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DEFINABLE FIBER BUNDLES AND AFFINENESS OF DEFINABLE C^r MANIFOLDS

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1. INTRODUCTION

Semialgebraic sets and semialgebraic maps have been studied and results on them can be seen in [1]. Let $\mathcal{M}$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. Every definable category on $\mathcal{M}$ is a generalization of the semialgebraic category and the definable category on $\mathcal{R}$ coincides with the semialgebraic one [22].

Some of recent results concerning o-minimal categories are [4], [5], [6], [7], [8], [10], [11], [12], [13], [14], [15], [16], [18], [21]. Semialgebraic G sets and semialgebraic G vector bundles are studied in [2], [19], [20].

In this note, we are concerned with homotopy property of definable fiber bundles and affineness of definable C^r manifolds. Throughout this article, the term "definable" means "definable with parameters in $\mathcal{M}$" and definable maps are assumed to be continuous.

The homotopy property for topological vector bundles is established in [9]. Its semialgebraic version, its equivariant semialgebraic version and its equivariant fiber bundle version are known in 12.7.7 [1], [2] and 2.10 [17], respectively.

We have the following as a definable fiber bundle version of this property.

Theorem 1.1 (1.1 [15]). *Let $\eta = (E, p, X, F, K)$ be a definable fiber bundle over a definable set X with fiber F and structure group K. If two definable maps $f, h : Y \to X$ between definable sets are homotopic and Y is compact, then $f^*(\eta)$ and $h^*(\eta)$ are definably fiber bundle isomorphic.*

Let X and Y be definable sets. Two definable maps $f, h : X \to Y$ are called *definably homotopic* if there exists a definable map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = h(x)$ for all $x \in X$. By 1.2 [11], if two definable maps between definable sets are homotopic, then they are definably homotopic. Hence two definable maps in Theorem 1.1 are definably homotopic.

We say that $\mathcal{M}$ is *polynomially bounded* if for every function $f : \mathbb{R} \to \mathbb{R}$ definable in $\mathcal{M}$, there exist a natural number k and a real number x_0 such that $|f(x)| \leq x^k$ for any $x > x_0$. Otherwise, $\mathcal{M}$ is called *exponential*. One of typical examples of polynomially bounded structures is $\mathcal{R}$. By a result of C. Miller [18], if $\mathcal{M}$ is exponential, then the exponential function $\mathbb{R} \to \mathbb{R}, x \mapsto e^x$ is definable. We call $\mathcal{M}$ *exponentially bounded* if for every function $h : \mathbb{R} \to \mathbb{R}$ definable in $\mathcal{M}$, there exist a natural number l and a

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real number x_1 such that $|h(x)| \leq exp_l(x)$ for any $x > x_1$, where $exp_l(x)$ denotes the lth iterate of the exponential function, e.g. $exp_2(x) = e^{e^x}$.

Theorem 1.2 (1.1 [10]). *If $\mathcal{M}$ is exponentially bounded and $0 \leq r < \infty$, then every definable C^r manifold is affine.*

2. Definable sets, definable fiber bundles and definable C^r manifolds

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, (f_i)_{i\in I}, (R_j)_{j\in J}, (c_k)_{k\in K})$ be a structure expanding $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$, where $+$ (respectively $\cdot$) : $\mathbb{R}^2 \to \mathbb{R}$ is the additive (respectively the multiplicative) function of $\mathbb{R}$, each $f_i : \mathbb{R}^{n(i)} \to \mathbb{R}, n(i) \in \mathbb{N} \cup \{0\}$ is a function, each $R_j \subset \mathbb{R}^{n(j)}, n(j) \in \mathbb{N}$ is a relation, and each c_k is a constant. We say that f (respectively R) is an *m-place function symbol* (respectively an *m-place relation symbol*) if $f : \mathbb{R}^m \to \mathbb{R}$ is a function (respectively $R \subset \mathbb{R}^m$ is a relation).

A *term* is a finite string of symbols obtained by repeated applications of the following two rules:

1. Variables are terms.
2. If f is an m-place function symbol of $\mathcal{M}$ and $t_1, \dots, t_m$ are terms, then the concatenated string $f(t_1, \dots, t_m)$ is a term.

