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HOMEOMORPHISM GROUPS OF FINITE TOPOLOGICAL SPACES

SUSUMU KONO AND FUMIHIRO USHITAKI †

ABSTRACT. As being pointed out by several authors, finite topological spaces have more interesting topological properties than one might at first expect. In this article, we study the homeomorphism groups of finite topological spaces as finite topological groups. In particular, we obtain a short exact sequence of finite topological groups which contains Homeo(X).

1. INTRODUCTION

Let X be a finite set, and let X_n denote the n-point set $\{x_1, x_2, \cdots, x_n\}$. Let $\mathcal{T}$ be a topology on X, that is, $\mathcal{T}$ is a family of subsets of X which satisfies:

(1) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$;
(2) $A, B \in \mathcal{T} \Rightarrow A \cup B \in \mathcal{T}$;
(3) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$.

A finite set X with a topology is called a *finite topological space* or *finite space* briefly. A finite topological group is also defined canonically, but it is not assumed to satisfy any separation axioms. We say that a finite topological space $(X, \mathcal{T})$ is a finite T_0-space if it satisfies the T_0-separation axiom.

As several authors have pointed out, finite topological spaces have more interesting topological properties than one might at first expect. It is remarkable that for every finite topological space X, there exists a simplicial complex K such that X is weak homotopy equivalent to $|K|$ ([5]), and that the classification of finite topological spaces by homotopy type is reduced to a certain homeomorphism problem ([14]). Some relations with simple homotopy theory are revealed in [8]. Group actions on finite spaces have been also studied by several authors ([1], [3], [15]). In [15], Stong proved rather surprising results for the equivariant homotopy theory for finite T_0-spaces. One can find a survey of the theory of the finite topological spaces from topological viewpoints in [2].

For discussing the theory of topological transformation groups on a finite topological space $(X, \mathcal{T})$, it is necessary to consider Homeo(X), the homeomorphism group of X. The purpose of the present article is to study the homeomorphism groups of finite topological spaces as finite topological groups. Concerning its topological structure, Proposition 3.3 and Corollary 3.7 say that Homeo(X) decomposes into the disjoint union of connected components equipped with trivial topologies which are homeomorphic to each other.

According to [5], for every finite space X, there exists a quotient space $\hat{X}$ of X such that $\hat{X}$ is homotopic to X and satisfies T_0-separation axiom. Then, in Theorem 4.7, we have the following spliting exact sequence

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$$1 \longrightarrow \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]) \xrightarrow{\iota} \mathrm{Homeo}(X) \xrightarrow{\pi} \mathrm{Homeo}_X(\hat{X}) \longrightarrow 1,$$

where $\mathrm{Homeo}_X(\hat{X})$ is a subgroup of $\mathrm{Homeo}(X)$.

The rest of this article is organized as follows. Section 2 gives a brief introduction to the theory of finite topological spaces. In section 3, we investigate finite topological groups and the homeomorphism groups of finite topological spaces from a topological viewpoint. Section 4 is devoted to proving Theorem 4.7 which is our main result of this article. In the last section, we present a couple of examples including the homeomorphism groups of finite topological groups.

2. Preliminaries

Let $(X_n, \mathcal{T})$ be a finite topological space. Let U_i denote the minimal open set which contains x_i, that is, U_i is the intersection of all open sets containing x_i. We see that $\{U_1, U_2, \cdots U_n\}$ is an open basis of $\mathcal{T}$. For $\mathcal{T}$, we define an $n \times n$ - matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & x_j \in U_i \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is called the *topogenous matrix* of $(X_n, \mathcal{T})$ by Shiraki in his works on finite topological spaces ([12], [13]). If a matrix A is the topogenous matrix of some finite topological space, A is simply called a *topogenous matrix*. This matrix has been investigated by several authors ([4], [6], [7], [9], [10], [11], [12], [13]) for enumerating the possible topologies on X_n or creating some topological invariants of finite topological spaces. The following theorem by H. Sharp Jr. is fundamental.

Theorem 2.1 ([10] : Theorem 4). *A matrix $A = (a_{ij})$ is a topogenous matrix if and only if A satisfies the following conditions.*
(1) $a_{ij} = 0$ *or* 1.
(2) $a_{ii} = 1$.
(3) $A^2 = A$, *where matrix multiplication involves Boolean arithmetic.*

Let X be a finite topological space. We define an equivalence relation $\sim$ on X by

$$x_i \sim x_j \quad \text{if} \quad U_i = U_j.$$

Let $\hat{X}$ be the quotient space $X/\sim$, and $\nu_X : X \to \hat{X}$ the quotient map. We note that

$$\nu_X(x_i) = U_i \cap C_i,$$

where C_i is the smallest closed set containing x_i. From now on, we denote $\nu_X(x) \in \hat{X}$ by $[x]$. For simplisity we will often use the notation $[x]$ for $\nu_X^{-1}([x])$ which is a subset of X. The following theorem bridges the gap between general finite topological spaces and finite T_0-spaces.

