数理解析研究所講究録 1290 巻 2002 年 31-41

NONSOLVABLE GENERAL LINEAR GROUPS ARE GAP GROUPS

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1. Introduction

In the theory of transformation groups, for a given smooth manifold, it is a problem what a subspace is obtained as a fixed point set of a smooth action on the manifold. If the smooth manifold is a disk or a euclidean space, Oliver [O2] has completely decided. The problem in the case when the smooth manifold is a sphere is studied by many person. A finite group G is *an Oliver group*, if G has no series of subgroups of the form $P \triangleleft H \triangleleft G$ where $|\pi(P)| \leq 1$, $|\pi(G/H)| \leq 1$ and H/K is cyclic. Here $\pi(G)$ is the set of primes dividing the order of G. Recall each nonsolvable group is an Oliver group and an Oliver group acts on a disk without fixed points. Laitinen and Morimoto has shown that a finite group G is a Oliver group if and only if G acts on a sphere with one fixed point. They gave a proof by using the equivariant surgery theory ([LM]). The equivariant surgery theory has been developed only on G-manifolds satisfying the weak gap condition (cf. [P], [PR], [M], [LüMa]). If a finite group is a gap group defined as below, we can apply the equivariant surgery theory and discuss whether a given subspace is realized as a fixed point sets of some smooth action on a sphere.

Let G be a finite group. Let $\mathcal{P}(G)$ be the set of all subgroups of prime power order (possibly 1) and set

$$\mathcal{D}(G) = \{ (P, H) \mid P < H \leq G \text{ and } P \in \mathcal{P}(G) \}.$$

For a prime p, let $O^p(G)$ be the smallest normal subgroup of G such that the index $[G : O^p(G)]$ is a power of p, namely

$$O^p(G) = \bigcap_H \{ H \mid H \trianglelefteq G \text{ and } [G : H] \text{ is a power of } p \}.$$

If the order $|G|$ of G is not divisible by p then $O^p(G)$ coincides with G. Let $\mathcal{L}(G)$ be the set of all subgroups of G which includes $O^p(G)$ for some prime p. A real (resp. complex) G-module should be understood to be a finite dimensional real (resp. complex) G-representation space. Let V be a G-module. We say that V is *$\mathcal{L}(G)$-free*, if $V^H = 0$ for all $H \in \mathcal{L}(G)$. An $\mathcal{L}(G)$-free G-module V is called a *gap G-module* if $\dim V^P > 2 \dim V^H$ for all $(P, H) \in \mathcal{D}(G)$. A finite group G not of prime power order is called a *gap group* if there is a gap G-module. Note that complexification of a gap real module is a gap complex module and realization of a gap complex module is also a gap real module. Any nonsolvalble perfect group is a gap group. However the symmetric group Σ_5 is not a gap group ([MY]). Doverman and Herzog [DH] has shown that symmetric groups Σ_n for $n > 5$ are all gap groups. In [MSY], we studied basic property which is useful to construct a gap module and in [Su1] we completely decided whether a product of symmetric groups is a gap group or not. The purpose of the paper is to decide whether general linear groups $GL(n, q)$ and projective linear groups $PGL(n, q)$ are gap groups. The result is as follows.

2000 *Mathematics Subject Classification.* 57S17, 20C15.
Key words and phrases. gap group, gap module, representation, linear group.

Theorem. *Let $n > 1$ be an integer and q be a power of a prime. The general linear group $GL(n, q)$ is a gap group if and only if $(n, q) \neq (2, 2), (2, 3)$. The projective general linear group $PGL(n, q)$ is a gap group if and only if either $n > 2$ or $n = 2$, $q \neq 2, 3, 5, 7, 9, 17$.*

2. Modules and conjugacy classes

Let G be a finite group not of prime power order. We construct an $\mathcal{L}(G)$-free gap G-module W to show the main theorem by using modules as below.

We set

$$\mathcal{D}^2(G) = \{(P, H) \in \mathcal{D}(G) \mid [H : P] = [HO^2(G) : PO^2(G)] = 2 \text{ and } PO^q(G) = G \text{ for all odd primes } q\}$$

and define a function $d_V : \mathcal{D}(G) \to \mathbb{Z}$ by

$$d_V(P, H) = \dim V^P - 2 \dim V^H$$

for a G-module V. In the proof by Laitinen and Morimoto [LM] that a finite group G has a one fixed point smooth action on a sphere S (that is, $S^G = \{x\}$) if and only if G is an Oliver group, they used a G-module

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \in \pi(G)} (\mathbb{R}[G] - \mathbb{R})^{O^p(G)}$$

to apply an equivariant surgery theory ([LM]). Morimoto [M] generalized this result. The module has a property that $d_{V(G)}(P, H) \geq 0$ for any $(P, H) \in \mathcal{D}(G)$ and $d_{V(G)}(P, H) > 0$ for any $(P, H) \in \mathcal{D}(G) \setminus \mathcal{D}^2(G)$ with $P \notin \mathcal{L}(G)$. We define an $\mathcal{L}(G)$-free G-module $V_{\mathcal{L}(G)}$ from G-module V by

$$V_{\mathcal{L}(G)} = (V - V^G) - \bigoplus_{p \in \pi(G)} (V - V^G)^{O^p(G)}.$$

For distinct primes p and q, $O^p(G)O^q(G) = G$ implies $V^{O^p(G)} \cap V^{O^q(G)} = V^G$. Then the direct sum is a G-submodule of $V - V^G$. In other words, regarding V as a G-submodule of $m\mathbb{R}[G]$ for sufficient large integer m, the module $V_{\mathcal{L}(G)}$ coincides with $V \cap mV(G)$. Clearly $\mathbb{R}[G]_{\mathcal{L}(G)} = V(G)$. For a subgroup K of G, we set

$$V(K; G) = \left(\mathrm{Ind}_K^G(\mathbb{R}[K] - \mathbb{R})\right)_{\mathcal{L}(G)}.$$

Given a gap subgroup K of G, we denote by $W(K; G)$ the induction $\mathrm{Ind}_K^G X$ for arbitrary gap K-module X. We should remark that the choice of X does not influence a construction of gap G-modules W to show the main theorem. By [MSY, Lemma 1.7], it holds $d_{W(K;G)}(P, H) \geq 0$ for any $(P, H) \in \mathcal{D}(G)$ and $d_{W(K;G)}(P, H) > 0$ if a conjugacy class of some element of H outside of P intersects with K.