Note that if $m = 0$, then the second rule says that constant symbols (0-place function symbols) are terms.

A *formula* is a finite string of symbols $s_1 \dots s_k$, where each s_i is either a variable, a function symbol, a relation symbol, one of the logical symbols $=, \neg, \vee, \wedge, \exists, \forall$, one of the brackets (,), or comma ,. Arbitrary formulas are generated inductively by the following three rules:

1. For any two terms t_1 and t_2, $t_1 = t_2$ and $t_1 > t_2$ are formulas.
2. If R is an m-place relation symbol and $t_1, \dots, t_m$ are terms, then $R(t_1, \dots, t_m)$ is a formula.
3. If ϕ and ψ are formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are formulas. If ϕ is a formula and v is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

A subset X of $\mathbb{R}^n$ is *definable* (in $\mathcal{M}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and elements $b_1, \dots, b_m \in \mathbb{R}$ such that $X = \{(a_1, \dots, a_n) \in \mathbb{R}^n | \phi(a_1, \dots, a_n, b_1, \dots, b_m) \text{ is true in } \mathcal{M}\}$.

Let $K \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^m$ be definable sets. We say that a continuous map $f : K \to L$ is *definable* (in $\mathcal{M}$) if the graph of f $(\subset K \times L \subset \mathbb{R}^n \times \mathbb{R}^m)$ is definable. A definable map $f : K \to L$ is called a *definable homeomorphism* if there exists a definable map $h : L \to K$ such that $f \circ h = id$ and $h \circ f = id$.

An *open interval* means something of the form $(a, b), a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{\infty\}$. We call $\mathcal{M}$ *o-minimal* (*order-minimal*) if every definable subset of $\mathbb{R}$ is a finite union of points and open intervals. Remark that $\mathcal{R}$ is o-minimal [22]. For example, $\mathcal{N} = (\mathbb{R}, +, \cdot, <, \mathbb{Z})$ is an expansion of $\mathcal{R}$ but not o-minimal because a definable subset $\mathbb{Z}$ of $\mathbb{R}$ in $\mathcal{N}$ is not a finite union of points and open intervals.

Notice that one can consider a definable category in a structure which is not o-minimal. But this category does not have satisfactory properties.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be definable open sets and $0 < r \leq \omega$. A C^r map $f : U \to V$ is called a *definable C^r map* if it is definable.

Let $A \subset \mathbb{R}^n$ be a definable set and $0 < r \leq \omega$. A definable map $f : A \to \mathbb{R}^m$ is a *definable C^r map* if there exist a definable open set $U \subset \mathbb{R}^n$ and a definable C^r map $F : U \to \mathbb{R}^m$ such that $A \subset U$ and $f = F|A$.

The following theorem states some of useful properties of definable sets and definable maps.

Theorem 2.1. (1) [Definable C^r cell decomposition (e.g. 7.3.3.2 [4])]. *Suppose that $0 \leq r < \infty$.*

(a) *For any definable set $A_1, \dots, A_k \subset \mathbb{R}^n$, there exists a decomposition of $\mathbb{R}^n$ into definable C^r cells partitioning $A_1, \dots, A_k$.*

(b) *For any definable function $f : A \to \mathbb{R}$, $A \subset \mathbb{R}^n$, there exists a decomposition into definable C^r cells partitioning A such that each restriction $f|C : C \to \mathbb{R}$ is a definable C^r map for each $C \subset A$ of the decomposition.*

(2) [Definable triangulation (e.g. (8.2.9 [4])]. *Let $S \subset \mathbb{R}^n$ be a definable set and $S_1, \dots, S_k$ definable subsets of S. Then there exist a finite simplicial complex K in $\mathbb{R}^n$ and a definable map $\phi : S \to \mathbb{R}^n$ such that ϕ maps S and each S_i definably homeomorphically onto a union of open simplexes of K. If S is compact, then we can take $K = \phi(S)$.*

(3) [Piecewise definable trivialization (e.g. 9.1.2 [4])]. *Let X and Y be definable sets and $f : X \to Y$ a definable map. Then there exist a finite partition $\{T_i\}_{i=1}^k$ of Y into definable sets and definable homeomorphisms $\phi_i : f^{-1}(T_i) \to T_i \times f^{-1}(y_i)$ such that $f|f^{-1}(T_i) = p_i \circ \phi_i$, $(1 \leq i \leq k)$, where $y_i \in T_i$ and $p_i : T_i \times f^{-1}(y_i) \to T_i$ denotes the projection.*

An equivariant version and an equivariant C^r version of Theorem 2.1 (3) are proved in [14].

