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Fully Nonlinear Oblique Derivative Problems for Singular Degenerate Parabolic Equations

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1. Capillary Boundary Condition

Following a recent joint work with M.-H. Sato, we discuss about the nonlinear Neumann type boundary value problems for singular parabolic partial differential equations.

Motion of hypersurfaces $\{\Gamma_t\}_{t\geq 0}$, which is confined in the closure of a bounded domain $\Omega\subset \mathbf{R}^n$, arises in many applications, and has been studied extensively in the past.

A characteristic of such a motion $\{\Gamma_t\}_{t\geq 0}$ is the needed description of the behavior of its boundary $\partial\Gamma_t$ and a typical situation is that $\partial\Gamma_t$ stays on $\partial\Omega$ and satisfies an appropriate geometrical condition.

A typical example of such a geometrical condition is the Capillary boundary condition (or the prescribed contact angle condition), which we address here.

We adapt here the level set approach, so the hypersurface is given as a level set $\Gamma_t=\{x\mid u(x)=c\}$ of a function $u\in C(\overline{\Omega})$, with $c\in\mathbf{R}$. If the contact angle between $\partial\Omega$ and Γ_t is given by $\gamma\in(0,\pi/2]$, then

$$|Du(x)|^{-1}Du(x)\cdot\nu(x)=\cos\gamma\quad \text{for } x\in\Gamma_t\cap\partial\Omega,$$

or equivalently,

$$\frac{\partial u}{\partial\nu}=\cos\gamma|Du|\quad \text{for } x\in\Gamma_t\cap\partial\Omega.$$

Here $\nu(z)$ denotes the unit outer normal vector of Ω at $z\in\partial\Omega$.

If every hypersurface $\{x\mid u(x)=c\}$, with $c\in\mathbf{R}$, is moved by its mean curvature, then the function u, which now depends not only on the space variable x but also on the time variable $t\geq 0$, should satisfy

$$\begin{cases} u_t=\operatorname{tr}\left[(I-\overline{Du}\otimes\overline{Du})D^2u\right] & \text{for } (t,x)\in(0,\infty)\times\Omega,\\ \dfrac{\partial u}{\partial\nu}=\cos\gamma|Du| & \text{for } (t,x)\in(0,\infty)\times\partial\Omega,\end{cases}$$

where $\gamma : \partial\Omega \to (0, \pi/2]$ and $\overline{p} := p/|p|$ for $p \neq 0$.

Thus, the fundamental mathematical task is to establish the existence and uniqueness of a solution of the initial-boundary value problem

$$\begin{cases} u_t = \operatorname{tr}\left[(I - \overline{Du} \otimes \overline{Du})D^2 u\right] & \text{for } (t,x) \in (0,\infty) \times \Omega, \\ \dfrac{\partial u}{\partial \nu} = \cos\gamma |Du| & \text{for } (t,x) \in (0,\infty) \times \partial\Omega, \\ u(x,0) = g(x) & \text{for } x \in \overline{\Omega}. \end{cases}$$

We present here a comparison and an existence theorems obtained in [IS] which are applicable to the above initial-boundary value problem.

2. Main Results

In what follows we deal with the following boundary value problem

$$u_t + F(t,x,u,Du,D^2u) = 0 \quad \text{in } (0,T) \times \Omega, \tag{1}$$

$$B(x, Du) = 0 \quad \text{in } (0,T) \times \partial\Omega, \tag{2}$$

where $T > 0$ is a fixed number.

We always assume that Ω is a bounded domain in $\mathbf{R}^n$ with C^1 boundary. However, the results below are still valid for certain Lipschitz domains Ω under appropriate interpretations.

Let us give a list of the assumptions on F and B . Henceforth, for $p, q \in \mathbf{R}^n \setminus \{0\}$ we write

$$\rho(p,q) = \left[(|p| \wedge |q|)^{-1}|p - q|\right] \wedge 1.$$

Here and below, we use the notation: $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Let $\mathcal{S}^n$ denote the space of $n \times n$ real symmetric matrices equipped with the usual ordering.

