standard dynamic programming argument, see for example [4], are needed in order to deduce from (2.17) that d is a viscosity solution of the eikonal equation (2.7). Namely we consider for $\delta > 0$ the differential inclusion

$$\dot{X}^\delta(t) \in \partial H_0(X^\delta(t), 0) + B(0, \delta)$$

and the corresponding regularized distances d^δ and show by a stability argument that $d \equiv \sup_{\delta > 0} d^\delta$ is actually a viscosity solution of (2.7).
This last step relies, of course, on the well - known duality relation

$$H_0(x,p) = \sup_{q \in \partial H_0(x,0)} p \cdot q$$

Concerning Theorem 2.2, let us first proceed heuristically by assuming that (2.7) has smooth solutions $d(x) = d(x,y)$ and look for solutions of (2.8) of the form

$$v^y(x,t) = g(y) + t\Psi\left(\frac{d(x;y)}{t}\right)$$

where $y \in \mathbb{R}^N$ plays the role of a parameter and Ψ is a smooth function to be appropriately selected. A simple computation shows that

$$v_t^y(x,t) = \Psi(\tau) - \tau\Psi'(\tau) \ , \ D_x v^y(x,t) = \Psi'(\tau) D_x d(x,y)$$

where $\tau = \frac{d(x,y)}{t}$. If v^y has to be a solution of (2.8), then necessarily

$$\Psi(\tau) - \tau\Psi'(\tau) + \Phi(H_o(x, \Psi'(\tau) D_x d)) = 0 \ .$$

For strictly increasing Ψ, the positive homogeneity of H_0, see assumption (2.2), and the fact that d solves (2.7) yield

$$\Psi(\tau) - \tau\Psi'(\tau) + \Phi(\Psi'(\tau)) = 0 \ . \tag{2.18}$$

Since the solution of the Clairaut's differential equation (2.18) is $\Psi = \Phi^*$, by the above heuristics we are lead to look at the following family of special solutions

$$v^y(x,t) = g(y) + t\Phi^*\left(\frac{d(x;y)}{t}\right) \tag{2.19}$$

of (2.8). It is not hard to realize that the envelope procedure originally proposed by E. Hopf [12], namely to take

$$\inf_{y \in \mathbb{R}^N} v^y(x,t)$$

which defines indeed the Hopf -Lax function (2.12), preserves, at least at points of differentiability of u, the fact that each v^y satisfies (2.8) and also enforces the matching of the initial condition in the limit as t tends to 0^+.

The rigorous implementation of the Hopf's method in our setting is made up of three basic steps. The first one is to show, for non smooth convex Φ, that the functions v^y defined in (2.19), which are in general just Hölder - continuous, do satisfy (2.8) for each y in the viscosity sense. This requires, in particular, to work with regular approximations of Φ, namely

$$\Phi_\delta(s) = \Phi(s) + \frac{\delta}{2}s^2 \ , \ \delta > 0$$

and to use the standard reciprocity formula of convex analysis

$$\Phi_\delta\left((\Phi_\delta^*)'(s)\right) + \Phi_\delta^*(s) = s(\Phi_\delta^*)'(s) \ .$$

The second step of the proof is to show that u is lower semicontinuous and solves (2.8) in the viscosity sense. A crucial tool to achieve this is the use of the stability properties of viscosity solutions with respect to inf and sup operations.
Observe that, since $p \longrightarrow \Phi\left(H_0(x,p)\right)$ is convex, it is enough at this purpose to check that

$$\lambda + H(x,\eta) = 0 \quad \forall(\eta,\lambda) \in D^- u(x,t)$$

at any (x,t), where $D^- u(x,t)$ is the subdifferential of u at (x,t), see [6].
The third step is to check the initial condition (2.9) and the fact that u is bounded below by a function of linear growth; this is performed much in the same way as in [2]. The uniqueness assertion is a consequence of a result in [6].

Let us conclude this section by exhibiting a few examples of Hamiltonians H_0 to which Theorem 2.1 and Theorem 2.2 do apply.