Theorem 2.2 ([5] : Theorem 4). *Let X and Y be finite topological spaces. Then the following hold.*
(1) *The quotient map $\nu_X : X \to \hat{X}$ is a homotopy equivalence.*
(2) *The quotient space $\hat{X}$ is a finite T_0-space.*

(3) *For each continuous map* $\varphi : X \to Y$, *there exists a unique continuous map* $\hat{\varphi} : \hat{X} \to \hat{Y}$ *such that* $\nu_Y \varphi = \hat{\varphi} \nu_X$.

3. Finite topological groups and the homeomorphism groups of finite topological spaces

In this section, we propose some basic properties on finite topological groups and the homeomorphism groups of finite topological spaces.

Definition 3.1. A finite set G is called a *finite topological group* if G satisfies the following conditions.

(1) G is a group.
(2) The maps $\alpha : G \times G \to G$ and $\beta : G \to G$ defined by $\alpha(g, h) = gh$ and $\beta(g) = g^{-1}$ are continuous. Here $G \times G$ is equipped with the product topology.

Remark 3.2. (1) In the definition of topological groups it is usually assumed to be a Hausdorff space. However, we do not require the T_2-separation axiom on finite topological groups. We note that every finite Hausdorff space has the discrete topology.
(2) From now on, for a finite topological group G, the minimal open set which contains an element g will be denoted by U_g as well as U_x the minimal open neighbourhood of x in finite topological space X.

Let G be a finite topological group. For given element $g \in G$, the map $L_g : G \to G$ defined by $L_g(h) = gh$ is called the *left transformation map* by g, and the map $R_g : G \to G$ defined by $R_g(h) = hg$ is called the *right transformation map* by g. We see that L_g and R_g are homeomorphisms of G onto itself. On the topological structures of finite topological spaces, the following result holds.

Proposition 3.3. *Let G be a finite topological group, g an element of G. Let U_g denote the minimal open set which contains g. Then, the following hold.*

(1) *For $g, h \in G$, U_g is homeomorphic to U_h.*
(2) *For $g, h \in G$, $U_g \cap U_h \neq \emptyset$ implies $U_g = U_h$.*
(3) *U_g has the trivial topology.*
(4) *Let e be the unit of G. There exists a subset $\{e, g_1, \cdots, g_{k-1}\}$ of G such that G has the decomposition into the connected components as follows:*

$$G = U_e \cup U_{g_1} \cup \cdots \cup U_{g_{k-1}} \quad \textit{(disjoint union)}.$$

Proof. (1) Since $L_h \circ L_{g^{-1}}(U_g)$ is an open set which contains h, we obtain $U_h \subset L_h \circ L_{g^{-1}}(U_g)$. Simillarly we have $U_g \subset L_g \circ L_{h^{-1}}(U_h)$. Hence it holds that $U_h = L_h \circ L_{g^{-1}}(U_g)$.
(2) If $U_g \cap U_h \neq \emptyset$, take any element $k \in U_g \cap U_h$. Then (1) follows that $U_g = U_k = U_h$.
(3) It is an immediate consequence of (2).
(4) By (2), there exists a decomposition as

$$G = U_e \cup U_{g_1} \cup \cdots \cup U_{g_{k-1}} \quad \text{(disjoint union)}.$$

By (3), each component is connected. A subset of a finite topological space is a connected component if and only if it is an open and closed connected subset. Hence, the above is a decomposition into the connected components. $\square$

For studying topological groups, it is useful to investigate the connected component G_0 which contains e. Proposition 3.3 implies that $G_0 = U_e$. Moreover we have:

Theorem 3.4. *Let G be a finite topological group. Let e be the unit of G. and U_e the minimal open set which contains e. Then, U_e is a closed and open normal subgroup of G.*

