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GEOMETRIC ASPECTS OF HÖLDER AND L^p ESTIMATES

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Abstract

In this note we will discuss some of the geometry of Hölder and L^p estimates for elliptic equations. We will also show that a probabilistic view point for L^p estimates.

1. Introduction

We will use standard notations. $B_r = \{x \in \mathbb{R}^n : |x| < r\}, Q_r = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : -r < x_i < r\}$ and $B_r(x) = B_r + x$, $Q_r(x) = Q_r + x$. For any measurable set A, $|A|$ is its measure. For any integrable function u, we denote the average of u as

$$\bar{u}_A = ⨍_A u = \frac{1}{|A|} \int_A u.$$

The classical Hölder estimates of elliptic equations is the following Schauder estimates, see [5], 1909 or [7], 1934.

Theorem 1 (Korn-Schauder). *If u is a solution of*

$$\Delta u = f \qquad \text{in } B_2 \tag{1}$$

then

$$|\mathbf{D}^2 u|_{C^\alpha(B_1)} \le C\left(|f|_{C^\alpha(B_2)} + \|u\|_{L^\infty(B_2)}\right) \qquad \text{for any } 0 < \alpha < 1. \tag{2}$$

There are many proofs for this theorem and we will sketch some of the proofs here but we will emphasis the interplay between the geometry of the equation and the geometry of the functions.

The classical Calderón–Zygmund estimates established in [3] 1952.

Theorem 2 (Calderón–Zygmund). *If u is a solution of (1) then*

$$\int_{B_1} |\mathbf{D}^2 u|^p \le C\left(\int_{B_2} |f|^p + \int_{B_2} u^p\right) \qquad \text{for any } 1 < p < +\infty. \tag{3}$$

These estimates are among the most fundamental estimates for elliptic equations.

The classical proof of Calderón–Zygmund estimates, uses the singular integrals

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} w_{ij}(y) f(x-y)\, dy \tag{4}$$

where w_{ij} is a homogeneous function of degree $-n$ with cancellation conditions. The approach involves an L^2–L^2 estimate and an L^1 to weak-L^1 estimate. See details in the book of Stein [9].

Our approach is more elementary. It gives an unified proof for elliptic, parabolic and subelliptic operators. Our proof is built upon geometrical intuitions. Our basic tools in this approach are the standard estimates for the the Vitali covering lemma and Hardy–Littlewood maximal function.

Our approach is very much influenced by [2] and the early works in [1] and [10], in which the Calderón–Zygmund decompositions were used. Here we will use the Vitali covering lemma. Analytically the difference between the Calderón–Zygmund decomposition and Vitali covering lemma is not quite essential but subtle. One is on cubes and the latter is on balls. However we hope that Vitali covering lemma can easily adapted to more complicated situations since balls can be easily defined on manifolds.

2. The geometry of functions and sets

Hölder spaces. We should start out with a geometric description of Hölder space which is the key to visualize the estimates.

First of all, the geometry of $\|u\|_{L^\infty(B_1)} \leq 1$ is that the graph of u is in the box $B_1 \times [-1, 1]$. This gives a very mild control of u.

The Hölder norm of u is actually very geometrical. Let us recall that u is *Hölder with* $[u]_{C^\alpha} \leq 1$ if

$$|u(x) - u(y)| \leq |x - y|^\alpha \qquad \text{for all } x \text{ and } y, \text{ say, in} B_1. \tag{5}$$

Geometrically, the graph of u is not a box anymore rather than a surface which is away from spikes: $|x|^\alpha$. That is, if (x_0, y_0) is on the graph of u, then all the points (x, y) with $y - y_0 > |x - x_0|^\alpha$ is not on the graph.

Now let see how this help us to understand the PDE.

Actually, one important observation already comes from these considerations. The estimates of u in Hölder is actually saying that u's graph is more and more concentrated to a single value. The concentration is in a precise controlable fashion. Similarly, estimates of u in $C^{1,\alpha}$ or $C^{2,\alpha}$ will say that u's graph is more and more look like a linear function or a second order polynomial.

This is the geometry of Hölder spaces.

Let us examine the local geometry of the equation. Equation (1) is translation invariant and scaling invariant as:

$$\Delta u(x + x_0) = f(x + x_0)$$

and

$$\Delta u(rx) = r^2 f(rx).$$

The first invariance says that all estimates at different points are equivalent and the second one says that the equation satisfies similar equations in different scales and the right hand side are increasingly regular (or small) as $r \to 0$. We also see that the scaling limit is a harmonic function.