Example 1. A class of examples is given by Hamiltonians of the form

$$H_0(x,p) = |A(x)p|_\alpha$$

where $A(x)$ is a symmetric positive definite $N \times N$ matrix with suitable conditions on the x - dependence and $|p|_\alpha = (\sum_{i=1}^N |p_i|^\alpha)^{\frac{1}{\alpha}}$, $\alpha \geq 1$.
Condition (2.5) is trivially satisfied and the minimum length of commutators needed is $k = 1$ for all $x \in \mathbb{R}^N$. On the other hand, condition (2.4) is fulfilled, for sufficiently small $\epsilon > 0$, with $M = N$ and $\sigma(x) = A(x)$. In this setting, the eikonal equation (2.7) is solved by a Riemannian metric, see [13], [18].

Example 2. A different kind of examples is provided by degenerate Hamiltonians of the form

$$H_0(x,p) = |A(x)p|_\alpha$$

where $A(x)$ is an $N \times M$ matrix with $M < N$ whose columns satisfy the Chow - Hörmander rank condition (2.5). A simple convex duality argument shows that assumption (2.4) holds with $\sigma(x) = A^*(x)$ for sufficiently small $\epsilon > 0$. The associated Carnot - Carathéodory metrics and their relations with eikonal equations have been recently investigated in a different functional setting in [16]. An interesting particular case (here $N = 3$ to simplify notations) is

$$H_0(x,p) = \left((p_1 - \frac{x_2}{2}p_3)^\alpha + (p_2 + \frac{x_1}{2}p_3)^\alpha\right)^{\frac{1}{\alpha}}$$

A simple computation shows that condition (2.5) holds with $k = 2$. Note that the control system (2.15) reduces in the present case to that of the well - known Brockett's system in nonlinear control theory, see [9]. Our Hopf - Lax formula (2.12) coincides in this case with the one the recently found for this example with $\alpha = 2$ in [14].

3 Hopf - Lax formula and convolutions

An interesting but not evident relationship exists between the Hopf - Lax formula, the inf - convolution in the sense of Yosida - Moreau and the classical integral convolution procedure. Let us illustrate this with reference to the Cauchy problem

$$u_t(x,t) + \frac{1}{2}|Du|^2 = 0 \text{ in } \mathbb{R}^N \times (0,+\infty) \tag{3.1}$$

$$u(x,0) = g(x) \text{ in } \mathbb{R}^N. \tag{3.2}$$

In this case

$$H_0(x,p) = \frac{1}{2}|p|_2,\ \Phi(s) = \frac{1}{2}s^2 \equiv \Phi^*(s),\ d(x,y) = |x-y|_2$$

and the Hopf - Lax function (2.12) becomes then

$$u(x,t) = \inf_{y \in \mathbb{R}^N} \left[g(y) + \frac{|x-y|^2}{2t} \right] \tag{3.3}$$

that is the inf - convolution of the initial datum g, see for example [4] for further informations. Assume that g is continuous and bounded and consider the parabolic regularization of the Cauchy problem (3.1), (3.2), that is

$$u_t^\epsilon - \epsilon \Delta u^\epsilon + \frac{1}{2}|Du^\epsilon|^2 = 0\ ,\ \ u^\epsilon(x,0) = g(x) \tag{3.4}$$

where ϵ is a positive parameter. A direct computation shows that if u^ϵ is a smooth solution of the above, then its Hopf - Cole transform

$$w^\epsilon = e^{-\frac{u^\epsilon}{2\epsilon}} \tag{3.5}$$

satisfies the linear heat problem

$$w_t^\epsilon - \epsilon \Delta w^\epsilon = 0 \ , \quad w^\epsilon(x,0) = g^\epsilon(x) = e^{-\frac{g(x)}{2\epsilon}} \ . \tag{3.6}$$

By classical linear theory, see [11] for example, its solution w^ϵ can be expressed as the convolution $w^\epsilon = \Gamma \star g^\epsilon$ where Γ is the fundamental solution of the heat equation, that is

$$w^\epsilon(x,t) = (4\pi\epsilon t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4\epsilon t}} e^{-\frac{g(x)}{2\epsilon}} dy$$

Hence, by inverting (3.5), the function

$$u^\epsilon(x,t) = -2\epsilon \ \log\left((4\pi\epsilon t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4\epsilon t}} e^{-\frac{g(x)}{2\epsilon}} dy \right) \tag{3.7}$$

turns out to be a solution of the quasilinear problem (3.4).