A group G is a *definable group* if G is a definable set and the group operations $G \times G \to G$ and $G \to G$ are definable. A subgroup of a definable group is a *definable subgroup* of it if it is a definable subset of it.

Let G be a definable group. A *definable set with a definable G action* is a pair (X, θ) consisting of a definable set X and a group action $\theta : G \times X \to X$ such that θ is a definable map. This action is not necessarily linear.

A *definable space* is an object obtained by pasting finitely many definable sets together along open definable subsets, and definable maps between definable spaces are defined similarly (see Chap. 10 [4]). Definable spaces are generalizations of semialgebraic spaces in the sense of [3].

Definition 2.2. (1) A topological fiber bundle $\eta = (E, p, X, F, K)$ is called a *definable fiber bundle* over X with fiber F and structure group K if the following two conditions are satisfied:

(a) The total space E is a definable space, the base space X is a definable set, the structure group K is a definable group, the fiber F is a definable set with an effective definable K action, and the projection $p : E \to X$ is a definable map.

(b) There exists a finite family of local trivializations $\{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i$ of η such that each U_i is a definable open subset of X, $\{U_i\}_i$ is a finite open covering of X. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,x}(z) = \pi_i \circ \phi_i(z)$, where π_i stands for the projection $U_i \times F \to F$. For any i and j with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations *definable.*

Definable fiber bundles with compatible definable local trivializations are identified.

(2) Let $\eta = (E, p, X, F, K)$ and $\zeta = (E', p', X', F, K)$ be definable fiber bundles whose definable local trivializations are $\{U_i, \phi_i\}_i$ and $\{V_j, \psi_j\}_j$, respectively. A definable map $\overline{f} : E \to E'$ is said to be a *definable fiber bundle morphism* if the following two conditions are satisfied:

(a) There exists a definable map $f : X \to X'$ such that $f \circ p = p' \circ \overline{f}$.

(b) For any i, j such that $U_i \cap f^{-1}(V_j) \neq \emptyset$ and for any $x \in U_i \cap f^{-1}(V_j)$, the map $f_{ij}(x) := \psi_{j,f(x)} \circ \overline{f} \circ \phi_{i,x}^{-1} : F \to F$ lies in K, and $f_{ij} : U_i \cap f^{-1}(V_j) \to K$ is a definable map.

A definable fiber bundle morphism $\overline{f} : E \to E'$ is called a *definable fiber bundle isomorphism* if $X = X'$, $f = id_X$ and there exists a definable fiber bundle morphism $\overline{f'} : E' \to E$ such that $f' = id_X$, $\overline{f} \circ \overline{f'} = id$, and $\overline{f'} \circ \overline{f} = id$. We say that η is *definably trivial* if η is definably fiber bundle isomorphic to the trivial bundle $(X \times F, proj, X, F, K)$, where $proj : X \times F \to X$ denotes the projection onto the first factor.

(3) A continuous section $s : X \to E$ of a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *definable section* if for any i, the map $\phi_i \circ s|U_i : U_i \to U_i \times F$ is a definable map.

(4) We say that a definable fiber bundle $\eta = (E, p, X, F, K)$ is a *principal definable fiber bundle* if $F = K$ and the K action on F is defined by the multiplication of K.

Definition 2.3. Suppose that $0 \leq r \leq \omega$.

(1) A definable subset X of $\mathbb{R}^n$ is called a *d-dimensional definable C^r submanifold of* $\mathbb{R}^n$ if for any $x \in X$ there exists a definable C^r diffeomorphism (a definable homeomorphism if $r = 0$) ϕ_x from some open definable neighborhood U_x of the origin in $\mathbb{R}^n$ onto some open definable neighborhood V_x of x in $\mathbb{R}^n$ such that $\phi_x(0) = x, \phi(\mathbb{R}^d \cap U_x) = X \cap V_x$. Here $\mathbb{R}^d$ denotes the subset of $\mathbb{R}^n$ those which the last $(n - d)$ components are zero.