If $\mathcal{L}(G) \cap \mathcal{P}(G) \neq \emptyset$, taking P an element of $\mathcal{L}(G) \cap \mathcal{P}(G)$, the group G is not a gap group since $d_V(P, G) = 0$ for any $\mathcal{L}(G)$-free module V. Hence $PGL(2, 2) \cong GL(2, 2) \cong D_6$ is not a gap group. (Remark that any dihedral group D_{2n} is not a gap group ([Su1]).) If there is an $\mathcal{L}(G)$-free G-module V such that $d_V(P, H) > 0$ for any $(P, H) \in \mathcal{D}^2(G)$, then $V \oplus (\dim V + 1)V(G)$ is an $\mathcal{L}(G)$-free gap G-module and thus we may construct such a module V. Let $(P, H) \in \mathcal{D}^2(G)$. Then H acts on $P\backslash G/K$ via $(h, PgK) \mapsto PhgK$. By [MSY, Lemma 2.1], we have

$$d_{V(K;G)}(P, H) = \left|(P\backslash G/K)^H\right| - \left|(O^2(G)P\backslash G/K)^H\right|.$$

We estimate the number of elements of the fixed point set $(P\backslash G/K)^H$.

Lemma 2.1. *Let K be a subgroup of G and L be a subgroup such that $K \leq L \leq N_G(K)$. If (P, H) is an element of $\mathcal{D}^2(G)$ with $(H \setminus P) \cap K \neq \emptyset$, then it holds*

$$|(P\backslash G/K)^H| \geq \frac{|L||K \cap P|}{|K||L \cap P|}.$$

Proof. By the proof of [MSY, Lemma 2.2], it holds

$$\left|(P\backslash G/K)^H\right| \geq \frac{|L|}{|L \cap PK|}.$$

Since an assignment $(L \cap P) \times K \to L \cap PK$ which (p, k) sends to pk is surjective, we obtain

$$\frac{|L \cap P||K|}{|K \cap P|} = |L \cap PK|$$

which concludes the proof. □

We review quite briefly about conjugacy classes of elements in $GL(n, q)$. Schur [Sc] and Jordan [J] studied independently the character of $GL(2, q)$. Let x_2 be an element of order $q^2 - 1$ of $GL(2, q)$. Let $GF(n)$ be a finite field consisting of n elements. $GF(q^2)$ is a two dimensional vector space over $GF(q)$. Since the multiplicative group $GF(q^2)^*$ is a cyclic group of order $q^2 - 1$, let σ be a generator of it. As viewing $GL(2, q)$ as $GL(GF(q^2))$, we define a map x_2 from $GF(q^2)$ to itself by $x_2(\gamma) = \sigma\gamma$. Then it is easy to see that the order of x_2 is $q^2 - 1$ and ${x_2}^{q+1}$ lies in the center $Z(GL(2, q))$. Furthermore x_2 is conjugate to $\begin{pmatrix} \sigma & 0 \\ 0 & \sigma^q \end{pmatrix}$ in $GL(2, q^2)$. It is also known that $N_{GL(2,q)}(\langle x_2 \rangle)$ is of order $2(q^2 - 1)$. Let $\rho = \sigma^{q+1}$ be a primitive element of $GF(q)$.

Conjugacy classes of linear groups has been studied (cf. [D, St]). Any element of $GL(2, q)$ is conjugate to one of the following elements in $GL(2, q)$:

$$\alpha_a = \begin{pmatrix} \rho^a & 0 \\ 0 & \rho^a \end{pmatrix}, \quad \beta_a = \begin{pmatrix} \rho^a & 0 \\ 1 & \rho^a \end{pmatrix}, \quad \gamma_{b,c} = \begin{pmatrix} \rho^b & 0 \\ 0 & \rho^c \end{pmatrix}, \quad {x_2}^d$$

where $0 \leq a < q - 1$, $0 \leq b < c < q - 1$ and $1 \leq d \leq q^2 - 1$ with $d \not\equiv 0 \mod q + 1$. Note that ${x_2}^a$ and ${x_2}^b$ are conjugate if and only if $b \equiv qa \mod q^2 - 1$.

Let n be an integer, τ a primitive element of $GF(q^n)$ and x_n an element of $GL(n, q)$ of order $q^n - 1$ conjugate to the diagonal matrix

$$diag(\tau, \tau^q, \tau^{q^2}, \cdots, \tau^{q^{n-1}})$$

in $GL(2, q^n)$. Any element of $GL(3, q)$ is conjugate to one of the following elements in $GL(3, q)$:

$$\begin{pmatrix} \alpha_a & 0 \\ 0 & \rho^a \end{pmatrix}, \begin{pmatrix} \alpha_a & 0 \\ 0 & \rho^b \end{pmatrix}, \begin{pmatrix} \beta_a & 0 \\ 0 & \rho^a \end{pmatrix}, \begin{pmatrix} \beta_a & 0 \\ 0 & \rho^b \end{pmatrix}, \begin{pmatrix} \gamma_{a,b} & 0 \\ 0 & \rho^c \end{pmatrix},$$

$$\begin{pmatrix} \rho^a & 0 \\ 0 & {x_2}^b \end{pmatrix}, \quad {x_3}^a, \quad \kappa_a = \begin{pmatrix} \rho^a & 0 & 0 \\ 1 & \rho^a & 0 \\ 0 & 1 & \rho^a \end{pmatrix}$$

