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Long time averaged reflection force and homogenization of oscillating Neumann boundary conditions.

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1 Introduction

We are interested in solving the homogenization of oscillating Neumann boundary conditions, by using the ergodic type problem on the boundary, namely the existence and uniqueness of the long time averaged reflection force.
Let us begin with the ergodic problem on the boundary. Our claim is that there exists a unique number d such that the following problem is solvable in the framework of the viscosity solution.

$$F(x,\nabla u,\nabla^2 u)=0 \qquad \text{in} \quad \Omega, \tag{1}$$

$$d+<\nabla u,\gamma(x)>-g(x)=0 \qquad \text{on} \quad \partial\Omega, \tag{2}$$

where Ω is a domain in $\mathbf{R}^n$, F is a fully nonlinear uniformly elliptic Hamilton-Jacobi-Bellman (HJB in short) operator:

$$F(x,\nabla u,\nabla^2 u)=\sup_{\alpha\in \mathrm{A}}\{-\sum_{i,j=1}^{n}a_{ij}^{\alpha}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}-\sum_{i=1}^{n}b_i^{\alpha}(x)\frac{\partial u}{\partial x_i}\}, \tag{3}$$

satisfying the conditions below. A is a set of controls, and by denoting $n\times n$ matrices $A^{\alpha}=(a_{ij}^{\alpha}(x))_{ij}$ $(\alpha\in \mathrm{A})$, there exist $n\times m$ matrices σ^{α} such that

$$\begin{aligned}&A^{\alpha}(x)=\sigma^{\alpha}(\sigma^{\alpha})^{t}(x) &&\text{any}\quad x\in\Omega,\quad \alpha\in \mathrm{A},\\ &\lambda_1 I\leq A^{\alpha}(x)\leq \Lambda_1 I &&\text{any}\quad x\in\Omega,\quad \alpha\in \mathrm{A},\end{aligned} \tag{4}$$

where $0 < \lambda_1 \leq \Lambda_1$ positive constants, I the $n \times n$ identity matrix. There exists a positive constant $L > 0$ such that

$$\begin{aligned} |a_{ij}^\alpha(x) - a_{ij}^\alpha(y)| &\leq L|x-y| \quad \text{any} \quad 1 \leq i,j \leq n, \quad x \in \Omega, \quad \alpha \in A, \\ |b_i^\alpha(x) - b_i^\alpha(y)| &\leq L|x-y| \quad \text{any} \quad 1 \leq i \leq n, \quad x \in \Omega, \quad \alpha \in A. \end{aligned} \tag{5}$$

There also exists a positive constant γ_0, such that for the outward unit normal vector $\mathbf{n}(x)$ $(x \in \partial\Omega)$, $\gamma(x)$ satisfies

$$< \gamma(x), \mathbf{n}(x) > \quad \geq \gamma_0 > 0 \qquad \text{any} \quad x \in \partial\Omega. \tag{6}$$

The domain Ω is assumed to be either one of the following:

$$\text{Bounded open domain in} \quad \mathbf{R}^n \qquad \text{with} \quad C^{3,1} \quad \text{boundary,} \tag{7}$$

or

$$\begin{aligned} &\text{Half space in} \quad \mathbf{R}^n, \quad \text{periodic in the first} \quad n-1 \quad \text{variables with} \quad C^{3,1} \quad \text{boundary} \\ &: \{(x', x_n) | \quad \text{periodic in} \quad x' = (x_1, ..., x_{n-1}) \in (\mathbf{R}/\mathbf{Z})^{n-1}, \quad x_n \geq f_1(x')\}, \\ &\text{where} \quad f_1 \in C^{3,1}((\mathbf{R}/\mathbf{Z})^{n-1})). \end{aligned} \tag{8}$$

(In the latter case (8), a supplement boundary condition at $x_n = \infty$ will be added to (1)-(2).)

The following example implies the qualitative meaning of the number d.

Example 1.1. *Let Ω be a domain in* (7)*, and $g(x)$ be a Lipschitz continuous function on $\partial\Omega$. Assume that there exists a number d such that the following problem has a viscosity solution.*

$$-\Delta u = 0 \qquad \text{in} \quad \Omega,$$

$$d+ < \nabla u, \mathbf{n}(x) > -g(x) = 0 \qquad \text{on} \quad \partial\Omega.$$

Then,

$$d = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} g(x) dS.$$

Proof of Example 1.1. In the Green's first identity:

$$\int_\Omega \Delta u v dx + \int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n} dS,$$

we put $v = 1$, and get $d|\partial\Omega| = \int_{\partial\Omega} g(x) dS$.