(F1) $F \in C([0,T] \times \overline{\Omega} \times \mathbf{R} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n)$.

(F2) There exists a constant $\gamma \in \mathbf{R}$ such that for each $(t,x,p,X) \in [0,T] \times \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n$ the function $u \mapsto F(t,x,u,p,X) - \gamma u$ is non-decreasing on $\mathbf{R}$.

(F3) For each $R > 0$ there exists a continuous function $\omega_R : [0,\infty) \to [0,\infty)$ satisfying $\omega_R(0) = 0$ such that if $X, Y \in \mathcal{S}^n$ and $\mu_1, \mu_2 \in [0,\infty)$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then

$$\begin{aligned} F(t,x,u,p,X) - F(t,y,u,q,-Y) \geq & -\omega_R(\mu_1(|x-y|^2 + \rho(p,q)^2) \\ & + \mu_2 + |p-q| + |x-y|(|p| \vee |q| + 1)) \end{aligned}$$

for all $t \in [0,T]$, $x, y \in \overline{\Omega}$, $u \in \mathbf{R}$, with $|u| \le R$, and $p, q \in \mathbf{R}^n \setminus \{0\}$.

(B1) $B \in C(\mathbf{R}^n \times \mathbf{R}^n) \cap C^{1,1}(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$.

(B2) For each $x \in \mathbf{R}^n$ the function $p \mapsto B(x,p)$ is positively homogeneous of degree one in p, i.e., $B(x, \lambda p) = \lambda B(x,p)$ for all $\lambda \ge 0$ and $p \in \mathbf{R}^n$.

(B3) There exists a positive constant θ such that $\langle \nu(z), D_pB(z,p)\rangle \ge \theta$ for all $z \in \partial\Omega$ and $p \in \mathbf{R}^n \setminus \{0\}$.

Theorem 1. *Suppose that (F1)–(F3) and (B1)–(B3) hold. Let $u \in \mathrm{USC}([0,T) \times \overline{\Omega})$ and $v \in \mathrm{LSC}([0,T) \times \overline{\Omega})$ be, respectively, viscosity sub- and supersolutions of (1)–(2). If $u(0,x) \le v(0,x)$ for $x \in \overline{\Omega}$, then $u \le v$ on $(0,T) \times \overline{\Omega}$.*

Under the above assumptions

$$-\infty < F_*(t,x,u,0,0) = F^*(t,x,u,0,0) < \infty$$

holds for all $(t,x,u) \in [0,T] \times \overline{\Omega} \times \mathbf{R}$.

Key observations for the proof of Theorem 1 are in the following lemmas.

Lemma 1. *Assume that (B1) and (B3) hold. For any $\varepsilon \in (0,1)$ there exists a function $\psi \in C^\infty(\overline{\Omega})$ satisfying the properties:*

$D\psi(x) \ne 0$ *for all* $x \in \partial\Omega$,

$\psi(x) \ge 0$ *for all* $x \in \overline{\Omega}$,

$\langle \nu(x), D\psi(x)\rangle \ge (1-\varepsilon)|D\psi(x)|$ *for all* $x \in \partial\Omega$,

and

$$\langle D_pB(x,p), D\psi(x)\rangle \ge 1 \quad \text{for all } (x,p) \in \partial\Omega \times (\mathbf{R}^n \setminus \{0\}).$$

Lemma 2. *Assume that (B1)–(B3) hold. There are a function $w \in C^{1,1}(\overline{\Omega} \times \overline{\Omega})$ and a positive constant C such that for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,*