Any element of $GL(4, q)$ is conjugate to one of the following twenty-two types in $GL(4, q)$:

$$\begin{pmatrix}\alpha_a & 0\\ 0 & \alpha_a\end{pmatrix}, \begin{pmatrix}\alpha_a & 0\\ 0 & \alpha_b\end{pmatrix}, \begin{pmatrix}\alpha_a & 0\\ 0 & \beta_a\end{pmatrix}, \begin{pmatrix}\alpha_a & 0\\ 0 & \beta_b\end{pmatrix}, \begin{pmatrix}\alpha_a & 0\\ 0 & \gamma_{a,b}\end{pmatrix}, \begin{pmatrix}\alpha_a & 0\\ 0 & \gamma_{b,c}\end{pmatrix},$$

$$\begin{pmatrix}\gamma_{a,b} & 0\\ 0 & \gamma_{c,d}\end{pmatrix}, \begin{pmatrix}\beta_a & 0\\ 0 & \gamma_{a,b}\end{pmatrix}, \begin{pmatrix}\beta_a & 0\\ 0 & \gamma_{b,c}\end{pmatrix}, \begin{pmatrix}\beta_a & 0\\ 0 & \beta_a\end{pmatrix}, \begin{pmatrix}\beta_a & 0\\ 0 & \beta_b\end{pmatrix},$$

$$\begin{pmatrix}\gamma_{a,b} & 0\\ 0 & {x_2}^c\end{pmatrix}, \begin{pmatrix}\alpha_a & 0\\ 0 & {x_2}^b\end{pmatrix}, \begin{pmatrix}\beta_a & 0\\ 0 & {x_2}^b\end{pmatrix}, \begin{pmatrix}{x_2}^a & 0\\ 0 & {x_2}^a\end{pmatrix}, \begin{pmatrix}{x_2}^a & 0\\ 0 & {x_2}^b\end{pmatrix},$$

$$\begin{pmatrix}\rho^a & 0\\ 0 & \kappa_a\end{pmatrix}, \begin{pmatrix}\rho^a & 0\\ 0 & \kappa_b\end{pmatrix}, \begin{pmatrix}\rho^a & 0\\ 0 & {x_3}^b\end{pmatrix}, \begin{pmatrix}{x_2}^a & 0\\ 1_2 & {x_2}^a\end{pmatrix}, \quad {x_4}^a, \quad \begin{pmatrix}\rho^a & 0 & 0 & 0\\ 1 & \rho^a & 0 & 0\\ 0 & 1 & \rho^a & 0\\ 0 & 0 & 1 & \rho^a\end{pmatrix}$$

In general each element of $GL(n,q)$ is conjugate to one of the following types (cf. [D, G, Sc]): $diag(X_1, X_2, \cdots, X_r)$ $(r \geq 1)$, where $1_{d_i} \in GL(d_i, q)$ is the identity matrix and

$$X_i = \begin{pmatrix}{x_{d_i}}^{a_i} & & & \\ 1_{d_i} & {x_{d_i}}^{a_i} & & \\ & \ddots & \ddots & \\ & & 1_{d_i} & {x_{d_i}}^{a_i}\end{pmatrix}.$$

We denote by ϕ: $GL(n,q) \to PGL(n,q)$ the canonical projection.

Proposition 2.2. *If either $n > 1$ is odd or q is even, then nonsolvable general linear groups $GL(n,q)$ and nonsolvable projective linear groups $PGL(n,q)$ are gap groups.*

Proof. The nonsolvable group $PSL(n,q)$ is a simple group and so is a gap group as $O^2(PSL(n,q))$ is whole $PSL(n,q)$. Since $[PGL(n,q) : PSL(n,q)] = GCM(n, q-1)$ is odd, the group $PGL(n,q)$ is a gap group by [MSY, Lemma 1.7] and so is $GL(n,q)$ by [Su1, Theorem 5.2]. □

Recall that $PSL(n,q)$ is a simple group unless $(n,q) = (2,2), (2,3)$ if $n > 1$. In the case where $n = 1$, the group $GL(1,q)$ is a cyclic group of order $q-1$ and $PGL(1,q)$ is the trivial group. Thus $GL(1,q)$ is a gap group if and only if the number $q-1$ is divisible by three distinct primes (cf. [MSY, Theorem 0.2]).

We close this section after we define some notation. For a partition $(n_1, \cdots, n_r)$ of n, that is $n_1 + \cdots + n_r = n$, we denote by $GL(n_1, \cdots, n_r; q)$ the canonical subgroup $GL(n_1, q) \times \cdots \times GL(n_r, q)$ of $GL(n,q)$. For a positive integer n, we denote by $n_{[2]}$ the largest number, which is a power of 2 and divides n. Let $n^{[2]} = n/n_{[2]}$.

3. $PGL(2,q)$ for $q = 3,5,7,9,17$ and $GL(2,3)$

In this section, $PGL(2,q)$ for $q = 3,5,7,9,17$ and $GL(2,3)$ are not gap groups. The characters of these groups are wellknown. For a subgroup K of G, the dimension of the fixed point set V^K is able to get from a character of V (cf. [MY]) and thus for $(P,H) \in \mathcal{D}^2(G)$, the number $d_V(P,H)$ is obtainable from the character of V as follows (cf. [Su1]):

$$d_V(P,H) = -\frac{1}{|P|} \sum_{h \in H \setminus P} \chi_V(h).$$

Here the symbol χ_V is a character of V. Let D be a dimension matrix over $\mathcal{D}^2(G)$, namely an entry of D is $d_V(P,H)$ where elements (P,H) of $\mathcal{D}^2(G)$ and $\mathcal{L}(G)$-free irreducible modules V are corresponding to columns and rows respectively. By [Su1], G is not a gap group, if there is an nonzero vector $\mathbb{y} \geq \mathbb{0}$ such that ${}^t\mathbb{y}D = \mathbb{0}$.

Consider $G = GL(2,3)$ of order 48. Any element of $GL(2,q)$ is conjugate to one of the following elements:

$$\begin{pmatrix}\rho^a & 0\\ 0 & \rho^b\end{pmatrix}, \quad \begin{pmatrix}\rho^a & 0\\ 1 & \rho^a\end{pmatrix}, \quad {x_2}^c,$$

where $1 \leq a \leq b < q$ and $1 \leq c < q^2$. The element x_2 is conjugate to $\begin{pmatrix}\sigma & 0\\ 0 & \sigma^q\end{pmatrix}$ in $GL(2,q^2)$ and thus x_2 and ${x_2}^q$ are conjugate in G, where σ is a primitive element of $GL(q^2)$. The character table of $GL(2,3)$ is as follows (cf. [St]):