Thus, d is a kind of the averaged quantity on $\partial\Omega$. For general Hamiltonians F, the way to construct the number d and $u(x)$ in (1)-(2) is the following. Here we assume that (7) holds. (The case (8) is more complicated, and will be treated in Section 3 below.) For any $\lambda > 0$, consider

$$F(x, \nabla u_\lambda, \nabla^2 u_\lambda) = 0 \qquad \text{in} \quad \Omega, \tag{9}$$

$$\lambda u_\lambda + < \nabla u, \gamma(x) > -g(x) = 0 \qquad \text{on} \quad \partial\Omega. \tag{10}$$

The regularity of u_λ ($\lambda \in (0,1)$) which will be shown in § 2 yields, for any fixed $x_0 \in \Omega$

$$\lim_{\lambda \downarrow 0} \lambda u_\lambda(x) = d \qquad \text{uniformly in} \quad \overline{\Omega}, \tag{11}$$

and by taking a subsequence $\lambda' \downarrow 0$,

$$\lim_{\lambda' \downarrow 0}(u_{\lambda'}(x) - u_{\lambda'}(x_0)) = u(x) \qquad \text{uniformly in} \quad \overline{\Omega}. \tag{12}$$

The limit number d is unique in the sense that with which (1)-(2) has a viscosity solution. The above limit function $u(x)$ is one of such solutions. (The solution of (1)-(2) is not unique, for $u + C$ (C constant) is also a solution.) We shall show in § 2 these facts. Now, the meaning of the number d can be stated by using (11). For any fixed measurable function $\alpha(t) : [0, \infty) \to A$ (control process), let (X_t^α, A_t^α) be the stochastic process defined by

$$\begin{aligned} X_t^\alpha &= x + \int_0^t \sigma^\alpha(X_s^\alpha) dW_s + \int_0^t b^\alpha(X_s^\alpha) ds - \int_0^t \gamma(X_s^\alpha) dA_s \qquad t \geq 0, \\ A_t^\alpha &= \int_0^t 1_{\partial\Omega}(X_s^\alpha) dA_s \quad \text{is continuous, non decreasing in} \quad t \geq 0, \end{aligned} \tag{13}$$

where $b^\alpha = (b_i^\alpha)_i$, $1_{\partial\Omega}(\cdot)$ a characteristic function on $\partial\Omega$, W_t ($t \geq 0$) an $m-$dimensional Brownian motion. The study of the existence and the uniqueness of (X_t^α, A_t^α) is called the Skorokhod problem, and its solvability is known under the preceding assumptions. We refer the readers to P.-L. Lions and A.S. Sznitman [29], P.-L. Lions, J.M. Menaldi and A.S. Sznitman [27], and P.-L. Lions [26]. Let

$$J_\lambda^\alpha(x) = E_x \int_0^\infty e^{-\lambda t} g(X_t^\alpha) 1_{\partial\Omega}(X_t^\alpha) dA_t,$$

and define

$$u_\lambda(x) = \inf_{\alpha(\cdot)} J_\lambda^\alpha(x) \qquad \text{in} \quad \Omega, \tag{14}$$

where the infimum is taken over all possible control processes. It is known that u_λ is the unique solution of (9)-(10). (See, P.-L. Lions and N.S. Trudinger [30], and M.I. Freidlin and A.D. Wentzell [20].) Thus,

$$d = \lim_{\lambda \downarrow 0} \inf_{\alpha(\cdot)} \lambda E_x \int_0^\infty e^{-\lambda t} g(X_t^\alpha) 1_{\partial\Omega}(X_t^\alpha) dA_t, \tag{15}$$

if the right hand side of (11) exists, which represents the fact that the number d is the long time averaged reflection force on the boundary. (Each time the tragectory reaches to $\partial\Omega$, it gains the force $g(x)$ and is pushed back in the direction of $-\gamma(x)$.) We remark the similarity of the convergence (11) to the so-called ergodic problem for HJB equations. That is, by considering,

$$\lambda u_\lambda(x) + F(x, \nabla u_\lambda, \nabla^2 u_\lambda) = 0 \qquad \text{in} \quad \Omega,$$

$$< \nabla u_\lambda(x), \gamma(x) >= 0 \qquad \text{on} \quad \partial\Omega,$$

it is known that an unique number d' exists such that

$$\lim_{\lambda \downarrow 0} \lambda u_\lambda(x) = d' \qquad \text{uniformly in} \quad \Omega.$$

We refer the readers to M. Arisawa and P.-L. Lions [7], M. Arisawa [1], [2], A. Bensoussan [11] for the various types (operators and boundary conditions) of ergodic problems. As the above ergodic problem "in the domain", the existence of d in (2) "on the boundary" relates to the ergodicity of the stochastic process (13).

Next, we turn our interests to the homogenization. The unique existence of d in (1)-(2) plays an essential role to study the homogenization of oscillating Neumann boundary conditions. The simplest example is as follows.