Proof. Since U_e is a connected component, it is sufficient to show that it is a normal subgroup of G. Since both $\alpha(U_e \times U_e)$ and $\beta(U_e)$ are connected subset which contains e, we have $\alpha(U_e \times U_e) \subset U_e$ and $\beta(U_e) \subset U_e$, that is, U_e is a subgroup of G. For any $g \in G$, by a similar discussion as above, we have

$$gU_eg^{-1} = L_g \circ R_{g^{-1}}(U_e) \subset U_e,$$

that is, U_e is normal. □

Corollary 3.5. *Let I_r be an $r \times r$-matrix whose all entries are equal to 1. Let G be a finite topological group, and A the topogenous matrix of G. Then, A is equivalent to the matrix of the form*

$$E_k \otimes I_r = \begin{pmatrix} I_r & & & \\ & I_r & & \\ & & \ddots & \\ & & & I_r \end{pmatrix}$$

for some integers r and k, that is, there exists a permutation matrix P such that ${}^tPAP = E_k \otimes I_r$. Conversely, if the topogenous matrix of a finite topological space X is equivalent to $E_k \otimes I_r$ for some integers r and k, we can define a finite topological group structure on X.

Proof. Decompose G as

$$G = U_{g_0} \cup U_{g_1} \cup \cdots \cup U_{g_{k-1}} \quad \text{(disjoint union)},$$

as in Proposion 3.3, where $g_0 = e$. Suppose that U_{g_i} has r elements for $0 \leqq i \leqq k-1$. Put $U_{g_i} = \{g_{i1}, \dots, g_{ij}, \dots, g_{ir}\}$. If we regard g_{ij} as the $(ri+j)$-th element of G, the topogenous matrix A of G is $E_k \otimes I_r$.

Conversely, if the topogenous matrix of a finite topological space X is equivalent to $E_k \otimes I_r$ for some integers r and k, X is decomposed into the disjoint union of connected components with trivial topology as

$$X = U_1 \cup U_2 \cup \cdots \cup U_k \quad \text{(disjoint union)},$$

where $\#U_i = r$ for each $1 \leqq i \leqq k$. Let $C_{kr} =< t >$ be a finite cyclic group of order kr which is generated by t. We define subsets of C_{kr}, $V_1, V_2, \dots, V_k$ by $V_i = \{t^i, t^{i+k}, \dots, t^{i+(r-1)k}\}$. We now consider a topological space C_{kr} with the topology generated by $\{V_1, V_2, \dots, V_k\}$. Then, we see that it is a finite topological group and C_{kr} is isomorphic to X as topological spaces. □

Now, we consider the topologies of the homeomorphism group of a finite topological space. When it is equipped with the compact open topology, it becomes not only a topological space, but also a finite topological group.

Proposition 3.6. *The topological space* Homeo(X) *is a topological group, that is, the maps* α : Homeo$(X) \times$ Homeo$(X) \to$ Homeo(X) *and* β : Homeo$(X) \to$ Homeo(X) *defined by* $\alpha(g, f) = g \circ f$ *and* $\beta(f) = f^{-1}$ *are continuous. Moreover, the canonical action* $\theta :$ Homeo$(X) \times X \to X$ *defined by* $\theta(f, x) = f(x)$ *is continuous.*

Proof. For a subset K of X and an open subset U of X, set

$$O(K, U) = \{f \in \mathrm{Homeo}(X) | f(K) \subset U\}.$$

We note that $O(K, U) = \cap_{x \in K} O(\{x\}, U)$.

First, we show the continuousity of α. It is sufficient to prove that $\alpha^{-1}(O(\{x\}, U))$ is an open subset of Homeo$(X) \times$ Homeo(X) for every point $x \in X$ and every open subset U. Suppose that $(f, g) \in \alpha^{-1}(O(\{x\}, U))$. Set $V = g^{-1}(U)$. Then, we see that

$$(g, f) \in O(V, U) \times O(\{x\}, V) \subset \alpha^{-1}(O(\{x\}, U)),$$

that is, (g, f) has an open neighbourhood in $\alpha^{-1}(O(\{x\}, U))$.

Next, we prove that β is a continuous map. Suppose that $f \in \beta^{-1}(O(\{x\}, U))$, where $x \in X$ and U is an open subset of X. Set $V = f(U)$. Then we have $f \in O(U, V)$ and $x \in V$. Suppose $g \in O(U, V)$. Since g is a homeomorphism, we have $g(U) = V$, thus we obtain that $g^{-1}(x) \in U$. This implies $g \in \beta^{-1}(O(\{x\}, U))$, which means $O(U, V)$ is an open neighbourhood of f included in $\beta^{-1}(O(\{x\}, U))$.

Let U be an open set of X. For any $(f, x) \in \theta^{-1}(U)$, by putting $W = f^{-1}(U)$, we have

$$(f, x) \in O(W, U) \times W \subset \theta^{-1}(U),$$

which implies the continuity of θ. $\square$

The following corollary is an immediate result of Proposition 3.3 and Proposition 3.6.