It is natural to expect that the solutions u^ϵ of (3.4) should converge, as $\epsilon \to 0^+$, to the solution of

$$u_t + \frac{1}{2}|Du|^2 = 0 \ \ , \ \ u(x,0) = g(x)$$

given by (3.3).

We have indeed the following result which shows, in particular, how the inf - convolution can be regarded, roughly speaking, as a singular limit of integral convolutions:

Theorem 3.1 *Assume that g is bounded. Then,*

$$\lim_{\epsilon \to 0^+} -2\epsilon \ \log\left((4\pi\epsilon t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4\epsilon t}} e^{-\frac{g(x)}{2\epsilon}} dy \right) = \inf_{y \in \mathbb{R}^N} \left[g(y) + \frac{|x-y|^2}{2t} \right] \tag{3.8}$$

The proof can be obtained by a direct application of a general large deviations result by S.N. Varadhan. Consider at this purpose the family of probability measures $P_{x,t}^\epsilon$ defined on Borel subsets of $\mathbb{R}^N$ by

$$P_{x,t}^\epsilon(B) = (4\pi\epsilon t)^{-\frac{N}{2}} \int_B e^{-\frac{|x-y|^2}{4\epsilon t}} dy$$

and the function

$$I_{x,t}(y) = \frac{|x-y|^2}{4t} .$$

It is not hard to check that, for all fixed x and t, the family $P^\epsilon_{x,t}$ satisfies the large deviation principle, see Definition 2.1 in [19], with rate function $I_{x,t}$.
By Theorem 2.2 in [19], then

$$\lim_{\epsilon \to 0^+} \epsilon \log \left(\int_{\mathbb{R}^N} e^{\frac{F(y)}{\epsilon}} dP^\epsilon_{x,t}(y) \right) = \sup_{y \in \mathbb{R}^N} [F(y) - I(y)]$$

for any bounded continuous function F. The choice $F = -\frac{g}{2}$ in the above shows then the validity of the limit relation (3.8).

The same convergence result can be proved also by purely PDE methods. Uniform estimates for the solutions of (3.4) and compactness arguments show the existence of a limit function u solving (3.1), (3.2) in the viscosity sense. Uniqueness results for viscosity solutions allow then to identify the limit u as the Hopf - Lax function, see [13], [4].

The way of deriving the Hopf - Lax function via the Hopf - Cole transform and the large deviations principle is closely related to the Maslov's approach [15] to Hamilton - Jacobi equations based on idempotent analysis. In that approach, the base field $\mathbb{R}$ of ordinary calculus is replaced by the semiring $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ with operations $a \oplus b = \min\{a, b\}$, $a \odot b = a + b$. A more detailed description of this relationship is beyond the scope of this paper; let us only observe in this respect that the nonsmooth operation $a \oplus b$ has the smooth approximation

$$a \oplus b = \lim_{\epsilon \to 0^+} -\epsilon \log \left(e^{-\frac{a}{\epsilon}} + e^{-\frac{b}{\epsilon}} \right).$$

A final remark is that the Hopf - Cole transform can be also used to deal with the parabolic regularization of more general Hamilton - Jacobi equations such as

$$u_t + \frac{1}{2}|\sigma(x)Du|^2 = 0$$

where σ is a given $N \times M$ matrix satisfying (2.5), provided the regularizing second order operator is chosen appropriately. Indeed, if one looks at the regularized problem

$$u^\epsilon_t - \epsilon \,\mathrm{div}\,(\sigma^*(x)\sigma(x)Du^\epsilon) + \frac{1}{2}|\sigma(x)Du^\epsilon|^2 = 0 ,$$

then the Hopf - Cole transform $w^\epsilon = e^{-\frac{u^\epsilon}{2\epsilon}}$ solves the linear subelliptic equation

$$w_t^\epsilon - \epsilon \operatorname{div}(\sigma^*(x)\sigma(x)Dw^\epsilon) = 0 \ .$$

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