Example 1.2. *Let c, g, $f_1(x, \xi_1)$ be functions defined in $(x, \xi_1) \in \mathbf{R}^2 \times \mathbf{R}\backslash\mathbf{Z}$ (periodic in ξ_1 with period 1). Assume that $f_1 \geq 0$, and that there exists a constant $c_0 > 0$ such that $c > c_0 > 0$. For any $\varepsilon \geq 0$, let*

$$\Omega_\varepsilon = \{(x_1, x_2) | \quad \varepsilon f_1(x, \frac{x_1}{\varepsilon}) \leq x_2 \leq b, \quad |x_1| \leq a\},$$

$$\Gamma_\varepsilon = \{(x_1, x_2) | \quad x_2 = \varepsilon f_1(x, \frac{x_1}{\varepsilon})\} \cap \partial\Omega_\varepsilon.$$

Let $u_\varepsilon(x)$ $(\varepsilon > 0)$ be the solution of

$$-\Delta u_\varepsilon = 0 \qquad \text{in} \qquad \Omega_\varepsilon, \tag{16}$$

$$< \nabla u_\varepsilon(x), \mathbf{n}_\varepsilon(x) > + c(x, \frac{x_1}{\varepsilon}) u_\varepsilon = g(x, \frac{x_1}{\varepsilon}) \qquad \text{on} \quad \Gamma_\varepsilon, \tag{17}$$

$$u_\varepsilon = 0 \qquad \text{on} \quad \partial\Omega_\varepsilon \backslash \Gamma_\varepsilon, \tag{18}$$

where $\mathbf{n}_\varepsilon(x)$ is the outward unit normal to Γ_ε. Then, as $\varepsilon \downarrow 0$, u_ε converges to a unique functiont $u(x)$ uniformly in $\overline{\Omega_0}$, which is the solution of

$$-\Delta u = 0 \qquad \text{in} \quad \Omega_0,$$

$$< \nabla u(x), \nu(x) > + \overline{L}(x, u, \nabla u) = 0 \qquad \text{on} \quad \Gamma_0, \tag{19}$$

$$u = 0 \quad on \quad \partial\Omega_0 \backslash \Gamma_0,$$

where ν is the outward unit normal to Γ_0, and $\overline{L}$ is defined as follows.
Let $O(x) = \{(\xi_1, \xi_2) | \quad \xi_2 \geq f_1(x, \xi_1), \quad \xi_1 \in \mathbf{R} \backslash \mathbf{Z}\}$. Then, for any fixed $(x, r, p) \in \Omega \times \mathbf{R} \times \mathbf{R}^2$, there exists a unique number $d(x, r, p)$ such that

$$-\Delta_\xi v \equiv -(\frac{\partial^2 v}{\partial \xi_1^2} + \frac{\partial^2 v}{\partial \xi_2^2}) = 0 \quad in \quad O(x),$$

$$d(x,r,p) + < \nabla_\xi v, \gamma(\xi) > -(\sqrt{1 + (\frac{\partial f_1}{\partial \xi_1})^2} g - \sqrt{1 + (\frac{\partial f_1}{\partial \xi_1})^2} cr - p_1 \frac{\partial f_1}{\partial \xi_1}) = 0 \quad on \quad \partial O(x),$$

where $\gamma(\xi) = (\frac{\partial f_1}{\partial \xi_1}, -1)$ $(\xi \in \partial O(x))$, and

$$\overline{L}(x, r, p) = -d(x, r, p). \tag{20}$$

In A. Friedman, B. Hu, and Y. Liu [21], a similar problem to the above example (linear, three scales case) was treated by the variational approach. We shall extend the result (including Example 1.2.) to nonlinear problems by using the existence of the long time averaged reflection number d in (1)-(2). As Example 1.2 indicates, the effective limit boundary condition (19) is defined by using the long time averaged number in (20). Our present approach was inspired by the classical method of formal asymptotic expansions of A. Bensoussin, J.L. Lions, and G. Papanicolaou [12]. This approach is closely related to the ergodic problem for HJB equations described in the preceding part of this introduction. For the application of the ergodic problem ([7], [1], [2]) to obtain the effective P.D.E. in the domain, we refer the readers to M. Arisawa [3], [4], M. Arisawa and Y. Giga [6], L.C. Evans [17], [18], and P.-L. Lions, G, Papanicolaou, and S.R.S. Varadhan [28]. As far as we know, there is no existing reference which treats the homogenization of the oscillating Neumann boundary conditions from the view point of the ergodic problem.

We consider the solvability of PDEs in the framework of viscosity solutions. (See M.G. Crandall and P.-L. Lions [15], M.G. Crandall, H. Ishii and P.-L. Lions [14], and W.H. Fleming and H.M. Soner [19].) We use the notation $B(x, r)$ $(x \in \Omega, \quad r > 0)$ for the open ball centered at x with radius $r > 0$.