Corollary 3.7. *There exists a subset* $\{id, f_1, \ldots, f_{k-1}\}$ *of* Homeo(X) *such that* Homeo(X) *decomposes as*

$$\mathrm{Homeo}(X) = U_{id} \cup U_{f_1} \cup \cdots \cup U_{f_{k-1}} \quad \textit{(disjoint union)},$$

where U_{id} *and each* U_{f_i} *are connected components of* Homeo(X).

In the following proposition, we treat one of the special cases of Corollary 3.7.

Proposition 3.8. *Let* X *be a finite topological space. The homeomorphism group* Homeo(X) *has the discrete topology if and only if* X *is a* T_0*-space.*

Proof. Let X be a finite T_0-space. By Corollary 3.7, it suffices to show that Homeo(X) satisfies the T_0-separation axiom. Let f and g be different homeomorphisms on X. Then, there exists a point $x \in X$ such that $f(x) \neq g(x)$. We may assume that there exists an open neighbourhood U of $f(x)$ which does not contain $g(x)$ without loss of generality. Then, we see that $O(\{x\}, U)$ is an open set containing f, but not g.

Conversely, suppose X does not satisfy the T_0-separation axiom. Then, there exist different points z and y of X such that $z \in U_y$ and $y \in U_z$. We note that $U_y = U_z$. Define a map $f : X \to X$ by

$$f(x) = \begin{cases} z & (x = y) \\ y & (x = z) \\ x & (\text{otherwise}). \end{cases}$$

Then, f is a homeomorphism, but is not the identity map on X. Suppose that $id_X \in O(K, U)$, where $K \subset X$ and U is an open subset of X. Then, $K = id_X(K) \subset U$ and $f(K) \subset \bigcup_{x \in K} U_x \subset U$. Hence $f \in O(K, U)$. This means that every open neighbourhood of id_X contains f, that is, $f \in U_{id_X}$. Similarly we have $id_X \in U_f$. Thus Homeo(X) does not satisfy the T_0-separation axiom. □

The following proposition also holds as usual.

Proposition 3.9. *Let X be a finite topological space, and G a topological group. Let $\varphi : G \times X \to X$ be a continuous action of G on X. Then, there exists unique continuous homomorphism $\Phi : G \to$ Homeo(X) such that $\varphi = \theta \circ (\Phi \times id_X)$.*

Proof. Since every element $g \in G$ defines a homeomorphism $\Phi(g)$ on X by $\Phi(g) = \varphi(g, x)$ where $x \in X$, we obtain a map $\Phi : G \to$ Homeo(X). The equality

$$\Phi(gh)(x) = \varphi(gh, x) = \varphi(g, \varphi(h, x)) = \Phi(g)(\Phi(h)(x)) = \Phi(g) \circ \Phi(h)(x)$$

shows that Φ is a group homomorphism. Suppose that $g \in \Phi^{-1}(O(\{x\}, U))$ where $x \in X$ and U is an open subset of X. Since φ is continuous, there exists an open neighbourhood W of g and an open neighbourhood V of x such that $\varphi(W \times V) \subset U$. Since for $h \in W$ it holds that $\Phi(h)(x) = \varphi(h, x) \in \varphi(W \times V) \subset U$, we have $h \in \Phi^{-1}(O(\{x\}, U))$. Thus we have $g \in W \subset \Phi^{-1}(O(\{x\}, U))$, which implies that Φ is continuous. By definition, we obtain

$$\varphi(g, x) = \Phi(g)(x) = \theta(\Phi(g), x) = \theta \circ (\Phi \times id_X)(g, x)$$

for every $(g, x) \in G \times X$.

Such a map $\Phi : G \to$ Homeo(X) is uniquely determined since if a map $\Phi' : G \to$ Homeo(X) satisfies $\varphi = \theta \circ (\Phi' \times id_X)$, it holds that

$$\Phi'(g)(x) = \theta(\Phi'(g), x) = \theta \circ (\Phi' \times id_X)(g, x) = \varphi(g, x) = \Phi(g)(x)$$

for every $(g, x) \in G \times X$. □

Proposition 3.9 indicates that if a topological group G acts on a finite topological space effectively, then it must be a finite topological group, and that the compact open topology is the weakest topology which makes the action of Homeo(X) on X continuous.

4. The structure of Homeo(X)

Now we consider the group structure of Homeo(X). We prepare the following lemma in order to reduce the problem of Homeo(X) to Homeo($\hat{X}$).

