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LABELED CONFIGURATION SPACES AND GROUP-COMPLETION

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1. Statement of the Results

In [7] we assigned to any pointed space Y and any topological abelian monoid M the configuration space $C^M(Y)$ of finite subsets of Y with labels in M. As a set $C^M(Y)$ consists of those pairs (S,σ), where S is a finite subset of the complement of the basepoint in Y and σ is a map $S\to M$. But (S,σ) is identified with (S',σ') if $S\subset S'$, $\sigma'|S=\sigma$, and $\sigma'(x)=0$ when $x\notin S$. The topology of $C^M(Y)$ depends not only on the topology of Y and M but also on the partial monoid structure of M.

Take $\mathbf{R}^\infty\ltimes X=\mathbf{R}^\infty\times X/\mathbf{R}^\infty\times *$ as Y and let $C^M(\mathbf{R}^\infty,X)$ be the subspace of $C^M(\mathbf{R}^\infty\ltimes X)$ consisting of those (S,σ) such that S can be embedded into $\mathbf{R}^\infty$ by the projection $\mathbf{R}^\infty\ltimes X\to\mathbf{R}^\infty$. In other words,

$$C^M(\mathbf{R}^\infty,X)=C^{X\wedge M}(\mathbf{R}^\infty),$$

where $X\wedge M$ is endowed with the partial monoid structure such that the sum of non-zero elements $(x_1,a_1),\cdots,(x_k,a_k)$ exists in $X\wedge M$ if and only if $x_1=\cdots=x_k$ and the sum $a_1+\cdots+a_k$ exists in M.

Let us write $E^M(X)=\Omega C^M(\mathbf{R}^\infty,\Sigma X)$. Then the results of [7] imply the following.

(1) The inclusion $C^M(\mathbf{R}^\infty,X)\to C^M(\mathbf{R}^\infty\ltimes X)$ is a homotopy equivalence if X is a euclidean neighborhood retract.

(2) The natural map $C^M(\mathbf{R}^\infty,X)\to E^M(X)$ is a group-completion, that is, induces an isomorphism of Pontrjagin ring

$$H_\bullet(C^M(\mathbf{R}^\infty,X))[\pi^{-1}]\cong H_\bullet(E^M(X))$$

where $\pi=\pi_0C^M(\mathbf{R}^\infty,X)\subset H_\bullet(C^M(\mathbf{R}^\infty,X))$.

(3) $E^M(X)$ is an infinite loop space, and the correspondence $X\to\pi_\bullet E^M(X)$ defines a generalized homology theory.

Among examples, we have

(1) If $M=\mathbf{N}$ is the set of positive integers then $C^M(\mathbf{R}^\infty,X)$ is equivalent to the free abelian monoid generated by X modulo the relation $*=0$. In this case we have $\pi_\bullet E^M(X)=H_\bullet(X)$ by the Dold-Thom theorem [2].

More generally, if M is a topological abelian group M then

$$\pi_\bullet E^M(X) = \bigoplus_i H_{\bullet+i}(X, \pi_i M)$$

is the homology theory defined by the generalized Eilenberg-Mac Lane spectrum $\bigvee K(\pi_i M, i)$.

(2) If M is the subset $\{1\}$ in the additive group $\mathbf{Z}$ then $\pi_\bullet E^M(X) = \pi_\bullet^S X$ is the stable homotopy of X. This is a consequence of the Barratt-Priddy-Quillen theorem.

(3) Let $M = \mathrm{Gr}(\mathbf{R}^\infty)$ be the Grassmannian of finite dimensional subspaces of $\mathbf{R}^\infty$, regarded as a partial monoid such that $V_1 + \cdots + V_k$ exists if and only if $V_i \perp V_j$ holds for $i \neq j$. Then $\pi_\bullet E^M(X) = ko_\bullet(X)$ is the connective homology theory associated to the real K-theory $KO_\bullet$. (See [6].)

In this note we give an alternative construction of group-completion by using the combinatorial structure of $C^M(\mathbf{R}^\infty, X)$. More precisely, we will see that the partial monoid structure of $C^M(\mathbf{R}^\infty, X)$ enables us to define an analogue of the classifying space (for topological monoids) which gives rise to a group-completion that, unlike $E^M(X) = \Omega C^M(\mathbf{R}^\infty, \Sigma X)$, depends only on $C^M(\mathbf{R}^\infty, X)$.

For each $k \geq 0$, let $BC^M(\mathbf{R}^\infty, X)_k$ be the subspace of $C^M(\mathbf{R}^\infty, X)^k$ consisting of those k-tuples $((S_1, \sigma_1), \ldots, (S_k, \sigma_k))$ such that for every $J \subset \{1, \ldots, k\}$ and $v \in \bigcup_{j \in J} S_j$ the sum $\sum_{j \in \Lambda(v)} \sigma_j(v)$ exists in $X \wedge M$, where $\Lambda(v) = \{j \mid v \in S_j\}$. Such a k-tuple will be called admissible. With respect to the evident face and degeneracy operators $BC^M(\mathbf{R}^\infty, X)_\bullet$ is a simplicial space, whose realization is denoted by $BC^M(\mathbf{R}^\infty, X)$.