2 Existence and uniqueness of the long time averaged reflection in the bounded domain.

In this section, the existence and uniqueness of the number d in (1)-(2) is shown in the case that Ω satisfies (7). The Hamiltonian $F(x, \nabla u, \nabla^2 u)$, given in (3), positively homogeneous in degree one, is assumed to satisfy (4) and (5); the vector field γ on $\partial\Omega$ is assumed to satisfy (6). For the existence, we further assume that

$$|a_{ij}^{\alpha}, |\nabla a_{ij}^{\alpha}|, |\nabla^2 a_{ij}^{\alpha}|, |b_i^{\alpha}|, |\nabla b_i^{\alpha}|, |\nabla^2 b_i^{\alpha}| \le K \qquad \text{any} \quad x \in \Omega, \quad 1 \le i,j \le n, \quad \alpha \in A, \tag{21}$$

where $K > 0$ is a constant, and that γ, g can be extendable in a neighborhood U of $\partial\Omega$ to twice continuously differentiable functions so that

$$|\nabla\gamma|, |\nabla^2\gamma|, |\nabla^2 g|, |\nabla^2 g| \le K \qquad \text{in} \quad U, \tag{22}$$

where $K > 0$ is the constant in (21). For the existence of d, we approximate (1)-(2) by (9)-(10) ($\lambda \in (0,1)$) and examine the regularity of u_λ, uniformly in λ. In order to have (11)-(12), we need the following estimates.

Theorem 2.1. *Assume that Ω is (7), and that (4), (6), (21) and (22) hold. Then there exists a unique solution $u_\lambda \in C^{1,1}(\overline{\Omega}) \cap C^{2,\beta}(\Omega)$ of (9)-(10), where $\beta > 0$ depends on n and Λ_1/λ_1. Moreover for any fixed $x_0 \in \Omega$, there exists a constant $C > 0$ such that the following estimates hold.*

$$|u_\lambda - u_\lambda(x_0)|_{L^\infty(\overline{\Omega})} \le C \qquad \textit{any} \quad \lambda \in (0,1), \tag{23}$$

$$|\nabla u_\lambda|_{L^\infty(\overline{\Omega})} \le C \qquad \textit{any} \quad \lambda \in (0,1), \tag{24}$$

$$|\nabla u_\lambda|_{1;\overline{\Omega}} \le C \qquad \textit{any} \quad \lambda \in (0,1). \tag{25}$$

Remark 2.1 One can replace the conditions (21)-(22) to other conditions ,for example those in [23], to have

$$|u_\lambda(x) - u_\lambda(y)| \le C|x-y|^\theta \qquad \text{any} \quad x, y \in \overline{\Omega}, \quad \lambda \in (0,1),$$

where $C > 0$, $\theta \in (0,1)$ are independent on $\lambda > 0$. The proof of this inequality can be done in a similar way to [23], but by taking account of the Neumann type boundary conditions, and also by using the estimate (23).

Proof of Theorem 2.1. For each $\lambda > 0$, the existence and uniqueness of $u_\lambda \in C^{1,1}(\overline{\Omega}) \cap C^{2,\beta}(\Omega)$ is established in P.-L. Lions and N.S. Trudinger [30]. We are to show the uniform (in $\lambda \in (0,1)$) regularity (23)-(25). The estimates (24)-(25) follow from (23) by using a similar argument in [30]. ([16], [24], [25].) Here, we only prove (23), and refer [5] for further details.

Proof of (23) We prove by a contradiction argument. Let $x_0 \in \Omega$ be fixed. Assume, as $\lambda > 0$ goes to 0

$$|u_\lambda - u_\lambda(x_0)|_{L^\infty(\overline{\Omega})} \to \infty.$$

$$\varepsilon_\lambda \equiv |u_\lambda - u_\lambda(x_0)|^{-1}_{L^\infty(\overline{\Omega})} \qquad \lambda \in (0,1),$$

and let $v_\lambda \equiv \varepsilon_\lambda(u_\lambda - u_\lambda(x_0))$. Then,

$$|v_\lambda|_{L^\infty(\overline{\Omega})} = 1, \quad v_\lambda(x_0) = 0 \qquad \text{any} \quad \lambda \in (0,1).$$

From (3), v_λ satisfies $F(x, \nabla v_\lambda, \nabla^2 v_\lambda) = 0$ in Ω, and from (4) the Krylov-Safonov inequality (see [13] for instance) leads: for any compact set $V \subset\subset \Omega$, there exists a constant $M_V > 0$ such that

$$|\nabla v_\lambda|_{L^\infty(\overline{V})} \leq M_V \qquad \text{any} \quad \lambda \in (0,1). \tag{26}$$

We denote

$$v^*(x) = \limsup_{\lambda\downarrow 0, y\to x} v_\lambda(y), \quad v_*(x) = \liminf_{\lambda\downarrow 0, y\to x} v_\lambda(y).$$

Then, since $v_\lambda(x_0) = 0$ $(\forall\lambda \in (0,1))$, from (26) we have

$$v^*(x_0) = v_*(x_0) = 0, \tag{27}$$

$$|v^*|_{L^\infty(\overline{\Omega})} = 1, \quad \text{or} \quad |v_*|_{L^\infty(\overline{\Omega})} = 1. \tag{28}$$

From (2), v_λ satisfies

$$< \nabla v_\lambda, \gamma(x) >= \varepsilon_\lambda g - \lambda(v_\lambda + \varepsilon_\lambda u_\lambda(x_0)),$$

and by the comparison result for (9)-(10)

$$|\lambda u_\lambda(x_0)|_{L^\infty(\overline{\Omega})} \leq C \qquad \text{any} \quad \lambda \in (0,1),$$

where $C > 0$ is a constant. By letting $\lambda \downarrow 0$, v^* and v_* are viscosity solutions of

$$< \nabla v^*, \gamma(x) > \leq 0 \qquad \text{on} \quad \partial\Omega, \tag{29}$$

$$< \nabla v_*, \gamma(x) > \geq 0 \qquad \text{on} \quad \partial\Omega, \tag{30}$$

and $v(x) = v^*(x) = v_*(x)$ $(x \in \Omega)$ satisfies

$$F(x, \nabla v, \nabla^2 v) = 0 \qquad \text{in} \quad \Omega. \tag{10'}$$

(We refer the readers to [14] and G. Barles and B. Perthame [10] for this stability result.)