Now we put these two geometries together. The goal is to prove more concentration of the graph of u. And by the scaling, one can achieve that by showing the graph is more concentrated in B_{r_0} than that in B_1. An iteration of this very fact will imply more and more concentration of the graph in $B_{r_0^2}$, $B_{r_0^3}$ ... and so on. The scaling of the PDE enable us to perform this iterations and the Hólder regularity reduced to one step concentration only: from B_1 to B_{r_0}.

Now we see how these get implemented.

Lemma 1. *If u is a solution in B_1 of (1), and h is the harmonic function with $h = u$ on ∂B_1, then*

$$|u(x) - h(x)| \leq \frac{1}{2n}(1 - |x|^2)\|f\|_{L^\infty(B_1)}.$$

We will omit the proof since since it is the standard maximum principle.

The geometry of this lemma says the the graph of u is very close to a graph of a harmonic function. Actually, one can arrange it as close as one wants by arranging f small. An immediate consequence of the lemma is the following.

Corollary 1. *For any $0 < \alpha < 1$, there are positive universal constants $r_0 < 1$ and ϵ_0 so that if u is a solution in B_1 of (1) with $|u| \le 1$ and $|f| \le \epsilon_0$ then there is a constant A(we can take $A = h(0)$, h is the harmonic function in the previous lemma), so that*

$$|u(x) - A| \le r_0^\alpha \text{ for } x \in B_{r_0}.$$

An iteration of the corollary gives that there are constants A_k so that

$$|u(x) - A_k| \le r_0^{k\alpha} \text{ for } x \in B_{r_0^k}$$

with smallness conditions on f. A_k is clearly convergent as geometric series and the Hölder norm estimates follows.

One can also prove the $C^{2,\alpha}$ estimates using second order approximation instead of constant approximation.

Lemma 2. *For each $0 < \alpha < 1$, there are positive universal constants $r_0 < 1$ and ϵ_0 so that if u is a solution in B_1 of (1) with $|u| \le 1$ and $|f|_{L^\infty(B_1)} \le \varepsilon_0$, then there is a harmonic polynomial $h(x)$ so that*

$$\|u - p\|_{L^\infty(B_{r_0})} \le r_0^{2+\alpha} \|u\|_{L^\infty(B_1)}. \tag{6}$$

We will refer the readers to [1] or [10] for details.

A more important fact of C^α space is that one can show the decay of $u(rx)$ with respect to many other norms such as the L^p norms. This fact is beautifully stated in the Campanato Embedding theorem:

$$\sup_{x\in\Omega} \inf_{c\in\mathbb{R}^1} \sup_{0<r<1} \frac{1}{r^\alpha} \left(\fint_{B_r(x)\cap\Omega} |u - c|^p \right)^{\frac{1}{p}} \sim [u]_{C^\alpha}. \tag{7}$$

This theorem says that we can understand if function is in Hölder function not only in pointwise sense, but also in L^p norms. Here we notice that only the averages not the L^p norm measure the *invariant smallness* of a function.

The geometry of u in L^∞ is clear visible however, once equipped with the above theorem, we have enormouse freedom to visualize the Hölder norm of u in all kinds of different norms. This L^p picture of Hölder norm is particularly important for nonlinear equations, such as minimal surfaces and harmonic maps.

For example one can prove Schauder estimates by the standard energy estimates outlined below.

Lemma 3. *For each $0 < \alpha < 1$, there are positive universal constants $r_0 < 1$ and ϵ_0 such that if*

$$\fint_{B_1} |f|^2 \le \varepsilon_0^2$$

then there is a second order harmonic polynomial $p(x)$ so that

$$\fint_{B_{r_0}} |u - p|^2 \le r_0^{2(2+\alpha)}. \tag{8}$$

The proof of this is almost the same as above.

L^p spaces. The information carried by L^p norm of a function is not as local as by the Hölder norms. In contrast to Hölder spaces, there is no way to say a function is like a L^p function at a point.