Lemma 4.1. *Let X be a finite topological space. Then, the map $\varphi :$ Homeo(X) $\times \hat{X} \to \hat{X}$ defined by $\varphi(f, [x]) = [f(x)]$ is a continuous action of Homeo(X) on $\hat{X}$.*

Proof. Since a homeomorphism preserves the equivalence relation, φ is well-defined. Since in the following commutative diagram, $id \times \nu_X$ is able to be regarded as a quotient map, the continuity of $\nu_X \circ \theta_X$ implies the continuity of φ.

$$\begin{array}{ccc} \mathrm{Homeo}(X) \times X & \xrightarrow{\theta_X} & X \\ {\scriptstyle id \times \nu_X} \downarrow & & \downarrow {\scriptstyle \nu_X} \\ \mathrm{Homeo}(X) \times \hat{X} & \xrightarrow[\varphi]{} & \hat{X} \end{array}$$

Since

$$\begin{aligned}\varphi(f\circ g,[x]) &= [f\circ g(x)] = [f(g(x))]\\ &= \varphi(f,[g(x)]) = \varphi(f,\varphi(g,[x]))\end{aligned}$$

and

$$\varphi(id,[x]) = [id(x)] = [x],$$

φ is a Homeo(X)-action on $\hat{X}$. □

Lemma 4.2. *There exists unique continuous homomorphism* $\pi : \mathrm{Homeo}(X) \to \mathrm{Homeo}(\hat{X})$ *such that the following diagram commutes:*

$$\begin{array}{ccc}\mathrm{Homeo}(X)\times X & \xrightarrow{\theta_X} & X\\ {\scriptstyle \pi\times\nu_X}\big\downarrow & & \big\downarrow{\scriptstyle \nu_X}\\ \mathrm{Homeo}(\hat{X})\times \hat{X} & \xrightarrow[\theta_{\hat{X}}]{} & \hat{X}\end{array}$$

Proof. It follows from Lemma 4.1 that there exists unique homomorphism $\pi : \mathrm{Homeo}(X) \to \mathrm{Homeo}(\hat{X})$ such that $\varphi = \theta_{\hat{X}} \circ (\pi \times id_{\hat{X}})$, where φ is the map obtained in Lemma 4.1. Then,

$$\begin{aligned}\theta_{\hat{X}}\circ(\pi\times\nu_X)(f,x) &= \theta_{\hat{X}}\circ(\pi\times id_{\hat{X}})(f,[x])\\ &= \varphi(f,[x]) = [f(x)] = [\theta_X(f,x)] = \nu_X\circ\theta_X(f,x).\end{aligned}$$

Suppose that $\theta_{\hat{X}} \circ (\pi' \times \nu_X) = \nu_X \circ \theta_X$ for another map $\pi' : \mathrm{Homeo}(X) \to \mathrm{Homeo}(\hat{X})$. Then,

$$\begin{aligned}\pi'(f)([x]) &= \pi'(f)(\nu_X(x)) = \theta_{\hat{X}}(\pi'(f),\nu_X(x))\\ &= \theta_{\hat{X}}\circ(\pi'\times\nu_X)(f,x) = \nu_X\circ\theta_X(f,x) = \pi(f)([x]).\end{aligned}$$

This shows the uniqueness of φ. □

The product $\prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$ is identified with the set of maps $F : \hat{X} \to \coprod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$ with $F([x]) \in \mathrm{Homeo}([x])$ for every $[x] \in \hat{X}$. Let F be an element of $\prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$. Then, F defines a map $\iota(F) : X \to X$ by $\iota(F)(x) = F([x])(x)$, under above identification. For $F, G \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$,

$$\begin{aligned}\iota(GF)(x) &= GF([x])(x) = G([x])\circ F([x])(x)\\ &= G([x])(F([x])(x))\\ &= \iota(G)(\iota(F)(x)) = \iota(G)\circ\iota(F)(x)\end{aligned}$$

for every $x \in X$ since it holds that $\nu_X((F([x]))(x)) = \nu_X(x)$. This implies that $\iota(GF) = \iota(G)\circ\iota(F)$ for every $F, G \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$. Then, we have the following theorem.

Theorem 4.3. *The map* ι *is continuous and the sequence*

$$1 \longrightarrow \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]) \xrightarrow{\ \iota\ } \mathrm{Homeo}(X) \xrightarrow{\ \pi\ } \mathrm{Homeo}(\hat{X})$$

is an exact sequence of finite topological groups.