	$\chi_1^{(i)}$	$\chi_q^{(i)}$	$\chi_{q+1}^{(m,n)}$	$\chi_{q-1}^{(k)}$
$\begin{pmatrix}\rho^a & 0\\ 0 & \rho^a\end{pmatrix}$	$\epsilon^{2ia(q+1)}$	$q\epsilon^{2ia(q+1)}$	$(q+1)\epsilon^{(m+n)a(q+1)}$	$(q-1)\epsilon^{ka(q+1)}$
$\begin{pmatrix}\rho^a & 0\\ 1 & \rho^a\end{pmatrix}$	$\epsilon^{2ia(q+1)}$	0	$\epsilon^{(m+n)a(q+1)}$	$-\epsilon^{ka(q+1)}$
$\begin{pmatrix}\rho^a & 0\\ 0 & \rho^b\end{pmatrix}$	$\epsilon^{i(a+b)(q+1)}$	$\epsilon^{i(a+b)(q+1)}$	$\epsilon^{(ma+nb)(q+1)} + \epsilon^{(mb+na)(q+1)}$	0
$\begin{pmatrix}\sigma^c & 0\\ 0 & \sigma^{cq}\end{pmatrix}$	$\epsilon^{ic(q+1)}$	$-\epsilon^{ic(q+1)}$	0	$-\epsilon^{kc} - \epsilon^{kcq}$

Here $1 \leq a, b < q$, $a \neq b$, $1 \leq c < q^2$ with $c/(q+1) \notin \mathbb{Z}$, $1 \leq i < q$, $1 \leq i < j < q$, $1 \leq k < q^2 - 1$ with $k/(q+1) \notin \mathbb{Z}$, and $\epsilon^{q^2-1} = 1$. Irreducible modules $\chi_1^{(i)}$ are not $\mathcal{L}(G)$-free but the others are. Let H be a Sylow 2-subgroup of G and set $P = H \cap SL(2,3)$. Then (P,H) belongs to $\mathcal{D}^2(G)$ and $d_V(P,H)$ is zero for any $\mathcal{L}(G)$-free irreducible modules. Therefore $GL(2,3)$ is not a gap group.

Next consider $G = PGL(2,q)$ for $q = 3, 5, 7, 9, 17$. As $PGL(2,3)$ is isomorphic to the symmetric group Σ_4 on 4 letters, the group $PGL(2,3)$ is not a gap group. Any element of $PGL(2,q)$ is conjugate to one of the following elements:

$$\phi\begin{pmatrix}\rho^a & 0\\ 0 & 1\end{pmatrix}, \quad \phi\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \quad \phi({x_2}^b),$$

where $0 \leq a < q-1$ and $1 \leq b < q+1$. In the case when $a = b = 1$ these elements are of order $q-1$, q, and $q+1$ respectively. The character table of $PGL(2,q)$ is known (cf. [St]):

	χ_1	χ_1'	χ_q	χ_q'	$\chi_{q+1}^{(i)}$	$\chi_{q-1}^{(j)}$
$\phi\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	1	1	q	q	$q+1$	$q-1$
$\phi\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$	1	1	0	0	1	-1
$\phi\begin{pmatrix}\rho^a & 0\\ 0 & 1\end{pmatrix}$	1	$(-1)^a$	1	$(-1)^a$	$\epsilon^{ia(q+1)} + \epsilon^{-ia(q+1)}$	0
$\phi({x_2}^b)$	1	$(-1)^b$	-1	$(-1)^{b+1}$	0	$-\epsilon^{jb(q-1)} - \epsilon^{-jb(q-1)}$

Here $1 \le a < q-1$, $1 \le b < q+1$, $1 \le i \le (q-1)/2$, $1 \le j \le (q+1)/2$, and $\epsilon^{q^2-1} = 1$. Irreducible modules χ_1 and χ_1' are not $\mathcal{L}(G)$-free but the others are. Let $q = 3, 5, 7, 9, 17$. Note each $(q-1)/2$ and $(q+1)/2$ is a power of a prime or 1. Setting $H = C_{q\mp1}$ and $P = H \cap PSL(2, q)$, it holds

$$(d_{\chi_q}(P, H), d_{\chi_q'}(P, H), d_{\chi_{q+1}^{(1)}}(P, H), \ldots, d_{\chi_{q-1}^{(\frac{q-1}{2})}}(P, H)) = (\mp1, \pm1, 0, \cdots, 0),$$

respectively. Therefore

$$d_V\left(C_{\frac{q+1}{2}}, C_{q+1}\right) + d_V\left(C_{\frac{q-1}{2}}, C_{q-1}\right) = 0$$

for any $\mathcal{L}(PGL(2, q))$-free module V. This implies that $PGL(2, q)$ is not a gap group.

The question stated in [MSY] is false. There are many counterexamples. For example the group $PGL(2, 7)$ is as $O^2(PGL(2, 7)) = PSL(2, 7)$ is isomorphic to the alternating group Alt_5.

4. $GL(2, q)$ for $q \ge 5$ odd

The group $G = GL(2, q)$ is of order $q(q-1)(q^2-1)$. Suppose $q \ge 5$ is odd and we show that $G = GL(2, q)$ is a gap group. In the next section we show that $PGL(2, q)$ for $q \ne 3, 5, 7, 9, 17$ is a gap group. By this, $GL(2, q)$ is automatically a gap group for $q \ne 3, 5, 7, 9, 17$ since $GL(2, q)$ has a gap group $PGL(2, q)$ as a quotient group (cf. [Su1]). However we can construct an $\mathcal{L}(G)$-free gap G-module all together.