In order to examine the information carried by L^p norm of a function, let us recall the formula,

$$\int_\Omega |u|^p \, dx = p \int_0^\infty t^{p-1} \left|\{x \in \Omega : |u| > t\}\right| dt. \tag{9}$$

If $\int_\Omega |u|^p \, dx = 1$, one will have that

$$|\{x \in \Omega : |u| > \lambda\}| \leq \frac{1}{\lambda^p}, \tag{10}$$

i.e., the measure

$$|\{x :\in \Omega : |u| > \lambda\}|$$

is small for λ large. This tells us that if we randomly choose a point x, then the probability for $|u(x)| > \lambda$ is small for λ large. The identity (9) shows the decay of $|\{x :\in \Omega : |u| > \lambda\}|$ in a precise way and this decay is the only information carried by the L^p norm. We also observe that the faster this probability decays the bigger the p is.

Now let us discuss how we can show that a function is in L^p.

First we see that one has to prove the decay of $|\{|u| > \lambda\}|$. As in the Hölder estimates, we should prove this decay inductively. A reasonable argument of this sort is to prove:

$$|\{|u| > \lambda_0\}| \leq \varepsilon \, |\{|u| > 1\}| \, . \tag{11}$$

The smaller ϵ or $\lambda_0 - 1$ is, the faster the decay is. Here one should realize that this estimate should be scaled to

$$|\{|u| > \lambda_0 \lambda\}| \leq \varepsilon \, |\{|u| > \lambda\}| \tag{12}$$

with proper conditions on the data. As in the Hölder space case, one should expect that an inductive argument proves the decay.

The $W^{2,p}$ theory of (1) says that $\mathbf{D}^2 u$ is in L^p if Δu is. Hence a reasonable expectation of an inductive estimate could be

$$\left|\left\{|\mathbf{D}^2 u| > \lambda_0\right\}\right| \leq \varepsilon \left(\left|\left\{|\mathbf{D}^2 u| > 1\right\}\right| + |\{|f| > \delta_0\}|\right). \tag{13}$$

Here we can scale (13) to

$$\left|\left\{|\mathbf{D}^2 u| > \lambda_0 \lambda\right\}\right| \leq \varepsilon \left(\left|\left\{|\mathbf{D}^2 u| > \lambda\right\}\right| + |\{|f| \geq \delta_0 \lambda\}|\right) \tag{14}$$

which is the so-called *good-λ inequality.* One can easily show the L^p estimates if (13) were true for fixed $\lambda_1 > 1$ and ϵ small. (13), however, is not true. One reason for the failure of (13) is that the condition

$$|\mathbf{D}^2 u(x_0)| \leq 1 \tag{15}$$

is unstable in the setting of $W^{2,p}$ theory.

Although (13) is not true, its modification (19) below is true.

The key modification is provided by one of the treasures in analysis, the Hardy–Littlewood maximal function.

For a locally integrable function v defined in $\mathbb{R}^n$, its maximal function is defined as

$$\mathcal{M}v(x) = \sup_{r>0} \fint_{B_r(x)} |v| \, d\mathcal{L}^n. \tag{16}$$

We also use

$$\mathcal{M}_\Omega v(x) = \mathcal{M}(v\chi_\Omega)(x),$$

if v is not defined outside Ω or equivalently we replace or extend v by 0 outside Ω. We will drop the index Ω if Ω is understood clearly in the context. We can also define the maximal function by taking the supremum in cubes.

$$\bar{\mathcal{M}}v(x) = \sup_{Q_r(x)} \fint_{Q_r(x)} |v| \, d\mathcal{L}^n. \tag{17}$$

It is clear that,

$$\mathcal{M}v \le C\bar{\mathcal{M}}v \le C\mathcal{M}v.$$

We will use the maximal function $\mathcal{M}v$ defined in (16) on balls in this paper.

The basic theorem for Hardy–Littlewood maximal function is the following:

Theorem 3.

$$\begin{aligned} \|\mathcal{M}(v)(x)\|_{L^p(\Omega)} &\le C\|v\|_{L^p(\Omega)} \qquad \textit{for any } 1 < p \le +\infty. \\ |\{x \in \Omega : \mathcal{M}v(x) \ge \lambda\}| &\le \frac{C}{\lambda}\|v\|_{L^1(\Omega)}. \end{aligned}$$

The first inequality is call strong $p-p$ estimates and the second is call weak $1-1$ estimates. This theorem says that the measures of $\{|v(x)| > \lambda\}$ and $\{\mathcal{M}v(x) > \lambda\}$ decay roughly in the same way. However $\mathcal{M}u(x) \le 1$ is much more stable and geometrical than $|u(x)| \le 1$ if u is merely an L^p function. The reason is that $\mathcal{M}u$ is invariant with respect to scaling. Another aspect of the maximal function is that $\{\mathcal{M}u \ge \lambda\}$ and $\{|u| \ge \lambda\}$ have roughly the same measure.