Now we employ the strong maximum principle of M. Bardi and F. Da-Lio [8]. Remark that $F(x,p,R)$ given in (3), satisfying (4) and (21) enjoys the following two properties of (31) and (32).

(Scaling property) For any $x_0 \in \Omega$, for any $\eta > 0$, there exists a function ϕ: $(0,1) \to (0,\infty)$ such that

$$\overline{F}(x, \xi p, \xi R) \geq \phi(\xi)\overline{F}(x,p,R) \qquad \text{any} \quad \xi \in (0,1), \tag{31}$$

holds for any $x \in B(x_0, \eta)$, $0 < |p| \leq \eta$, $|R| \leq \eta$.

(Nondegeneracy property) For any $x_0 \in \Omega$, for any small vector $\nu \neq 0$, there exists a positive number r_0 such that

$$\overline{F}(x_0, \nu, I - r\nu \otimes \nu) > 0 \qquad \text{any} \quad r > r_0. \tag{32}$$

We cite the following result for our present and later purposes.

Lemma A. ([8]) *(Strong maximum priciple) Let $\Omega \subset \mathbf{R}^n$ be an open set and let u be an upper semicontinuous viscosity subsolution of*

$$\overline{F}(x, \nabla u, \nabla^2 u) = 0 \qquad in \quad \Omega,$$

which attains a maximum in Ω. Assume that $\overline{F}$ satisfies (31), (32), and

for any $x_0 \in \Omega$ there exists $\rho_0 > 0$ such that for any $\nu \in B(0, \rho_0) \backslash \{0\}$, (32) holds

for some $\quad r_0 > 0.$ (33)

Then, u is a constant.

We go back to the proof of (23). Assume that $|v^*|_{L^\infty(\overline{\Omega})} = 1$ holds in (28). (The another case of $|v_*|_{L^\infty(\overline{\Omega})} = 1$ can be treated similarly.) Thus from (27), v^* is not constant, and from (10)' and the strong maximum principle (Lemma A), v^* attains its maximum at a point $x_1 \in \partial\Omega$:

$$v^*(x_1) > v^*(x) \qquad \text{any} \quad x \in \Omega.$$

Since $\partial\Omega$ is $C^{3,1}$, the interior sphere condition (see D. Gilbarg and N.S. Trudinger [22]) is satisfied : there exists $y \in \Omega$ such that for $R = |x_1 - y|$

$$B(y, R) \in \Omega, \qquad x_1 \in \partial B(y, R).$$

Let

$$\phi(x) = e^{-cR^2} - e^{-c|x-y|^2} \qquad x \in \Omega,$$

where $c > 0$ is a constant large enough so that

$$\begin{aligned} F(x_1, \nabla\phi(x_1), \nabla^2\phi(x_1)) &= F(x_1, 2c(x_1 - y)e^{-c|x_1-y|^2}, 2ce^{-c|x_1-y|^2}(I - 2c(x_1 - y) \otimes (x_1 - y))) \\ &= 2ce^{-c|x_1-y|^2} F(x_1, x_1 - y, I - 2c(x_1 - y) \otimes (x_1 - y)) > 0 \end{aligned}$$

holds. (Here, we used (3), (32) and (33).) By the lower semicontinuity of F in x, there exists $r \in B(0, R)$ and $C' > 0$ such that

$$F(x, \nabla\phi(x), \nabla^2\phi(x)) \geq C' > 0 \qquad \text{in} \quad B(x_1, r) \cap \overline{\Omega}. \tag{34}$$

We claim that

$$v^*(x) - v^*(x_1) - \phi(x) \le 0 \qquad \text{in} \quad B(x_1, r) \cap \overline{\Omega}. \tag{35}$$

In fact, if $x \in B(y, R)^c$, $\phi(x) \ge 0$ and (35) holds. Assume that for $x' \in B(x_1, r) \cap B(y, R)$ (35) does not hold, and

$$v^*(x') - v^*(x_1) - \phi(x') = \max_{B(x_1, r) \cap B(y, R)} v^*(x) - v^*(x_1) - \phi(x).$$

Then by the definition of the viscosity solution,

$$F(x', \nabla\phi(x'), \nabla^2\phi(x')) \le 0,$$

which contradicts to (34). Therefore, (35) holds. By remarking that $\phi(x_1) = 0$, (35) indicates that $v^* - \phi$ takes its maximum at $x_1 \in \partial\Omega$. Since v^* satisfies (29) in the sense of viscosity solutions, either

$$< \phi(x_1), \gamma(x_1) > \le 0,$$

or

$$F(x_1, \nabla\phi(x_1), \nabla^2\phi(x_1)) \le 0$$

must be satisfied. However from the definition of ϕ, (6) and (34), both of the above are not satisfied. We got a contradiction, and proved (23).