Similarly, let $EC^M(\mathbf{R}^\infty, X)$ be the realization of the simplicial space $EC^M(\mathbf{R}^\infty, X)_\bullet$ such that

$$EC^M(\mathbf{R}^\infty, X)_k \subset C^M(\mathbf{R}^\infty, X)^k \times C^M(\mathbf{R}^\infty, X)$$

is the set of admissible $(k+1)$-tuples, and that the projection $EC^M(\mathbf{R}^\infty, X)_k \to BC^M(\mathbf{R}^\infty, X)_k$ is compatible with face and degeneracy maps. Then the fiber of the induced map $EC^M(\mathbf{R}^\infty, X) \to BC^M(\mathbf{R}^\infty, X)$ at the basepoint is $C^M(\mathbf{R}^\infty, X)$. As $EC^M(\mathbf{R}^\infty, X)$ is contractible, we obtain a natural map

$$C^M(\mathbf{R}^\infty, X) \to \Omega BC^M(\mathbf{R}^\infty, X).$$

The main result of this note is the following two theorems.

Theorem 1. *For any topological partial monoid M, the natural map*

$$C^M(\mathbf{R}^\infty, X) \to \Omega BC^M(\mathbf{R}^\infty, X)$$

is a group-completion.

Theorem 2. *Let M be a subset of a topological abelian group A and let $\pm M = M \cup -M \subset A$. Then the natural map*

$$C^M(\mathbf{R}^\infty, X) \to C^{\pm M}(\mathbf{R}^\infty, X),$$

induced by the inclusion $M \subset \pm M$, is a group-completion.

In particular, if $M = \{1\} \in \mathbf{Z}$ then $C^{\pm M}(\mathbf{R}^\infty, X)$ is nothing but the space of positive and negative particles $C^{\pm}(\mathbf{R}^\infty, X)$ introduced by Mcduff [3]. Thus we have

Corollary 3 (Caruso [1]). *For any pointed space X the space $C^{\pm}(\mathbf{R}^\infty, X)$ is weakly equivalent to $\Omega^\infty\Sigma^\infty X$.*

2. Proofs

Theorem 1 follows from Proposition 1.5 of [5], because the correspondence $\mathbf{k} \mapsto BC^M(\mathbf{R}^\infty, X)_k$ is a Γ-space such that the maps

$$BC^M(\mathbf{R}^\infty, X)_k \to BC^M(\mathbf{R}^\infty, X)^k$$

induced by the projections $p_s \colon \mathbf{k} \to \mathbf{1}$ are homotopy equivalences.

To prove Theorem 2, let $C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)}$ be the realization of the simplicial space $E_\bullet$ such that $E_k \subset C^M(\mathbf{R}^\infty, X)^k \times C^{\pm M}(\mathbf{R}^\infty, X)$ is the subset of admissible $(k+1)$-tuples. Let

$$\xi \colon C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)} \to BC^M(\mathbf{R}^\infty, X)$$

be the map induced by the projection $\xi_\bullet \colon E_\bullet \to BC^M(\mathbf{R}^\infty, X)_\bullet$. Then each ξ_k is a homology fibration since it is equivalent to the projection

$$C^M(\mathbf{R}^\infty, X)^k \times C^{\pm M}(\mathbf{R}^\infty, X) \to C^M(\mathbf{R}^\infty, X)^k.$$

As $C^M(\mathbf{R}^\infty, X)$ acts on $C^{\pm M}(\mathbf{R}^\infty, X)$ through homology equivalences, we see from [4, Proposition 4] that ξ is a homology fibration with fiber $C^{\pm M}(\mathbf{R}^\infty, X)$.

Assume that $C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)}$ is contractible. Then $C^{\pm M}(\mathbf{R}^\infty, X)$ is weakly equivalent to $\Omega BC^M(\mathbf{R}^\infty, X)$, and Theorem 2 follows from the commutative diagram

$$\begin{array}{ccccc}
C^M(\mathbf{R}^\infty, X) & \longrightarrow & EC^M(\mathbf{R}^\infty, X) & \longrightarrow & BC^M(\mathbf{R}^\infty, X) \\
\downarrow & & \downarrow & & \Vert \\
C^{\pm M}(\mathbf{R}^\infty, X) & \longrightarrow & C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)} & \longrightarrow & BC^M(\mathbf{R}^\infty, X)
\end{array}$$

together with Theorem 1.

Thus, to prove Theorem 2 we need only show

Lemma 4. *$C^{\pm M}(\mathbf{R}^\infty, X)_{C^M(\mathbf{R}^\infty, X)}$ is contractible.*

(My proof of this lemma is rather complicated, and is omitted here.)

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