Likewise we will replace (15) by

$$\left(\mathcal{M}|\mathbf{D}^2u|^2\right)(x) \le 1. \tag{18}$$

If $\mathcal{M}|\mathbf{D}^2u|^2(x_0) \le 1$, one would see that $\mathbf{D}^2u(x)$ is *really* ≤ 1 at x_0 in all scales in the sense of L^2.

In fact we will show that

$$\begin{aligned} \left|\{x \in B_1 : \mathcal{M}|\mathbf{D}^2u|^2 > \lambda_0^2\}\right| \le \varepsilon \big(\left|\{x \in B_1 : \mathcal{M}|\mathbf{D}^2u|^2 > 1\}\right| \\ +|\{x \in B_1 : \mathcal{M}(f^2) > \delta_0^2\}|\big) \end{aligned} \tag{19}$$

where δ_0 can be taken as small as possible since it is about the *data*.

The proof of (19) is based on Vitali lemma and its modification.

Lemma 4 (Vitali). *Let C be a class of balls in $\mathbb{R}^n$ with bounded radius. Then there is a finite or countable sequence $B_i \in C$ of disjoint balls such that*

$$\cup_{B\in C} B \subset \cup_i 5B_i,$$

where $5B_i$ is the ball with the same center as B_i and radius five times big.

We will use the following in this paper.

Theorem 4 (Modified Vitali). *Let $0 < \varepsilon < 1$ and let $C \subset D \subset B_1$ be two measurable sets with $|C| < \varepsilon|B_1|$ and satisfying the following property: for every $x \in B_1$ with $|C \cap B_r(x)| \ge \varepsilon|B_r|$, $B_r(x) \cap B_1 \subset D$. Then $|D| \ge \frac{1}{20^n\varepsilon}|C|$.*

Proof. Since $|C| < \varepsilon|B_1|$, we see that for almost every $x \in C$, there is an r_x so that $|C \cap B_{r_x}(x)| = \varepsilon|B_{r_x}|$ and $|C \cap B_r(x)| < \varepsilon|B_r|$ for all $1 > r > r_x$. By Vitali' covering lemma, there are $x_1, x_2, \cdots$, so that $B_{r_{x_1}}(x_1), B_{r_{x_2}}(x_2), \cdots$ are disjoint and $\cup_k B_{5r_{x_k}}(x_k) \cap B_1 \supset C$.

From the choice of $B_{r_{x_k}}$, we have

$$|C \cap B_{5r_{x_k}}(x_k)| < \epsilon|B_{5r_{x_k}}(x_k)| = 5^n\epsilon|B_{r_{x_k}}(x_k)| = 5^n|C \cap B_{r_{x_k}}(x_k)|.$$

We also notice that

$$|B_{r_{x_k}}(x_k)| \leq 4^n|B_{r_{x_k}}(x_k) \cap B_1|$$

since $x_k \in B_1$ and $r_{x_k} \leq 1$.

Putting everything together,

$$\begin{aligned}
|C| &= |\cup_k B_{5r_{x_k}}(x_k) \cap C| \\
&\leq \sum_k |B_{5r_{x_k}}(x_k) \cap C| \\
&\leq 5^n \sum_k \varepsilon|B_{r_{x_k}}(x_k)| \\
&\leq 20^n \sum_k \varepsilon|B_{r_{x_k}}(x_k) \cap B_1| \\
&= 20^n\varepsilon|\cup B_{r_{x_k}}(x_k) \cap B_1| \\
&\leq 20^n\varepsilon|D|.
\end{aligned}$$

This finishes the proof. □

The proof of (19) will be carried out in next section.

3. Elliptic Equations

Now we prove Theorem 2. We only need to prove it for $p > 2$ since the statement for $p < 2$ follows from the standard duality argument.

The starting point of the estimates is the following classical estimates. See [4], page 317.