Proof. By definition, it is clear that ι is a monomorphism. For any open set $O(K,U) \subset$ Homeo(X), we see that

$$\iota^{-1}(O(K,U)) = \begin{cases} \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]) & (\text{if } \bigcup_{x\in K} U_x \subset U) \\ \emptyset & (\text{otherwise}). \end{cases}$$

This shows that ι is continuous.

According to definition, we obtain

$$((\pi \circ \iota)(F))([x]) = [\iota(F)(x)] = [F([x])(x)] = [x] = id_{\hat{X}}([x])$$

for every $F \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$ and every $[x] \in \hat{X}$. Hence, it holds that $\pi \circ \iota(F) = id_{\hat{X}}$ for every $F \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$. Let f be an element of $\ker \pi$. Then, $f(x) \in [x]$ for every $x \in X$, thereby f defines an element $F \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$ by $F([x])(x) = f(x)$ for every $x \in X$. Then $\iota(F) = f$. □

Remark 4.4. For $[x] \in \hat{X}$, define a homomorphism $\iota_{[x]}$: Homeo$([x]) \to$ Homeo(X) by setting

$$(\iota_{[x]}(F_{[x]}))([y]) = \begin{cases} F_{[x]}(y) & (y \in [x]) \\ y & (\text{otherwise}), \end{cases}$$

where $F_{[x]} \in$ Homeo$([x])$ Then, $\alpha \circ (\iota_{[x]} \times \iota_{[y]}) = \alpha \circ (\iota_{[y]} \times \iota_{[x]})$ for every $[x], [y] \in \hat{X}$ and ι coincides with $\prod_{[x]\in\hat{X}} \iota_{[x]}$ followed by the composition.

Corollary 4.5. *Let X be a finite topological space. Let U_{id_X} be the identity component of* Homeo(X). *Then, we have*

$$U_{id_X} = \ker(\pi) = \mathrm{Im}(\iota) \cong \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]),$$

as finite topological groups.

Proof. By definition, $\hat{X}$ satisfies the T_0 separation axiom. It follows from Proposition 3.8 that Homeo$(\hat{X})$ has the discrete topology. Therefore the identity component U_{id_X} is contained in $\ker(\pi)$. Since $\prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$ has the trivial topology and connected, we have $\mathrm{Im}(\iota) \subset U_{id_X}$. Thus we obtain that $U_{id_X} = \ker(\pi) = \mathrm{Im}(\iota)$.

Since the map $\tilde{\iota} : \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]) \to \mathrm{Im}(\iota)$ defined by ι is an isomorphism between groups equipped with the trivial topology, ι is also a homeomorphism. □

Remark 4.6. Proposition 3.8 is a corollary of Corollary 4.5.

Set a subset Homeo$_X(\hat{X})$ of Homeo$(\hat{X})$ by

$$\mathrm{Homeo}_X(\hat{X}) = \left\{ f \in \mathrm{Homeo}(\hat{X}) \,\middle|\, \begin{array}{l} \#f([x]) = \#[x] \text{ for every } [x] \in \hat{X}, \\ \text{where the numbers are counted as subsets of } X \end{array} \right\}.$$

We see that Homeo$_X(\hat{X})$ is a subgroup of Homeo$(\hat{X})$. Between any $[x], [y] \subset X$ with $\#[x] = \#[y]$, we can construct a family of homeomorphisms

$$h_{[x],[y]} : [x] \to [y]$$

to satisfy the following conditions:

$$\begin{cases} h_{[y],[z]} \circ h_{[x],[y]} = h_{[x],[z]} \\ h_{[y],[x]} \circ h_{[x],[y]} = id_{[x]} \end{cases} \quad \text{for every } [x], [y], [z] \in \hat{X}.$$

For every $f \in \mathrm{Homeo}_X(\hat{X})$, define a map $\sigma(f) : X \to X$ by

$$(\sigma(f))(x) = h_{[x],f([x])}(x)$$

for every $x \in X$. Then, we have

$$\begin{aligned} (\sigma(f^{-1}) \circ \sigma(f))(x) &= \sigma(f^{-1})(h_{[x],f([x])}(x)) = h_{f([x]),f^{-1}(f([x]))}(h_{[x],f([x])}(x)) \\ &= h_{f([x]),[x]}(h_{[x],f([x])}(x)) = h_{f([x]),[x]} \circ h_{[x],f([x])}(x) \\ &= id_{[x]}(x) = x, \end{aligned}$$