Theorem 2.2. *Assume that Ω is (7), and that (4), (6), (21) and (22) hold. Then there exists a number d and a function $u(x) \in C^{1,1}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ ($\alpha \in (0,1)$) which satisfy (1)-(2).*

Proof of Theorem 2.2. From (23)-(25) and the Evans-Krylov estimate, we can extract a subsequence $\lambda' \downarrow 0$ such that there exist a number d and $u(x) \in C^{1,1}(\overline{\Omega}) \cap C^{2,\beta}(\Omega)$, and

$$\lim_{\lambda' \downarrow 0} \lambda' u_{\lambda'}(x) = d, \quad \lim_{\lambda' \downarrow 0} (u_{\lambda'} - u_{\lambda'})(x_0) = u(x) \qquad \text{uniformly on} \quad \overline{\Omega}. \tag{36}$$

From the usual stability result ([14]), it is clear that the pair (d, u) satisfies (1)-(2).

As for the uniqueness of the number d, we give the following theorem in which we consider (1)-(2) in the framework of viscosity solutions.

Theorem 2.3. *Assume that Ω is (7), and that (4), (5), (6) and (22) hold. Then, the number d such that (1)-(2) has a viscosity solution u is unique.*

Proof of Theorem 2.3. We argue by contradiction. Let (d_1, u_1) and (d_2, u_2) be two pairs satisfying (1)-(2) in the sense of viscosity solutions. We assume $d_1 > d_2$. We need the following Lemma, the proof of which is done by a contracdition argument, which we abbreviate. (See [5].)

Lemma 2.4. *Let $v = u_1 - u_2$. Then, v satisfies*

$$- M^+(\nabla^2 v) + \inf_{\alpha \in A}\{-\sum_{i=1}^n b_i^\alpha \frac{\partial v}{\partial x_i}\} \leq 0 \qquad in \quad \Omega, \tag{37}$$

$$< \nabla v, \gamma > \leq d_2 - d_1 < 0 \qquad on \quad \partial\Omega, \tag{38}$$

where

$$M^+(X) = \sup_{\lambda_1 I \leq A \leq \Lambda_1 I} Tr(AX) \qquad X \in \mathbf{S}^n. \tag{39}$$

By admitting the above Lemma, the proof of Theorem 2.3 is immediate. From the strong maximum principle (Lemma A), v, which is not constant, attains its maximum at some point $x_1 \in \partial\Omega$

$$v(x_1) > v(x) \qquad \text{any} \quad x \in \Omega.$$

However, as we have seen in the proof of (23) in Theorem 2.1, this is not compatible with $< \nabla v, \gamma > \leq d_2 - d_1$ on $\partial\Omega$, in the sense of viscosity solutions. Thus, we have proved that $d_1 = d_2$ must be hold.

3 Long time averaged reflection force in half spaces.

In this section, the existence and uniqueness of the number d in (1)-(2) is shown in the case that Ω satisfies (8), with a supplement boundary condition at $x_n = \infty$. We denote

$$\Omega = \{(x', x_n) | \quad x_n \geq f(x'), \quad x' \in (\mathbf{R}/\mathbf{Z})^{n-1}\},$$

$$\Gamma_0 = \partial\Omega = \{(x', x_n) | \quad x_n = f(x'), \quad x' \in (\mathbf{R}/\mathbf{Z})^{n-1}\},$$

where $f(x')$ is periodic in $x' \in (\mathbf{R}/\mathbf{Z})^{n-1}$ and is $C^{3,1}$. Our goal is to find a unique number d which admits a viscosity solution u of (1)-(2) such that

$$u \quad \text{is bounded.} \tag{40}$$

We list our results in the following without their proofs, which are in [5]. The first one is the uniqueness of d.

Theorem 3.1. *Assume that Ω is (8), and that (4), (5), (6) and (22) hold. Moreover, assume that*

$$b_n^\alpha(x) \leq 0 \qquad any \quad x \in \Omega, \quad \alpha \in A. \tag{41}$$

Then, the number d such that (1)-(2) and (40) has a viscosity solution u is unique.

Remark 3.1. (Counter example.) If we do not assume the boundary condition at infinity (40), d is not unique in general. For example, consider

$$-\Delta u = 0 \qquad \text{in} \quad \{x_n \geq 0\} \subset \mathbf{R}^n, \tag{42}$$

$$d + <\nabla u, \mathbf{n}(x)> = 0 \qquad \text{on} \quad \{x_n = 0\} \subset \mathbf{R}^n, \tag{43}$$

where $\mathbf{n}$ is the outward unit normal, and the solution u is periodic in $x' = (x_1, ..., x_{n-1})$. Then, for any c, $d \in R$, $u = -dx_n + c$ is the solution of (42)-(43). Thus, the number d in (43) is not unique. (Green's first identity does not hold in the half space.)