(i) $|x-y|^4 \le w(x,y) \le C|x-y|^4$,

$|D_xw(x,y)| \vee |D_yw(x,y)| \le C|x-y|^3$,

(ii) $B(x, D_xw(x,y)) \ge 0 \quad$ *if* $x \in \partial\Omega$,

$B(y, -D_yw(x,y)) \ge 0 \quad$ *if* $y \in \partial\Omega$,

(iii) $|D_xw(x,y) + D_yw(x,y)| \le C|x-y|^4$,

$\rho(D_xw(x,y), -D_yw(x,y)) \le C|x-y| \quad$ *if* $x \ne y$,

and for a. e. $(x,y) \in \overline{\Omega} \times \overline{\Omega}$,

(iv) $D^2w(x,y) \le C\left\{|x-y|^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + |x-y|^4 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right\}.$

Regarding the existence of a solution, the main result is:

Theorem 2. *Assume that (F1)–(F3) and (B1)–(B3) hold. Then for each* $g \in C(\overline{\Omega})$ *there is a (unique) viscosity solution* $u \in C([0,T) \times \overline{\Omega})$ *of (1)–(2) satisfying*

$$u(x,0) = g(x) \qquad \text{for } x \in \overline{\Omega}.$$

The uniqueness assertion above is an immediate consequence of Theorem 1. The standard technique based on the Perron method and the construction of sub- and supersolutions is applied to proving Theorem 2.

3. A brief comparison with previous results

One of features in the previous results is that the assumptions allow the function $F(p,X)$ to be discontinuous for $p=0$. In the case when F is continuous in its variables, there are already many comparison and existence results for viscosity solutions of second order degenerate parabolic PDE with boundary condition (1.2). A few of those which are concerned with viscosity solutions are those obtained in [L, I, B1]. [I, B1] are the first work which treated general nonlinear Neumann type boundary value problems for degenerate elliptic and parabolic partial differential equations in the viscosity solutions approach.

In the case of singular PDE like the mean curvature flow equation, [GS] is the first which treated the Neumann problem. More general Neumann type probems are dealt with in [S1, S2, B2]. The results in [B2] are close to Theorems 1 and 2 here. Indeed, the results in [B2] has a better feature compared with our results here. Indeed, the regularity assumption on B in [B2] is weaker than (B1). On the other hand, our regularity assumption on $\partial\Omega$ is weaker than that of [B2].

4. A class of functions F

We examine here that a class of functions F satisfy (F1)–(F3).

Let $A : \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \to M^{n\times m}$, where $M^{n\times m}$ denotes the space of real $n \times m$ matrices. Assume that A is a homogeneous function of degree zero, i.e.,

$$A(x,\lambda p) = A(x,p) \quad \text{for all } (x,p,\lambda) \in \overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}) \times (0,\infty), \tag{3}$$

and satisfies

$$\|A(x,p) - A(y,q)\| \le C_1(|x-y| + |p-q|) \quad \text{for all } x,y \in \overline{\Omega} \text{ and } p,q \in S^{n-1}, \tag{4}$$

where $C_1 > 0$ is a constant.

It follows that for all $x, y \in \overline{\Omega}$ and $p, q \in \mathbf{R}^n \setminus \{0\}$,

$$\begin{aligned}\|A(x,p) - A(y,q)\| &\le C_1\Big(|x-y| + \Big|\frac{p}{|p|} - \frac{q}{|q|}\Big|\Big) \\ &\le C_1\Big(|x-y| + \frac{|p-q|}{|p| \vee |q|}\Big) \\ &\le C_1(|x-y| + 2\rho(p,q)).\end{aligned}$$

Let $b \in C(\overline{\Omega}, \mathbf{R}^n)$ satisfy

$$|b(x) - b(y)| \le C_2|x-y| \quad \text{for all } x, y \in \overline{\Omega}. \tag{5}$$

Furthermore let $c, f \in C(\overline{\Omega}, \mathbf{R})$ be given. Define the function $F \in C(\overline{\Omega} \times \mathbf{R} \times (\mathbf{R}^n \setminus \{0\}) \times \mathcal{S}^n)$ by

$$F(x,u,p,X) = -\operatorname{tr}[A(x,p)A(x,p)^T X] + b(x)\cdot p + c(x)u + f(x).$$