Lemma 5. *If*

$$\begin{cases} \Delta u = f & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

then

$$\int_{B_1} |\mathbf{D}^2 u|^2 \leq C \int_{B_1} |f|^2.$$

Lemma 6. *There is a constant N_1 so that for any $\varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$ and if u is a solution of (1) in a domain $\Omega \supset B_4$, with*

$$\left\{\mathcal{M}\left(|f|^2\right) \leq \delta^2\right\} \cap \left\{\mathcal{M}\,|\mathbf{D}^2 u|^2 \leq 1\right\} \cap B_1 \neq \emptyset \tag{20}$$

then

$$\left|\left\{\mathcal{M}\,|\mathbf{D}^2 u|^2 > N_1^2\right\} \cap B_1\right| < \varepsilon\,|B_1|. \tag{21}$$

Proof. From condition (20), we see that there is a point $x_0 \in B_1$ so that

$$\fint_{B_r(x_0)} \left|\mathbf{D}^2 u\right|^2 \le 2^n \quad \text{and} \quad \fint_{B_r(x_0)} |f|^2 \le 2^n \delta^2, \tag{22}$$

for all $B_r(x_0) \subset \Omega$ and consequently we have

$$\fint_{B_4} \left|\mathbf{D}^2 u\right|^2 \le 1 \quad \text{and} \quad \fint_{B_4} |f|^2 \le \delta^2.$$

Then

$$\fint_{B_4} \left|\nabla u - \overline{\nabla u}_{B_4}\right|^2 \le C_1.$$

Let v be the solution of the following equation

$$\begin{cases} \Delta v = & 0 \\ v = & u - \left(\overline{\nabla u}\right)_{B_4} \cdot \mathbf{x} - \bar{u}_{B_4} \text{ on } \partial B_4. \end{cases}$$

Then by the minimality of harmonic function with respect to energy in B_4,

$$\int_{B_4} |\nabla v|^2 \le \int_{B_4} \left|\nabla u - \overline{\nabla u}_{B_4}\right|^2 \le C_1.$$

Now we can use the local $C^{1,1}$ estimates that there is a constant N_0 so that

$$\left\|\mathbf{D}^2 v\right\|^2_{L^\infty(B_3)} \le N_0^2. \tag{23}$$

At the same time we have,

$$\int_{B_3} \left|\mathbf{D}^2 (u - v)\right|^2 \le C \int_{B_4} f^2 \le C\delta^2.$$

From the weak $1-1$ estimate,

$$\begin{aligned} \lambda |\{x \in B_3 : \mathcal{M}_{B_3} |\mathbf{D}^2 (u - v)|^2(x) > \lambda\}| &\le \frac{C}{N_0^2} \int_{B_3} \left|\mathbf{D}^2 (u - v)\right|^2 \\ &\le \frac{C}{N_0^2} \int_{B_4} f^2 \\ &\le C\delta^2. \end{aligned}$$

Consequently,

$$|\{x \in B_1 : \mathcal{M}_{B_3} |\mathbf{D}^2 (u - v)|^2(x) > N_0^2\}| \le C\delta^2.$$

Now we claim that

$$\{x \in B_1 : \mathcal{M}|\mathbf{D}^2 u|^2 > N_1^2\} \subset \{x \in B_1 : \mathcal{M}_{B_3}|\mathbf{D}^2(u - v)|^2 > N_0^2\},$$

where $N_1^2 = max(4N_0^2, 2^n)$.

Actually if $y \in B_3$, then

$$\begin{aligned} \left|\mathbf{D}^2 u(y)\right|^2 &= \left|\mathbf{D}^2 u(y)\right|^2 - 2\left|\mathbf{D}^2 v(y)\right|^2 + 2\left|\mathbf{D}^2 v(y)\right|^2 \\ &\le 2\left|\mathbf{D}^2 u(y) - \mathbf{D}^2 v(y)\right|^2 + 2N_0^2. \end{aligned}$$

Let x be a point in $\left\{x \in B_1 : \mathcal{M}_{B_3}|\mathbf{D}^2(u - v)|^2(x) \le N_0^2\right\}$.