and similarly

$$(\sigma(f) \circ \sigma(f^{-1}))(x) = x$$

for every $x \in X$. Hence $\sigma(f)$ is a bijection. For every $x \in X$,

$$\nu_X(\sigma(f)(x)) = \nu_X(h_{[x],f([x])}(x)) = f([x]) = f(\nu_X(x)),$$

that is, it holds that $\nu_X \circ \sigma(f) = f \circ \nu_X$. Let U be an open subset of X. Since $U = \nu_X^{-1}(\nu_X(U))$, $\nu_X(U)$ is an open subset of $\hat{X}$ and $\sigma(f)$ is continuous because

$$\begin{aligned} \sigma(f)^{-1}(U) &= \sigma(f)^{-1}(\nu_X^{-1}(\nu_X(U))) = (\nu_X \circ \sigma(f))^{-1}(\nu_X(U)) \\ &= (f \circ \nu_X)^{-1}(\nu_X(U))) = \nu_X^{-1}(f^{-1}(\nu_X(U))) \end{aligned}$$

is an open subset of X. Since $\sigma(f)^{-1} = \sigma(f^{-1})$, $\sigma(f)^{-1}$ is also continuous, thereby, $\sigma(f)$ is a homeomorphism on X. Let f and g be elements of $\mathrm{Homeo}_X(\hat{X})$. Then,

$$\begin{aligned} \sigma(f \circ g)(x) &= h_{[x],f\circ g([x])}(x) = h_{[x],f(g([x]))}(x) \\ &= h_{g([x]),f(g([x]))} \circ h_{[x],g([x])}(x) \\ &= h_{g([x]),f(g([x]))}(h_{[x],g([x])}(x)) \\ &= \sigma(f)(\sigma(g)(x)) = (\sigma(f) \circ \sigma(g))(x) \end{aligned}$$

for every $x \in X$. This implies that $\sigma : \mathrm{Homeo}_X(\hat{X}) \to \mathrm{Homeo}(X)$ is a homomorphism. Since $\mathrm{Homeo}_X(\hat{X})$ has the trivial topology, σ is continuous.

Now we have prepared to state the following theorem.

Theorem 4.7. *Let X be a finite topological space. Then, the following hold.*

(1) $\mathrm{Homeo}_X(\hat{X}) = \mathrm{Im}(\pi)$, *where π is the homomorphism defined in Lemma 4.2.*

(2) *The sequence*

$$1 \longrightarrow \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]) \xrightarrow{\iota} \mathrm{Homeo}(X) \xrightarrow{\pi} \mathrm{Homeo}_X(\hat{X}) \longrightarrow 1$$

is exact, where the same symbol π is used for the map defined by π in Lemma 4.2.

(3) $\pi \circ \sigma = id_{\mathrm{Homeo}_X(\hat{X})}$.

Proof. Suppose that $f \in \mathrm{Homeo}(X)$. Then, f defines a bijection $f|_{[x]} : [x] \to f([x]) = \pi(f)([x])$ for every $[x] \in \hat{X}$. This implies that $\#\pi(f)([x]) = \#[x]$ for every $[x] \in \hat{X}$, and hence $\pi(f) \in \mathrm{Homeo}_X(\hat{X})$, that is, $\mathrm{Homeo}_X(\hat{X}) \supset \mathrm{Im}(\pi)$. Suppose that $f \in \mathrm{Homeo}_X(\hat{X})$. Then, $\sigma(f) \in \mathrm{Homeo}(X)$ and $\pi(\sigma(f)) = f$. Thus, we have $\mathrm{Homeo}_X(\hat{X}) \subset \mathrm{Im}(\pi)$, which completes the proof of the part(1) of the theorem.

The rest of the theorem is directly obtained from Lemma 4.2 and the part (1) of this theorem. □

For $f \in \mathrm{Homeo}_X(\hat{X})$, define a map $\rho(f) : \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]) \to \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$ by

$$\rho(f)(F) = \iota^{-1}(\sigma(f)\iota(F)\sigma(f^{-1})),$$

where $F \in \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$. Then, $\rho(f) \in \mathrm{Aut}(\prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]))$ and $\rho : \mathrm{Homeo}_X(\hat{X})$ $\mathrm{Aut}(\prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]))$ is a continuous homomorphism.