Let K be the normal subgroup generated by elements of $Z(G)$ and $SL(2, q)$. Then K has a quotient group $PSL(2, q)$ and is a gap group for $q \ge 4$, since $PSL(2, q)$ is simple. Let K_1 be the subgroup generated by two elements $\begin{pmatrix} \rho^{(q-1)^{[2]}} & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & \rho^{(q-1)^{[2]}} \end{pmatrix}$, and K_2 be a cyclic group generated by an element $\phi(x_2)^{(q^2-1)^{[2]}}$. Then the order of K_1 and K_2 are $((q-1)^2)_{[2]}$ and $(q^2-1)_{[2]}$ respectively. Set

$$W = 2V(K_1; G) \oplus V(K_2; G) \oplus 4W(K; G).$$

We claim that $V = W \oplus (\dim W + 1)V(G)$ is a gap module. It is sufficient to show that $d_W(P, H) > 0$ for any $(P, H) \in \mathcal{D}^2(G)$. Let $(P, H) \in \mathcal{D}^2(G)$. It holds $d_{W(K;G)}(P, H)$ is nonnegative and in particular positive if $(H \smallsetminus P) \cap gKg^{-1}$ is not empty for some $g \in G$. Since $|(PO^2(G)\backslash G/K_i)^H| = 1$, we have $d_{V(K_i;G)}(P, H) \ge -1$ in general and

$$d_{V(K_i;G)}(P, H) \ge \frac{|L_i||K_i \cap g^{-1}Pg|}{|K_i||L_i \cap g^{-1}Pg|} - 1,$$

for some subgroup L_i if $(H \smallsetminus P) \cap gK_ig^{-1} \ne \emptyset$ by Lemma 2.1. In particular, $d_{V(K_1;G)}(P, H) > 0$ yields $d_W(P, H) > 0$. In the case when $(H \smallsetminus P) \cap K \ne \emptyset$, we obtain $d_W(P, H) \ge -2 - 1 + 4 > 0$. We consider in the case when $(H \smallsetminus P) \cap K = \emptyset$. Any elements of G of order 2 is conjugate to either $h_1 = \begin{pmatrix} \rho^{\frac{q-1}{2}} & 0 \\ 0 & 1 \end{pmatrix}$ or $h_2 = \begin{pmatrix} \rho^{\frac{q-1}{2}} & 0 \\ 0 & \rho^{\frac{q-1}{2}} \end{pmatrix} \in K$. Set $L_1 = N_G(K_1)$ and let L_2 be the subgroup $N_G(K_2)$ of order $2(q^2-1)$. If $(q-1)_{[2]} \ne q-1$, that is, $q-1$ is not a power of 2, then P is not a 2-group and there is an elements of $H \smallsetminus P$ of order 2. Since

$$\frac{|L_1||K_1 \cap g^{-1}Pg|}{|K_1||L_1 \cap g^{-1}Pg|} \ge 2, \quad \frac{|L_2||K_2 \cap g^{-1}Pg|}{|K_2||L_2 \cap g^{-1}Pg|} \ge 2(q-1)_{[2]}(q+1) \ge 12,$$

it holds either $d_{V(K_1;G)}(P, H) \ge 1$ or $d_{V(K_2;G)}(P, H) \ge 11$. which yields $d_W(P, H) > 0$. In any cases, we have $d_W(P, H) > 0$. Therefore if $(q-1)^{[2]} > 1$, then $H \smallsetminus P$ has an element of order 2 and thus $d_W(P, H) > 0$ for any element (P, H) of $\mathcal{D}^2(G)$ which implies W is a gap module.

Now let $q-1$ be a power of 2. The element h_1 is an element of K as

$$h_1 \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}^{\frac{q-1}{4}} \in SL(2,q).$$

Let $(P,H) \in \mathcal{D}^2(G)$. Recall $d_W(P,H) > 0$ if the order of P is odd. Suppose P is a (nontrivial) 2-group. Note that L_1 is a 2-Sylow subgroup of G, and a Sylow 2-subgroup of L_2 is a cyclic group of order $2(q-1)$. Take an element $g \in G$ such that $g^{-1}Hg \leq L_1$. Recall that any element of G of 2 power order is conjugate to some element of either K_1 or K_2. It holds

$$\frac{|L_1||K_1 \cap g^{-1}Pg|}{|K_1||L_1 \cap g^{-1}Pg|} = \frac{2|K_1 \cap g^{-1}Pg|}{|g^{-1}Pg|}.$$

This number equals to 2 if $g^{-1}Pg = K_1 \cap g^{-1}Pg$. Assume that $g^{-1}(H \setminus P)g \cap K_1 \neq \emptyset$. If $g^{-1}Pg = K_1 \cap g^{-1}Pg$, then $d_{V_1(K_1;G)} > 0$ and thus $d_W(P,H) > 0$. We claim that $g^{-1}Pg > K_1 \cap g^{-1}Pg$ implies $h^{-1}(H \setminus P)h \cap K_2 \neq \emptyset$ for some $h \in G$. Suppose $K_1 \cap g^{-1}Pg \neq g^{-1}Pg$. Take an element α of $g^{-1}Pg \setminus K_1$ and an element $h \in g^{-1}(H \setminus P)g \cap K_1$. Then $h\alpha \in g^{-1}(H \setminus P)g \cap (L_1 \setminus K_1)$ and thus it is conjugate to an element of K_2. In the case where $k^{-1}(H \setminus P)k \cap K_2 \neq \emptyset$ for some $k \in G$, it holds

$$d_{V(K_2;G)}(P,H) \geq \frac{q+1}{2}, \quad d_W(P,H) \geq -2 + \frac{q+1}{2} - 1 = \frac{q-5}{2}.$$

Hence W is a gap module for $q > 5$. Finally we must consider in the case when $q = 5$. Assume that $g^{-1}(H \setminus P)g \cap K_2 \neq \emptyset$, $L_2 \cap g^{-1}Pg \neq K_2 \cap g^{-1}Pg$ for some $g \in G$ and $k^{-1}(H \setminus P)k \cap K_1 = \emptyset$ for any $k \in G$. Then H be a Sylow 2-subgroup of L_2 generated by $\begin{pmatrix} 0 & \rho \\ 1 & 0 \end{pmatrix}$ and $P = H \cap K$ up to conjugate. It follows that $(P\backslash G/K_2)^H$ consists of 6 elements

$$PeK_2,\ P\begin{pmatrix} 1 & \rho^3 \\ 1 & \rho^2 \end{pmatrix}K_2,\ P\begin{pmatrix} \rho^2 & \rho^3 \\ \rho^2 & \rho^2 \end{pmatrix}K_2,\ P\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}K_2,\ P\begin{pmatrix} \rho & 1 \\ 1 & \rho^2 \end{pmatrix}K_2,\ P\begin{pmatrix} \rho^3 & 1 \\ \rho^2 & \rho^2 \end{pmatrix}K_2.$$

Thus it holds $d_{V(K_2;G)}(P,H) = 5$. Therefore we also conclude that $d_W(P,H) > 0$ in all cases.