If $r \le 2$ we have $B_r(x) \subset B_3$ and

$$\sup_{r \le 2} \fint_{B_r(x)} |\mathbf{D}^2 u|^2 \le 2\mathcal{M}_{B_3}(|\mathbf{D}^2(u - v)|^2)(x) + 2N_0^2 \le 4N_0^2.$$

Now for $r > 2$, we have $x_0 \in B_r(x) \subset B_{2r}(x_0)$, we have

$$\fint_{B_r(x)} |\mathbf{D}^2 u|^2 \leq \frac{1}{|B_r|} \int_{B_{2r}(x_0)} |\mathbf{D}^2 u|^2 \leq 2^n,$$

where we have used (22). This says that $\mathcal{M}\left(|\mathbf{D}^2 u|^2\right)(x) \leq N_1^2$.

This establishes the claim.

Finally, we have

$$\begin{aligned}\left|\left\{x \in B_1 : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2\right\}\right| &\leq \left|\{x \in B_1 : \mathcal{M}_{B_3}(|\mathbf{D}^2(u-v)|^2 > N_0^2\}\right| \\ &\leq \frac{C}{N_0} \int f^2 \\ &\leq \frac{C\delta^2}{N_0^2} \\ &< C\delta^2 = \varepsilon\,|B_1|,\end{aligned}$$

by taking δ satisfying the last identity above. This completes the proof. □

An immediate consequence of the above lemma is the following corollary.

Corollary 2. *Assume u is a solution in a domain Ω and assume in a ball B so that $4B \subset \Omega$. If $\left|\left\{x : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2\right\} \cap B\right| \geq \varepsilon\,|B|$, then $B \subset \left\{x : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right)(x) > 1\right\} \cup \{\mathcal{M} f^2 > \delta^2\}$.*

The moral of Corollary 2 is that the set $\left\{x : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > 1\right\}$ is bigger than the set $\left\{x : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2\right\}$ modulo $\{\mathcal{M}(f^2) > \delta^2\}$ if $\left|\left\{x : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2\right\} \cap B\right| = \varepsilon\,|B|$. As said in the construction of the Vitali lemma, we will cover a good portion of the set $\left\{x : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2\right\}$ by disjoint balls so that in each of balls the density of the set is ε. As an application of Corollary 2, we will show the decay of the measure of the set $\left\{x : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2\right\}$.

The covering is a careful choice of balls as in Vitali covering lemma.

Corollary 3. *Assume that u is a solution in a domain $\Omega \supset B_4$, with the condition that $\left|\left\{x \in B_1 : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2\right\}\right| \leq \varepsilon |B_1|$. Then for $\varepsilon_1 = 20^n \varepsilon$,*

1. $\left|\left\{x \in B_1 : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2\right\}\right| \leq \varepsilon_1 \left(\left|\left\{x \in B_1 : \mathcal{M}\left(\mathbf{D}^2 u\right)^2(x) > 1\right\}\right| + \left|\left\{x \in B_1 : \mathcal{M}\left(|f|^2\right) > \delta^2\right\}\right|\right).$
2. $\left|\left\{x \in B_1 : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2 \lambda^2\right\}\right| \leq \varepsilon_1 \left(\left|\left\{x \in B_1 : \mathcal{M}\left(\mathbf{D}^2 u\right)^2 > \lambda^2\right\}\right| + \left|\left\{x \in B_1 : \mathcal{M}\left(|f|^2\right) > \delta^2 \lambda^2\right\}\right|\right).$
3. $\left|\left\{x \in B_1 : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > (N_1^2)^k\right\}\right| \leq \sum_{i=1}^k \varepsilon_1^i \left|\left\{x \in B_1 : \mathcal{M}\left(|f|^2\right) > \delta^2 (N_1^2)^{k-i}\right\}\right| + \varepsilon_1^k \left|\left\{x \in B_1 : \mathcal{M}\left(\left|\mathbf{D}^2 f\right|^2\right) > 1\right\}\right|.$

Proof. (1) is a direct consequence of Corollary 6 and Theorem 4 on

$$C = \left\{x \in B_1 : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > N_1^2\right\},$$

$$D = \left\{x \in B_1 : \mathcal{M}\left(\left|\mathbf{D}^2 u\right|^2\right) > 1\right\} \cup \left\{x \in B_1 : M\left(|f|^2\right) > \delta^2\right\}.$$

(2) is obtained by applying (1) to the equation $\Delta\left(\lambda^{-1}u\right) = \lambda^{-1}f$.