As an corollary of Theorem 4.7, we obtain:

Corollary 4.8. *Define a map*

$$\kappa : \mathrm{Homeo}(X) \to (\prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])) \rtimes_\rho \mathrm{Homeo}_X(\hat{X})$$

by $\kappa(f) = (\iota^{-1}(f \circ (\sigma(\pi(f^{-1})))), \pi(f))$, *where* $f \in \mathrm{Homeo}(X)$. *Then,* κ *is an isomorphism of finite topological groups.*

Example 4.9. Let $(X_8, \mathcal{T})$ be a finite topological space with the topology which has the following open basis.

$$\{\ \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_4, x_5, x_6\}, \{x_7\}, \{x_7, x_8\}\ \}.$$

Then, the quotient space $\hat{X}$ is the set of six points

$$\{[x_1] = [x_2],\ [x_3],\ [x_4] = [x_5],\ [x_6],\ [x_7],\ [x_8]\}$$

with the topology generated by a open basis

$$\{\{[x_1]\}, \{[x_1], [x_3]\}, \{[x_4]\}, \{[x_4], [x_6]\}, \{[x_7]\}, \{[x_7], [x_8]\}\}.$$

We see that

$$\mathrm{Homeo}(\hat{X}) \cong \mathfrak{S}_3,\ \mathrm{Homeo}_X(\hat{X}) \cong \mathbb{Z}_2,\ \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x]) \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

and consequently,

$$\mathrm{Homeo}(X) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong D_4,$$

where D_4 is a dihedral group of order 8.

Remark 4.10. There are infinitely many finite topological groups which are not isomorphic to any homeomorphism groups of finite spaces as topological groups. As an example, consider the case that $\mathrm{Homeo}(X)$ has the trivial topology, Proposition 3.8 and Theorem 4.7 follow that $\mathrm{Homeo}(X) \cong \prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$. We see that $\prod_{[x]\in\hat{X}} \mathrm{Homeo}([x])$ is isomorphic to a direct product of symmetric groups. Thus, for example, there are no finite spaces whose homeomorphism groups are isomorphic to finite cyclic groups of order $m \geqq 3$ with the trivial topology.

5. Some special cases

Let us consider some special cases in which the homeomorphism groups have rather simple structures.

Proposition 5.1. *Let G be a finite topological group with n points and k connected components. Put $\ell = \dfrac{n}{k}$. Then,*

$$\mathrm{Homeo}(X) \cong (\mathfrak{S}_\ell^{tri})^k \rtimes_\rho \mathfrak{S}_k^{dis}$$

as topological groups, where $\mathfrak{S}_\ell^{tri}$ denotes the ℓ-th symmetric group with the trivial topology and $\mathfrak{S}_k^{dis}$ denotes the m-th symmetric group with the discrete topology and $\rho : \mathfrak{S}_k^{dis} \to \mathrm{Aut}((\mathfrak{S}_\ell^{tri})^k)$ is the continuous homomorphism defined by

$$\rho(g)(\tau_1, \tau_2, \ldots, \tau_k) = (\tau_{g^{-1}(1)}, \tau_{g^{-1}(2)}, \ldots, \tau_{g^{-1}(k)})$$

for every $(\tau_1, \tau_2, \ldots, \tau_k) \in (\mathfrak{S}_\ell^{tri})^k$ and $g \in \mathfrak{S}_k^{dis}$ which is regarded as the set of all permutations on $\{1, 2, \ldots, k\}$.

Proof. According to Proposition 3.3, each connected component has ℓ number of points. For any $g \in G$, it holds that $\mathrm{Homeo}([g]) \cong \mathfrak{S}_\ell^{tri}$. Since $\hat{G}$ has trivial topology, we obtain that $\mathrm{Homeo}_G(\hat{G}) = \mathrm{Homeo}(\hat{G}) \cong \mathfrak{S}_k^{dis}$. By using Corollary 4.8, we complete the proof. □

Remark 5.2. The group obtained in Proposition 5.1 is so called the wreath product of $\mathfrak{S}_\ell^{tri}$ and $\mathfrak{S}_k^{dis}$. It is usually written as $\mathfrak{S}_\ell^{tri} \wr \mathfrak{S}_k^{dis}$.

More generally, we have the following result. We can easily prove it by a similar discussion as Proposition 5.1.

Proposition 5.3. *Let X be a finite topological space with n points. If the canonical action $\mathrm{Homeo}(X) \times X \to X$ is transitive, then there exist positive integers ℓ and k satisfing $k\ell = n$ such that*

$$\mathrm{Homeo}(X) \cong \mathfrak{S}_\ell^{tri} \wr \mathfrak{S}_k^{dis}$$

as topological groups.

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Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, 560-0043 Japan
E-mail address: *kono@math.sci.osaka-u.ac.jp*

Department of Mathematics
Kyoto Sangyo University
Kyoto, 603-8555 Japan
E-mail address: *ushitaki@ksuvx0.kyoto-su.ac.jp*