5. $PGL(2,q)$ for $q \geq 3$

Recall if q is even, a nonsolvable general projective linear group $PGL(n,q)$ is a gap group. Let $q \neq 1,3,5,7,9,17$ be a power of an odd prime. We show that $PGL(2,q)$ is a gap group. An element of $PGL(2,q)$ outside of $PSL(2,q)$ is of order $q-1$, q or $q+1$. Either $(q-1)/2$ or $(q+1)/2$ is not a power of a prime.

Lemma 5.1. (1) $3^{2^k} > 2^{k+2} + 1$ *for* $k \geq 2$.
(2) $3^{2^k} \equiv 2^{k+2} + 1 \mod 2^{k+3}$ *for* $k \geq 1$.
(3) *The equation* $2^n + 1 = 3^m$ *implies* $(n,m) = (1,1), (3,2)$.
(4) $2^n - 1 = 3^m$ *only if* $(n,m) = (1,0), (2,1)$.

Proof. (1) If $k > 2$, then $2^k > k+3$ implies $3^{2^k} > 3^{k+3} > 2 \cdot 2^{k+2} > 2^{k+2} + 1$. If $k = 2$, $3^{2^2} = 81 > 2^4 + 1 = 17$.

(2) We show the assertion by induction. It is clear that $3^2 = 2^3 + 1$ in the case when $k = 1$. Assuming $3^{2^k} = 2^{k+3}x + 2^{k+2} + 1$ for some integer $x \geq 0$, it holds $3^{2^{k+1}} = (3^{2^k})^2 = (2^{k+2}(2x+1)+1)^2 = 2^{2k+4}(2x+1)^2 + 2^{k+3}(2x+1) + 1 \equiv 2^{k+3} + 1 \mod 2^{k+4}$.

(3) If $m = 1$, then $2^n = 2$ and thus $n = 1$. Let $m > 1$. Since $3^m - 1 = (2+1)^m - 1 = 2(m + 2\sum_{j=2}^{m} {}_mC_j \cdot 2^{j-2})$, m is divisible by 2. Let $a_k = 3 \cdot 2^{k-1} - 2$ $(k \geq 1)$. It holds that $a_1 = 1$ and $a_{k+1} = 2a_k + 2$. Take $k \geq 1$ such that 2^{a_k} divides m but $2^{a_{k+1}}$ does not. Set $m = 2^{a_k}\ell$. We obtain $3^m - 1 = (3^{2^{a_k}})^\ell - 1 = (2^{a_k+3}x + 2^{a_k+2} + 1)^\ell - 1 = (2^{a_k+2}(2x+1)+1)^\ell - 1 = \ell \cdot 2^{a_k+2}(2x+1) + 2^{2a_k+4}y$. If y is positive, 2^{a_k+2} divides ℓ and thus $2^{a_{k+1}}$ divides m, which is contradiction. Thus $y = 0$, $\ell = 1$, $m = 2^{a_k}$. If $a_k > 1$, then $2x + 1 > 1$ is odd and thus $3^m - 1$ is not 2 power. Then $a_k = 1$, $k = 1$, $m = 2$ and $n = 3$.

(4) $3^m + 1 = (2+1)^m + 1 = 2(1+m) + 4z$, where $z = \sum_{j=2}^{m} {}_mC_j \cdot 2^{j-2}$. If $z = 0$, then $m = 0$ or 1, and $q = 2$ or 2^2 respectively. Suppose $z > 0$. Since 4 divides $3^m + 1$ (Note $3^m + 1 > 4$), m is odd, set $m = 2\ell + 1$ $(\ell > 0)$. It implies that $3^m + 1 = (2+1)(2^3+1)^\ell + 1 = 4 \neq 0 \mod 2^3$, which is contradiction. □

Proposition 5.2 (cf. [OP]). (1) *Let q be a power of 2. If $q-1$ and $q+1$ are prime power, possibly 1, then $q = 2, 4, 8$.*

(2) *Let $q > 1$ be odd prime power. If $\frac{q-1}{2}$ and $\frac{q+1}{2}$ are prime power, possibly 1, then $q = 3, 5, 7, 9, 17$.*

Proof. First note that $q(q^2-1)$ is divisible by 6 and $GCM(q-1, q+1) \leq 2$.

(1) Let $q = 2^b$. We show the assertion by dividing 2 cases. The first case is $q - 1 = 3^a$ and $q + 1 = p^c$, $(p \neq 2, 3)$. By Lemma 5.1 (4), it holds $b = 1, 2$ and thus $q = 2, 4$. In the case where $q - 1 = p^a$ and $q + 1 = 3^c$, $(p \neq 2, 3)$, it holds $b = 1, 3$ and thus $q = 2, 8$ by Lemma 5.1 (3). Therefore $q = 2, 4, 8$ only occurs.

(2) Since q is odd, either $q - 1$ or $q + 1$ is divisible by 4. We use Lemma 5.1 in each case. If $q - 1 = 2^a, q = 3^b, q + 1 = 2p^c$, we have $q = 3, 9$. If $q - 1 = 2^a, q = p^b, q + 1 = 2 \cdot 3^c$, it holds that $2^{a-1} + 1 = 3^c$ and $c = 1, 2$, $q = 5, 17$. If $q - 1 = 2 \cdot 3^a, q = p^b, q + 1 = 2^c$, we obtain $3^a + 1 = 2^{c-1}$ and $a = 0, 1$, $q = 3, 7$. If $q - 1 = 2 \cdot p^a, q = 3^b, q + 1 = 2^c$, then $3^b + 1 = 2^c$, $b = 0, 1$ and thus $q = 1, 3$. Therefore $q = 3, 5, 7, 9, 17$. □

Take subgroups

$$K_- = \left\langle \phi\begin{pmatrix} \rho^{(q-1)^{[2]}} & 0 \\ 0 & 1 \end{pmatrix} \right\rangle, \quad K_+ = \left\langle \phi(x_2)^{(q+1)^{[2]}} \right\rangle,$$

$$N_- = D_{2(q-1)} = \left\langle \phi\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix}, \phi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \quad N_+ = D_{2(q+1)}.$$