Next, for the existence of d we approximate (1)-(2) and (40) by

$$\begin{aligned} &F(x, \nabla u_\lambda^R, \nabla^2 u_\lambda^R) = 0 \qquad \text{in} \quad \Omega_R = \{(x', x_n) | \quad f(x') \leq x_n \leq R\}, \\ &<\nabla u_\lambda^R, \mathbf{n}(x)> = 0 \qquad \text{on} \quad \Gamma_R = \{(x', x_n) | \quad x_n = R\}, \\ &\lambda u_\lambda^R + <\nabla u_\lambda^R, \gamma(x)> - g(x) = 0 \qquad \text{on} \quad \partial\Omega = \Gamma_0 = \{x_n = f(x')\}, \end{aligned} \tag{44}$$

where $R > 0$ is large enough so that Γ_R and Γ_0 do not intersect, say $R \geq R_0$. As in § 2 (Theorem 2.1), we examine the regularity of u_λ^R uniformly in $\lambda \in (0, 1)$ and $R > R_0$. By combining this and the former uniqueness, we obtain the following.

Theorem 3.3. *Assume that Ω is (8), and that (4), (6), (21) and (22) hold. Then, there exists a unique number d such that (1)-(2) and (40) has a viscosity solution u.*

4 Homogenization of oscillating Neumann type boundary conditions.

In this section, we study the following homogenization problem.

$$G(x, \nabla u_\varepsilon, \nabla^2 u_\varepsilon) = \sup_{\alpha \in A}\{-\sum_{ij=1}^{2} a_{ij}^\alpha(x) \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} - \sum_{i=1}^{2} b_i^\alpha(x) \frac{\partial u_\varepsilon}{\partial x_i}\} = 0 \tag{45}$$

$$\text{in} \quad \Omega_\varepsilon = \{(x_1, x_2) | \quad -a \leq x_1 \leq a, \quad f_0(x_1) + \varepsilon f_1(x_1, \frac{x_1}{\varepsilon}) \leq x_2 \leq b\} \subset \mathbf{R}^2,$$

$$<\nabla u_\varepsilon, \mathbf{n}_\varepsilon> + c(x_1, \frac{x_1}{\varepsilon}) u_\varepsilon = g(x_1, \frac{x_1}{\varepsilon}) \tag{46}$$

$$\text{on} \quad \Gamma_\varepsilon = \{(x_1, x_2) | \quad -a \leq x_1 \leq a, \quad x_2 = f_0(x_1) + \varepsilon f_1(x_1, \frac{x_1}{\varepsilon})\},$$

$$u_\varepsilon = 0 \qquad \text{on} \quad \partial\Omega_\varepsilon \backslash \Gamma_\varepsilon, \tag{47}$$

where $\varepsilon > 0$, $a_{ij}^\alpha(x)$, $b_i^\alpha(x)$ are Lipschitz in x satisfying (5), $\mathbf{n}_\varepsilon(x)$ is the outward unit normal to Ω_ε,

$$c, \quad g, \quad f_1(x_1, \xi_1) \quad \text{are defined in} \quad \Omega_\varepsilon \times \mathbf{R}, \quad \text{periodic in} \quad \xi_1 \in \mathbf{R} \backslash \mathbf{Z}, \tag{48}$$

$$0 \leq f_1(x_1, \xi), \quad 0 < C < c(x, \xi_1) \qquad \text{in} \quad \Omega_\varepsilon \times \mathbf{R} \backslash \mathbf{Z}, \tag{49}$$

where $C > 0$ is a constant,

$$f_0'(\pm a) = 0, \quad \frac{\partial f_1}{\partial \xi_1}(\pm a, \xi_1) = 0, \tag{50}$$

denoting $A_\alpha = (a_{ij}^\alpha(x))_{1 \leq i,j \leq n}$,

$$\lambda_1 \leq A_\alpha \leq \Lambda_1 \qquad \text{any} \quad \alpha \in \mathrm{A}. \tag{51}$$

We are interested in the limit of u_ε of (45)-(47) as ε goes to 0. Remark that Example 1.2 is a special case of the above. For our nonlinear problem, we need further assumptions listed in the following.