(3) is an iteration of (2) by $\lambda = N_1, (N_1)^2, \ldots$. □

Theorem 5. *If $\Delta u = f$ in B_4 then*

$$\int_{B_1} |\mathbf{D}^2 u|^p \leq C \int_{B_4} |f|^p + |u|^p .$$

Proof. Without lose of generality, we may assume that $\|f\|_p$ is small and the measure $|\{x \in B_1 : \mathcal{M}|\mathbf{D}^2u|^2 > N_1^2\}| \leq \varepsilon|B_1|$ by multiplying the function by a small constant. We will show that $\mathcal{M}\left(|\mathbf{D}^2u|^2\right) \in L^{\frac{p}{2}}(B_1)$ from which it follows that $\mathbf{D}^2u \in L^p(B_1)$. Since $f \in L^p$, we have that $\mathcal{M}\left(|f|^2\right) \in L^{\frac{p}{2}}$ with small norm. Suppose

$$\|f\|_{L^p(B_4)} = \delta.$$

Then

$$\sum_{i=1}^{+\infty} (N_1)^{ip} \left|\left\{\mathcal{M}\left(|f|^2\right) > \delta^2 (N_1)^{2i}\right\}\right| \leq \frac{pN^p}{\delta^p(N-1)} \|\mathbf{f}\|^p_{L^p(B_1)} \leq C.$$

Hence

$$\begin{aligned}
\int_{B_1} |\mathbf{D}^2u|^p &\leq \int_{B_1} \left(\mathcal{M}\left(|\mathbf{D}^2u|^2\right)\right)^{\frac{p}{2}} dx \\
&= p\int_0^{\infty} \lambda^{p-1} \left|\{x \in B_1 : \mathcal{M}|\mathbf{D}^2u|^2 \geq \lambda^2\}\right| d\lambda \\
&\leq p\left(|B_1| + \sum_{k=1}^{+\infty} (N_1)^{kp} \left|\left\{x \in B_1 : \mathcal{M}\left|\mathbf{D}^2u\right|^2 > (N_1)^{2k}\right\}\right|\right) \\
&\leq p\left(|B_1| + \sum_{i=1}^{+\infty} N_1^{kp} \sum_{i=1}^{k} \varepsilon_1^i \left|\left\{x \in B_1 : \mathcal{M}\left|f\right|^2 \geq \delta^2 N_1^{2(k-i)}\right\}\right|\right. \\
&\qquad \left. + \sum_{k=1}^{\infty} N_1^{kp}\varepsilon^k \left|\left\{x : \mathcal{M}\left(|D^2u|^2\right) \geq 1\right\}\right|\right) \\
&\leq p\left(|B_1| + \sum_{i=1}^{+\infty} (N_1)^{ip} \varepsilon_1^i \sum_{k\geq i} N_1^{(k-i)p} \left|\left\{x \in B_1 : \mathcal{M}\left|f\right|^2 \geq \delta^2 N_1^{2(k-i)}\right\}\right|\right. \\
&\qquad \left. + \sum_{k=1}^{\infty} N_1^{kp}\varepsilon_1^k \left|\left\{x \in B_1 : \mathcal{M}\left|\mathbf{D}^2u\right|^2 \geq 1\right\}\right|\right) \\
&\leq C,
\end{aligned}$$

if we take ε_1 so that $N_1^p\varepsilon_1 < 1$ and the theorem follows. □

We remark that our methods can be adapted to prove the same result by using the Caldron Zygmund decomposition.

The advantage of Vitali covering lemma is that it holds on any manifolds whereas the Caldron Zygmund decomposition requires cubes which give clean cut in Euclidean spaces which are rare to find on manifolds.

4. $W^{1,p}$ ESTIMATES

One can easily adapt the methods in the preceding section to obtain $W^{1,p}$ estimates of the following type:

Theorem 6. *If*

$$\Delta u = \operatorname{div} \mathbf{f} = \sum_{i=1}^{n} \partial_i f_i \qquad \text{in } B_1$$

then

$$\int_{B_{\frac{1}{2}}} |\nabla u|^p \leq C_p \int_{B_1} |\mathbf{f}|^p + |u|^p \qquad \text{for } 1 < p < +\infty.$$

Theorem 6 is proved by the following elementary energy estimates lemma and the steps as in the previous section.

Lemma 7. *If*

$$\begin{cases} \Delta u = \operatorname{div} \mathbf{f} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

then

$$\int_{B_1} |\nabla u|^2 \leq \int_{B_1} |\mathbf{f}|^2.$$

For a proof of this elementary lemma, see [4], page 297.

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