Then the order of $K_\mp$, $N_\mp$ is $(q \mp 1)_{[2]}$, $2(q \mp 1)$ respectively. Any elements of G of 2 power order is conjugate to some element of either of K_- or K_+. Set $W_- = 2V(K_-; G) \oplus V(K_+; G)$ and $W_+ = V(K_-; G) \oplus 2V(K_+; G)$. We show either W_+ or W_- is a gap module. Note that any element of G of order 2 is conjugate to either $h_- = \phi\begin{pmatrix} \rho^{\frac{q-1}{2}} & 0 \\ 0 & 1 \end{pmatrix}$ or $h_+ = \phi(x_2)^{\frac{q+1}{2}}$. Furthermore note that $h_\mp \in PSL(2,q)$ and $h_\pm \notin PSL(2,q)$, if $q \mp 1$ is divisible by 4 respectively. Let $(P, H) \in \mathcal{D}^2(G)$. Since $|(PO^2(G)\backslash G/K_\pm)^H| = 1$, we have $d_{V(K_\pm;G)}(P, H) \geq -1$ and

$$d_{V(K_\pm;G)}(P, H) \geq \frac{|N_\pm||K_\pm \cap g^{-1}Pg|}{|K_\pm||N_\pm \cap g^{-1}Pg|} - 1 \geq 0.$$

if $(H \smallsetminus P) \cap gK_\pm g^{-1} \neq \emptyset$ for some $g \in G$. If there exist elements $\alpha \in (N_- \smallsetminus K_-) \cap g^{-1}Pg$ and $\beta \in g^{-1}(H \smallsetminus P)g \cap K_-$, then the element $\alpha\beta \in g^{-1}(H \smallsetminus P)g \cap (N_- \smallsetminus K_-)$ is conjugate to the element of K_+ of order 2. Similarly, if $(N_+ \setminus K_+) \cap g^{-1}Pg$ and $g^{-1}(H \setminus P)g \cap K_+$ are nonempty sets, then there exists an element $g^{-1}(H \setminus P)g \cap (N_- \setminus K_-)$ of order 2 which is conjugate to the element of K_+. Consider separating three cases.

The first case is where $q \mp 1 \geq 10$ is a power of 2. We claim $d_{W_\mp}(P, H) > 0$. Suppose $g^{-1}(H \setminus P)g \cap K_\pm \neq \emptyset$. Recall that $(q \pm 1)^{[2]} = (q \pm 1)/2$ is a composite odd integer. If P is of odd order, then $d_{V(K_\pm;G)}(P, H) \geq 6 - 1 = 5$ and if P is of 2 power order, then $d_{V(K_\pm;G)}(P, H) \geq 15 - 1 > 5$. Therefore it holds $d_{W_\mp}(P, H) \geq -2 + 5 > 0$. Next suppose $g^{-1}(H \setminus P)g \cap K_\mp \neq \emptyset$. Then P is a 2-group, since supposing P is of odd order, there exists an element of $g^{-1}(H \setminus P)g \cap K_\mp$ of order 2 which belongs to $PSL(2, q)$, contradiction. If $L_\mp \cap g^{-1}Pg = K_\mp \cap g^{-1}Pg$, then it holds $d_{V(K_\mp;G)}(P, H) \geq 2 - 1 = 1$ and $d_{W_\mp}(P, H) \geq 2 - 1 > 0$. By the above fact, $L_+ \cap g^{-1}Pg > K_+ \cap g^{-1}Pg$ does not occur and $L_- \cap g^{-1}Pg > K_- \cap g^{-1}Pg$ yields $d_{W_-}(P, H) \geq 0 + 5 > 0$.

The second case is where $q \mp 1 = 4k$ such that $k \geq 3$ is not a power of 2 and $(q \pm 1)/2$ is a power of an odd prime. We show $d_{W_\pm}(P, H) > 0$. First suppose $g^{-1}(H \setminus P)g \cap K_\pm \neq \emptyset$. If P is of odd order, then it holds $d_{V(K_\pm;G)}(P, H) \geq 2 - 1 = 1$ and if P is of even order, then $d_{V(K_\pm;G)}(P, H) \geq (q \pm 1)^{[2]} - 1 \geq 6 - 1 > 1$. Therefore it holds $d_{W_\pm}(P, H) \geq 2 - 1 > 0$. Next suppose $g^{-1}(H \setminus P)g \cap K_\mp \neq \emptyset$. Then P is a 2-group. If $L_\mp \cap g^{-1}Pg = K_\mp \cap g^{-1}Pg$, then it holds $d_{V(K_\mp;G)}(P, H) \geq 2(q \mp 1)^{[2]} - 1 \geq 5$ and $d_{W_\pm}(P, H) \geq 5 - 2 > 0$. If $L_- \cap g^{-1}Pg > K_- \cap g^{-1}Pg$, then it holds $d_{V(K_-;G)}(P, H) \geq 0$ and $d_{W_+}(P, H) \geq 0 + 2 > 0$ and it is impossible that $L_+ \cap g^{-1}Pg > K_+ \cap g^{-1}Pg$.

The third case is where $q \mp 1 = 4k$ such that $k \geq 3$ is not a power of 2 and $(q \pm 1)/2$ is a composite odd integer. We show $d_{W_\mp}(P, H) > 0$ in this case. Supposing $g^{-1}(H \setminus P)g \cap K_\pm \neq \emptyset$, if P is of odd order then it holds $d_{V(K_\pm;G)}(P, H) \geq 6 - 1 = 5$ and if P is of even order, then $d_{V(K_\pm;G)}(P, H) \geq (q \pm 1)^{[2]} - 1 \geq 15 - 1 > 5$. Therefore it holds $d_{W_\mp}(P, H) \geq 5 - 2 > 0$. Suppose $g^{-1}(H \setminus P)g \cap K_\mp \neq \emptyset$. Then P is a 2-group. If $L_\pm \cap g^{-1}Pg = K_\pm \cap g^{-1}Pg$, then it holds $d_{V(K_\mp;G)}(P, H) \geq 2(q \mp 1)^{[2]} - 1 \geq 5$ and $d_{W_\mp}(P, H) \geq 10 - 1 > 0$. If $L_- \cap g^{-1}Pg > K_- \cap g^{-1}Pg$, then it holds $d_{V(K_-;G)}(P, H) \geq 0$ and $d_{W_+}(P, H) \geq 0 + 10 > 0$ and it is impossible that $L_+ \cap g^{-1}Pg > K_+ \cap g^{-1}Pg$. (Similarly, $d_{W_\pm}(P, H) > 0$ holds.)