If $X, Y \in \mathcal{S}^n$ and $\mu_1, \mu_2 \in [0, \infty)$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \le \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then

$$\begin{aligned}&-\operatorname{tr}[A(x,p)A(x,p)^T X] - \operatorname{tr}[A(y,q)A(y,q)^T Y] \\ &\le C_3\|A(x,p) - A(y,q)\|^2 \le 4C_3C_1(|x-y|^2 + \rho(p,q)^2).\end{aligned}$$

Thus F satisfies condition (F3). Also, it is immediate to see that condition (F2) is satisfied with $\gamma \le \min_{\overline{\Omega}} c$.

If $A(x,p) = I - |p|^{-2}(p \otimes p)$, $b = 0$, and $c = f = 0$, then it is the case of the mean curvature flow equation and the above conditions on A, b, c, and f are valid.

More generally, let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty index sets, and let $A_{\alpha\beta} \in C(\overline{\Omega} \times (\mathbf{R}^n \setminus \{0\}), M^{n\times m})$, $b_{\alpha\beta} \in C(\overline{\Omega}, \mathbf{R}^n)$, $c_{\alpha\beta} \in C(\overline{\Omega})$, and $f_{\alpha\beta} \in C(\overline{\Omega})$, with $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$, be given. Assume that these sets of functions are uniformly bounded, that $\{c_{\alpha\beta}\}$ and $\{f_{\alpha\beta}\}$ are equi-continuous, that $\{A_{\alpha\beta}\}$ satisfies (3) and (4) with a uniform constant C_1, and that $\{b_{\alpha\beta}\}$ is equi-Lipschitz continuous (i.e., satisfies (5) with a uniform constant C_2. Define

$$F_{\alpha\beta}(x,u,p,X) = -\operatorname{tr}[A_{\alpha\beta}(x,p)A_{\alpha\beta}^T(x,p)X] + b_{\alpha\beta}(x)\cdot p + c_{\alpha\beta}(x)u + f_{\alpha\beta}(x),$$

and

$$F(x,u,p,X) = \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} F_{\alpha\beta}(x,u,p,X).$$

Then the function F satisfies (F1)–(F3).

5. Functions B

In this section we examine functions B which describes the boundary condition. Consider the function B of the form

$$B(x,p) = \mu(x) \cdot p - |C(x)p|,$$

where $\mu : \mathbf{R}^n \to \mathbf{R}^n$ is a $C^{1,1}$ vector field over $\mathbf{R}^n$ and $C : \mathbf{R}^n \to M^{n\times n}$ is a $C^{1,1}$ function satisfying $\det C(x) \neq 0$ in a neighborhood of $\partial\Omega$. It is clear that (B2) is satisfied. We can modify the definition of B so that the resulting function $\widetilde{B}$ satisfies (B1) and $\widetilde{B}(x,\cdot) = B(x,\cdot)$ for all x in a neighborhood of $\partial\Omega$.

As before let $\nu(x)$ denote the unit outer normal of Ω at $x \in \partial\Omega$. By calculation, we have

$$D_pB(x,p) = \mu(x) - \frac{C(x)^TC(x)p}{|C(x)p|} \quad \text{if } p \neq 0,$$

and we see that (B3) is equivalent to the condition

$$\mu(x) \cdot \nu(x) > \xi \cdot C(x)\nu(x) \quad \text{for all } (x,\xi) \in \partial\Omega \times S^{n-1}.$$

A particular case is when $\mu = \nu$ and $C(x) = a(x)I$ for some $a \in C^{1,1}(\mathbf{R}^n)$ such that $0 < a(x) < 1$ for $x \in \partial\Omega$, which corresponds to the Capillary condition. In this case the boundary regularity of Ω should be of class $C^{2,1}$ so that $\mu = \nu \in C^{1,1}(\mathbf{R}^n)$ is satisfied, which is one of requirements of Theorems 1 and 2. It is interesting to find that the results in [B2] need the same $C^{2,1}$ regularity of the boundary.

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