$$b_1^\alpha \equiv 0, \quad b_2^\alpha = a_{11}^\alpha f_0'' \qquad \text{any} \quad \alpha \in A, \quad x \in \Omega_\varepsilon, \tag{52}$$

$$\{a_{11}^\alpha(1 + f_0'^2) - 2a_{12}^\alpha f_0' + a_{22}^\alpha\}^2 \geq 4(a_{11}^\alpha a_{22}^\alpha - {a_{12}^\alpha}^2) \qquad \text{forall} \quad \alpha \in A, \quad x \in \Omega_\varepsilon, \tag{53}$$

and for

$$O(x_1) = \{(\xi_1, \xi_2) | \quad \xi_2 \geq f_1(x_1, \xi_1), \quad \text{periodic in} \quad \xi_1\},$$

$$\partial O(x_1) \quad is \quad C^{3,1}. \tag{54}$$

These assumptions come from the following formal asymptotic expansion of u_ε:

$$u_\varepsilon = u(x) + \varepsilon v(\frac{x_1}{\varepsilon}, \frac{x_2 - f_0(x_1)}{\varepsilon}) + O(\varepsilon^2), \tag{55}$$

where we are assuming that "the corrector" v depends only on $\xi_1 = \frac{x_1}{\varepsilon}$ and $\xi_2 = \frac{x_2 - f_0(x_1)}{\varepsilon}$ (ξ_1, ξ_2 are rescaled variables.) By introducing the formal derivatives of u_ε into (45)-(46), we get the so-called cell problem for v, which is nothing less than the ergodic problem on the boundary, studied in §2, 3. Let $(x, r, p) \in \Omega \times \mathbf{R} \times \mathbf{R}^2$ ($p = (p_1, p_2)$) be arbitrarily fixed, and define the following operators.

$$P_{x,r,p}^\alpha(D_\xi^2 v(\xi_1, \xi_2)) \equiv \tag{56}$$

$$\equiv -[a_{11}^\alpha \frac{\partial^2 v}{\partial \xi_1^2} + 2(a_{12}^\alpha - a_{11}^\alpha f_0') \frac{\partial^2 v}{\partial \xi_1 \partial \xi_2} + \{a_{11}^\alpha (f_0')^2 - 2a_{12}^\alpha f_0' + a_{22}^\alpha\} \frac{\partial^2 v}{\partial \xi_2^2}] \qquad \text{in} \quad O(x_1),$$

and

$$P_{x,r,p}(D_\xi^2 v(\xi_1, \xi_2)) \equiv \sup_{\alpha \in \mathrm{A}} \{P_{x,r,p}^\alpha(D_\xi^2 v(\xi_1, \xi_2))\} \qquad \text{in} \quad O(x_1). \tag{57}$$

We denote the outward unit normal to the boundary of $\Omega = \{(x_1, x_2) | \quad -a \leq x_1 \leq a, \quad x_2 \geq f_0(x_1)\}$ as

$$\nu = \frac{1}{\sqrt{1 + (f_0')^2}}(f_0', -1).$$

$$\gamma(\xi_1,\xi_2)=\frac{(f_0'+\frac{\partial f_1}{\partial \xi_1},-\{f_0'(f_0'+\frac{\partial f_1}{\partial \xi_1})+1\})}{\sqrt{1+(f_0')^2}}\qquad \text{on}\quad \partial O(x_1), \tag{58}$$

and for $(x,r,p)\in\Omega\times\mathbf{R}\times\mathbf{R}^2$

$$H(x,r,p,\xi)=\frac{1}{\sqrt{1+(f_0')^2}}\{-\sqrt{1+(f_0'+\frac{\partial f_1}{\partial \xi_1})^2}(c(x,\xi_1)r-g)-p_1\frac{\partial f_1}{\partial \xi_1}\}. \tag{59}$$

By using these notations, our cell problem obtained from (45)-(46) is: for any fixed $(x,r,p)\in\Omega\times\mathbf{R}\times\mathbf{R}^n$, find a unique number $d(x,p,r)$ such that the following problem has a viscosity solution (corrector) $v(\xi_1,\xi_2)$.

$$\begin{aligned}
&P_{x,r,p}(D_\xi^2 v(\xi_1,\xi_2))=0 \qquad \text{in}\quad O(x_1),\\
&d(x,r,p)+<\nabla_\xi v,\gamma>-H(x,r,p,\xi)=0 \qquad \text{on}\quad \partial O(x_1),\\
&v\quad \text{is bounded in}\quad \overline{O(x_1)}.
\end{aligned}\tag{60}$$

In fact, from Theorem 3.3, we know that $d(x,r,p)$ exists. Now, our main result is the following.

Theorem 5.1. *Assume that (48)-(54) hold. Then, there exists a unique function $u(x)$ such that*

$$\lim_{\varepsilon\downarrow 0}u_\varepsilon(x)=u(x)\qquad \textit{locally uniformly in}\quad \overline{\Omega},$$

which is the unique solution of

$$\sup_{\alpha\in A}\{-\sum_{i,j=1}^n a_{ij}^\alpha\frac{\partial^2 u}{\partial x_i\partial x_j}-\sum_{i=1}^n b_i^\alpha\frac{\partial u}{\partial x_i}\}=0\qquad \textit{in}\quad \Omega, \tag{61}$$

$$<\nabla u,\nu>+\overline{L}(x,u,\nabla u)=0\qquad \textit{on}\quad \Gamma_0, \tag{62}$$

and (47), where $d(x,r,p)$ is defined in (60).

The rigorous proof of the above theorem is done by the perturbed test function method, based on the maximum principle. (See [18], [5].)

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