Putting all together, this completes the proof.

6. *PGL(n, q)* for *n* ≥ 4 even and *q* ≥ 5 odd

We show that $PGL(4, q)$ is a gap group for $q \neq 3, 5, 7, 9, 17$. Recall $PGL(3, q)$ and $PGL(2, q)$ are gap groups and then so are $PGL(3, 1; q) \cong GL(3, q)$ and $PGL(2, 2; q)$. Let $(P, H) \in \mathcal{D}^2(PGL(4, q))$. Consider any element z of H outside of P of 2-power order. If z is not conjugate to an element of $\langle x_4 \rangle$, a conjugacy class of z intersects with a set $PGL(2, 2; q) \cup PGL(3, 1; q)$. Set

$$K_1 = \left\langle \phi(x_4)^{(q^4-1)^{[2]}} \right\rangle.$$

Note that $d_{V(K_1;G)}(P, H) \geq 2 - 1 > 0$, if the conjugacy class of z intersects with $\langle x_4 \rangle$. Therefore

$$V(K_1; G) \oplus 2W(PGL(2, 2; q); G) \oplus 2W(PGL(3, 1; q); G) \oplus 2V(G)$$

is a gap module.

Next we show that $PGL(4, q)$ is a gap group for $q = 3, 5, 7, 9, 17$. Let $G = PGL(4, q)$. The group $PGL(3, 1; q)$ is also a gap group. Note that $[PGL(4, q) : PSL(4, q)] = 2$. Let

$$K_2 = \left\langle \phi\begin{pmatrix} x_2 & \\ & 1_2 \end{pmatrix}, \phi\begin{pmatrix} 1_2 & \\ & x_2 \end{pmatrix} \right\rangle.$$

Any element of $PGL(4, q)$ of order a power of 2 which is not conjugate to an element of either $PGL(3, 1; q)$ or $PSL(4, q)$ is conjugate to an element of K_1 or K_2. The order of $N_G(K_2)/K_2$,

$N_G(K_1)/K_1$ is divisible by 4, $4\left((q^4-1)/(q-1)\right)^{[2]}$ (≥ 4) respectively. Thus the module

$$V(K_1;G)\oplus V(K_2;G)\oplus 3W(PGL(3,1;q);G)\oplus 3V(G)$$

is a gap module.

Now we show that $G = PGL(n,q)$ is a gap group by induction on $n \geq 4$. We have already shown it for $n = 4$. Suppose $n \geq 6$ and that $PGL(r,q)$ is a gap group for $3 \leq r < n$. Note that $PGL(j, n-j; q)$ is a gap group for $1 \leq j \leq n/2$ as $PGL(n_2; q)$ is a gap group. Consider an element z of G outside of $PSL(n,q)$ of order a power of 2. If the conjugacy class of z does not intersect with $PGL(j, n-j)$ for any $1 \leq j \leq n/2$, then z is conjugate to an element of $\langle x_n{}^{(q^n-1)^{[2]}}\rangle$. Therefore the module

$$V(\langle x_n{}^{(q^n-1)^{[2]}}\rangle;G)\oplus \bigoplus_{1\leq j\leq \frac{n}{2}} 2W(PGL(j,n-j;q);G)\oplus 2V(G).$$

is a gap G-module. □

We can construct a gap module for $GL(n,q)$ quite similarly. Also remark that $GL(n,q)$ is a gap group as it has a quotient gap group $PGL(n,q)$.

Proposition 6.1. *Let $d \geq 2$, $k > 2$, and q a power of an odd prime. Let y_k be an element of order $q^k - 1$ of $GL(k,q)$ and*

$$A = \begin{pmatrix} y_k & & & \\ 1_k & y_k & & \\ & \ddots & \ddots & \\ & & 1_k & y_k \end{pmatrix} \in PGL(kd,q).$$

The group $M[k,d]$ generated by $\phi(A)$ is a gap group. Furthermore, if $q+1$ is not a power of 2, then $M[2,d]$ is also a gap group. □

Proof. A cyclic group is a gap group if and only if its order is divisible by distinct two odd primes. Since the (2, 1)-entry of A^r is $ry_k{}^{r-1}$, the order of $\phi(A)$ is divisible by $q(q^k-1)/(q-1)$. Suppose $\frac{q^k-1}{q-1}$ is a power of 2. the number k is even, say $2m$, as $(q^k-1)/(q-1) = q^{k-1}+q^{k-2}+\cdots+1 \equiv k \mod 2$. Let $(q^m-1)/(q-1) = 2^a$ and $q^m+1 = 2^b$. Then $2^b - 2 = 2^a(q-1)$ and thus $a = 0$, $m = 1$, $k = 2$, and $q = 2^b - 1$ which is a contradiction. Hence $(q^k-1)/(q-1)$ is divisible by an odd prime and $q(q^k-1)/(q-1)$ is divisible by distinct two odd primes. □

7. Direct product with $PGL(n,q)$

We write a result with respect to direct groups with $PGL(n,q)$ without a proof. Recall that $PGL(2,2)$ is a dihedral group and $PGL(2,3)$ and $PGL(2,5)$ are isomorphic to symmetric groups Σ_4 and Σ_5 respectively. Direct product groups of these groups are considered in [MSY, Su1].

Let $p > 1$ and $q > 1$ be both powers of an odd prime, The group $PGL(2,q)\times C_p$ is not a gap group if and only if $q = 2,3$. Under $p \leq q$, the group $PGL(2,p)\times PGL(2,q)$ is not a gap group if and only if $(p,q) = (2,2),(2,3),(2,5),(2,7),(2,9),(2,17),(3,3)$. All direct product groups of $GL(2,3)$'s are not gap groups. It is also known when a direct product group of projective linear groups is a gap group. More general it is considered in [Su2] for $G_1\times G_2$ with $[G_1 : O^2(G_1)] = [G_2 : O^2(G_2)] = 2$.

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