(2) A *definable C^r manifold X of dimension* d is a C^r manifold with a finite system of charts $\{\phi_i : U_i \to \mathbb{R}^d\}$ such that for each i and j, $\phi_i(U_i \cap U_j)$ is an open definable subset of $\mathbb{R}^d$ and the map $\phi_j \circ \phi_i^{-1}|\phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism (a definable homeomorphism if $r = 0$). We call this atlas *definable C^r*. Definable C^r manifolds with compatible atlases are identified.

(3) Let X (respectively Y) be a definable C^r manifold with definable C^r charts $\{\phi_i : U_i \to \mathbb{R}^n\}_i$ (respectively $\{\psi_j : V_j \to \mathbb{R}^m\}_j$). A C^r map $f : X \to Y$ is said to be a *definable C^r map* if for any i and j $\phi_i(f^{-1}(V_j) \cap U_i)$ is open and definable in $\mathbb{R}^n$ and the map $\psi_j \circ f \circ \phi_i^{-1} : \phi_i(f^{-1}(V_j) \cap U_i) \to \mathbb{R}^m$ is a definable C^r map.

(4) Let X and Y be definable C^r manifolds. We say that X is *definably C^r diffeomorphic to* Y (*definably homeomorphic* to Y if $r = 0$) if one can find definable C^r maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h = id$ and $h \circ f = id$.

(5) A definable C^r manifold is said to be *affine* if it is definably C^r diffeomorphic (definably homeomorphic if $r = 0$) to a definable C^r submanifold of some $\mathbb{R}^l$.

3. Sketches of proofs

Theorem 1.1 is obtained from the following three results.

Lemma 3.1 ([15]). *Let A be a definable set, $X_1 = \{(x_1, x_2) \in A \times [0,1] | f_1(x_1) < x_2 \leq f_2(x_1)\}, X_2 = \{(x_1, x_2) \in A \times [0,1] | f_2(x_1) \leq x_2 < f_3(x_1)\}$ and $\eta = (E, p, X, F, K)$ a definable fiber bundle over $X = X_1 \cup X_2$, where $f_i : A \to [0,1]$, $(1 \leq i \leq 3)$, are definable functions with $f_1 < f_2 < f_3$. If $\eta|X_1$ and $\eta|X_2$ are definably trivial, then η is definably trivial.*

Lemma 3.2 ([15]). *Let X be a compact definable set and $\eta = (E, p, X \times [0,1], F, K)$ a definable fiber bundle over $X \times [0,1]$. Then there exists a finite definable open covering $\{U_i\}_i$ of X such that each $\eta|(U_i \times [0,1])$ is definable trivial.*

Theorem 3.3 ([15]). *Let X be a compact definable set, $r : X \times [0,1] \to X \times [0,1], r(x,t) = (x,1)$ and $\eta = (E, p, X \times [0,1], F, K)$ a definable fiber bundle over $X \times [0,1]$. Then there exists a definable fiber bundle morphism $\phi : E \to E$ with $p \circ \phi = r \circ p$.*

Let $e_n : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}$ be the function defined by $e_n(x) = \begin{cases} e^{-exp_{n-1}(1/x^2)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, where $exp_0(x) = x$. Then elementary computations show the following proposition.

Proposition 3.4 ([10]). *(1) For any polynomial function $P(x_1, \dots, x_n)$ in n variables,*

$$\lim_{x \to 0} P(\frac{1}{x}, exp_1(\frac{1}{x^2}), \dots, exp_{n-1}(\frac{1}{x^2}))e_n(x) = 0.$$

(2) Every e_n is a C^∞ function.

Since $\mathcal{M}$ is exponentially bounded, a similar proof of C.14 [7] proves the following proposition.

Proposition 3.5 ([7], [10]). *Let A be a non-empty compact definable subset of $\mathbb{R}^n$ and f, g two definable functions on A such that $f^{-1}(0) \subset g^{-1}(0)$. If $\mathcal{M}$ is exponentially bounded, then there exist a natural number k and a positive constant c such that $e_k(g) \leq c|f|$ on A.*

Theorem 1.2 is proved by using the above two propositions.

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